The isochrone model : a fundamental state for self-gravitating systems

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Collisionless Boltzmann (Vlasov) Equation and Modeling of Self-Gravitating Systems and Plasmas – CIRM

Luminy – October 29th to November 3rd 2017

Outline

1) Astrophysical systems in the context of the Vlasov-Poisson system

2) Isochrony & Self–gravitating systems (details & proofs in SPD 2017 Com. Math. Phys.)

3) The fundamental isochrone state

4) Some conclusions

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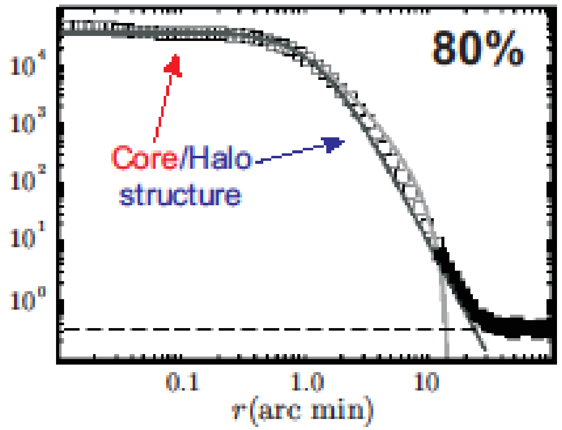
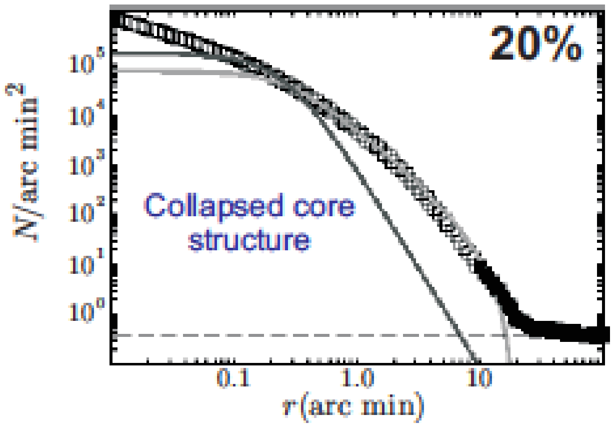
Astrophysical systems in the context of the Vlasov-Poisson system





Globular clusters

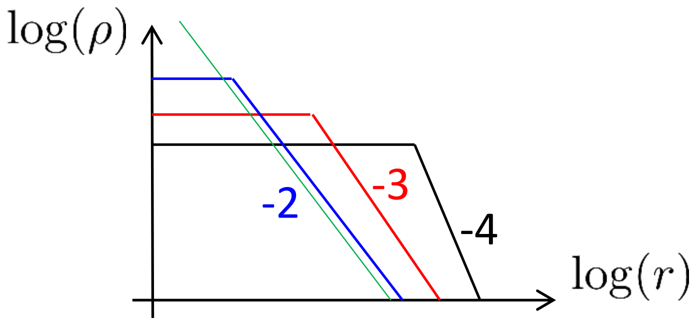
Spherical systems with $N=10^{\,4\text{-}5}$ particles

~ isolated systems in the galactic potential

« Standard model »
King = truncated isothermal sphere

$$f(E) \propto \mathrm{e}^{-\beta E} \times 1_{E<E_\ell}$$

Evolution effect due to dissipation
other long time scales : $N\,T_{\mathrm{d}}\;/\ln(N)$



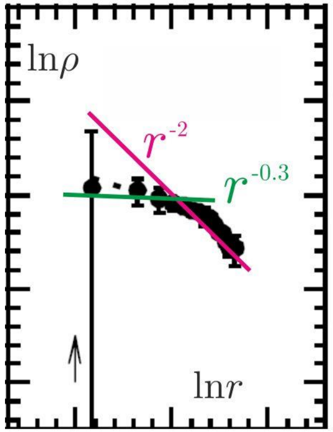
Astrophysical systems in the context of the Vlasov-Poisson system

