Derivation of the Vlasov equation

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The microscopic system

- N interacting particles (for example stars), Newtonian dynamics
- Trajectory in phase space:
  $X = (Q, P) = (q_1, q_2, \dots q_N, p_1, p_2, \dots p_N) \in \mathbb{R}^{6N}$
- q_j: position of particle j
  p_j momentum (=speed) of particle j
- Newtonian dynamics: $\dot{Q} = P$
  $\dot{P} = F(Q)$
  Force on j^{th} particle: $(F)_j = N^{-1} \sum_{k \neq j} f(q_j - q_k)$
- Macroscopic: law of motion for particle density
- Vlasov equation: $\frac{\partial}{\partial t}\rho + \nabla_q \rho \cdot p + \nabla_p \rho \cdot (\rho \star_q f) = 0$

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  Understand X_t as density: $\rho_t^{emp} = \sum_{j=1}^{N} \delta(x - x_j)$
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  $f(q) = \pm \frac{q}{|q|^3}$
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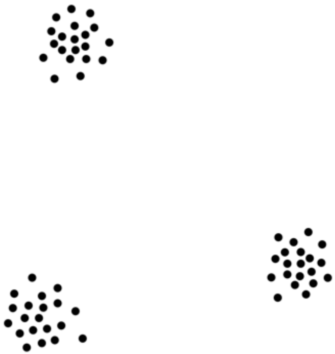
Dynamics of clusters

ϵ^{-1} cluster, ϵN particles each



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- Goal: compare trajectory X_t with density ρ_t.
- Idea: bring ρ_t to level of trajectories
- Vlasov equation: $\frac{\partial}{\partial t}\rho + \nabla_q \rho \cdot p + \nabla_p \rho \cdot (\rho \star_q f) = 0$
- Define $\dot{\overline{Q}} = \overline{P} \qquad \dot{\overline{P}} = \overline{F}(Q)$
- $(\overline{F})_j = \rho \star_q f(t, q_j)$
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Our results

- Niklas Boers, P.P. (2015): $f(q) = \pm \frac{q}{|q|^{3-\delta}}$, cut-off: $N^{-1/3}$ (= distance to nearest neighbour)
- Dustin Lazarovici, P.P. (2015): $f(q) = \pm \frac{q}{|q|^3}$, cut-off: $N^{-1/3+\delta}$
- $(q_j(0), p_j(0))$ independent and identically distributed
- Sample space $\Omega = \mathbb{R}^{6N}$, Probability measure $\mathbb{P}$ with density $\prod_{j=1}^N \rho(0, q_j, p_j)$
- $Q(t), P(t), \overline{Q}(t), \overline{P}(t)$ random variables
- For any t: $(\overline{q}_j(t), \overline{p}_j(t))$ independent and identically distributed (Probability density $\prod_{j=1}^N \rho(t, q, p)$)!
- Law of large numbers: empirical density of $(\overline{Q}(t), \overline{P}(t))$ converges in probability against $\rho(t, q, p)$. Show $(Q, P) \approx (\overline{Q}, \overline{P})$.

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Our results

- Niklas Boers, P.P. (2015):
  $f(q) = \pm\frac{q}{|q|^{3-\delta}}$, cut-off: $N^{-1/3}$ (= distance to nearest neighbour)
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Sub-Coulomb case

- $A_t := \left\|(Q_t, P_t) - (\overline{Q}_t, \overline{P}_t)\right\|_\infty \geq N^{-1/3}$
- Theorem: for any t holds: $\lim_{N\to\infty} \mathbb{P}(A) = 0$
  $\|\cdot\|_\infty$ maximum norm on $\mathbb{R}^{6N}$.
- $(\mathbb{P}(A) \leq C_\gamma N^{-\gamma}$ for any $\gamma > 0)$
- Define: $J(t) = \min\left\{N^{1/3}\left\|(Q,P) - (\overline{Q},\overline{P})\right\|_\infty, 1\right\}$
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$$J(t) = min\left\{N^{1/3}\left\|(Q,P)-(\overline{Q},\overline{P})\right\|_{\infty},1\right\}$$

- Expectation value easier to control than $\mathbb{P}$: $\mathbb{P}(A) = \mathbb{E}(\chi_A)$
- J_t has reached its maximum when $\left\|(Q,P)-(\overline{Q},\overline{P})\right\|_{\infty} = N^{-1/3}$.
- We only need to consider the case $\left\|(Q,P)-(\overline{Q},\overline{P})\right\|_{\infty} < N^{-1/3}$ (boundary condition)

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  Use the second order of the equation
- Derivation of Vlasov-Maxwell
- Derivation of other equations, e.g. Keller-Segel equation

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