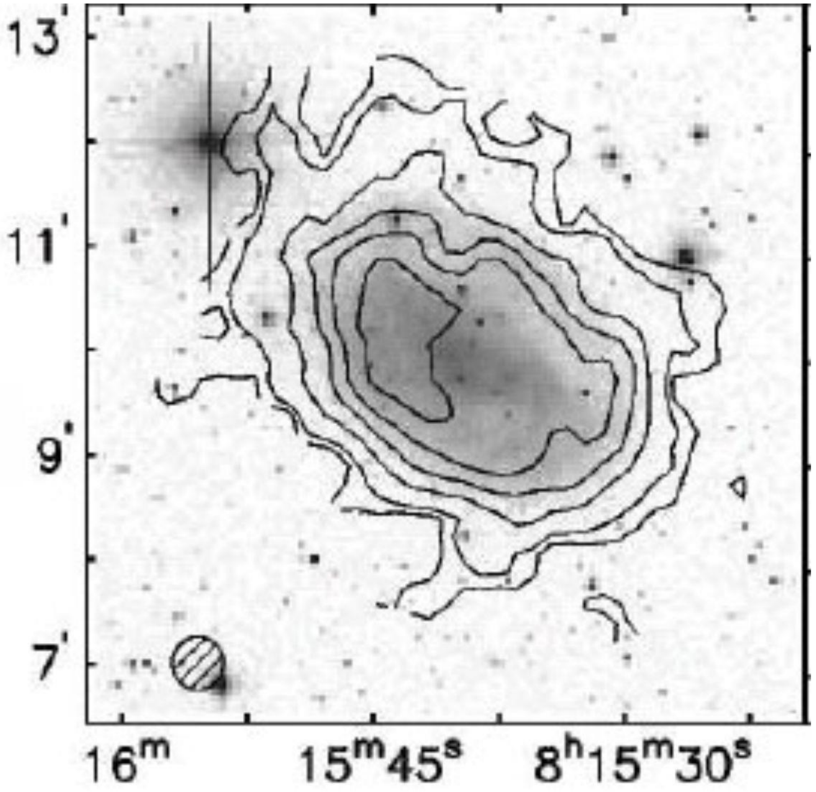
$$\rho(r) = \rho_o \left(\frac{r}{R_e}\right)^{-p_n} \exp\left[-b_n \left\{\left(\frac{r}{R_e}\right)^{1/n} - 1\right\}\right]$$

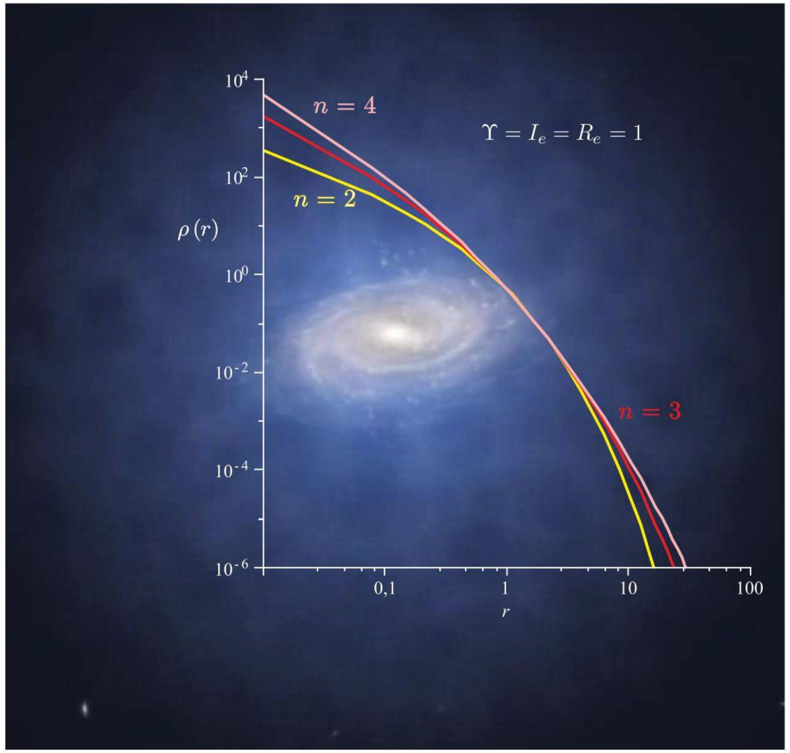
Deprojection of Sersic $R^{1/n}$ Law,... Einasto-Sersic

Galaxies (@ z=0)

UGC 4325
(typical)







Triple power law structure with cusp

Non isolated system

Hierachical formation process

Low Surface Brightness(LSB): 90% of all galaxies

Core Halo structure

Isolated systems

CDM Simulations & High Surface Brightness (HSB)

Spherical systems including the dark matter halo ?

$N = 10^{9\text{ - }12}$ particles

Theorized 1972 – 1st obs. 1986 – Large catalogue for 10 years

Theory not so restrictive !

Equilibrium

$f(E)$ or $f(E,L^2)$: spherical in the position space by Gidas, Ni, Niremberg 1984 Theorem (e.g. P&Aly, 1996)

$f(E,I)$: it depends on I

Violent relaxation : not clear yet…

A clever remark by Michel Hénon in 1958 …

Stability analysis

Jeans' instability for homogeneous too dense systems

Decreasing spherical systems generally stable if not too radial (See Binney&Tremaine)

Gravothermal catastrophe to understand long time evolution (ask P.H. Chavanis)

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A few words about isochrony (Hénon, 1958; SPD 2017 Com. Math. Phys.)

Spherical system : $\psi = \psi(|\vec{r}|)$

Each test particle with $E = m\xi$ and $\vec{L} = m\vec{\Lambda}$ moves in a plane.

Orbital equation

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = \xi - \psi(r) - \frac{\Lambda^2}{2r^2}$$

$r(t)$ is periodic for bounded & non radial orbits

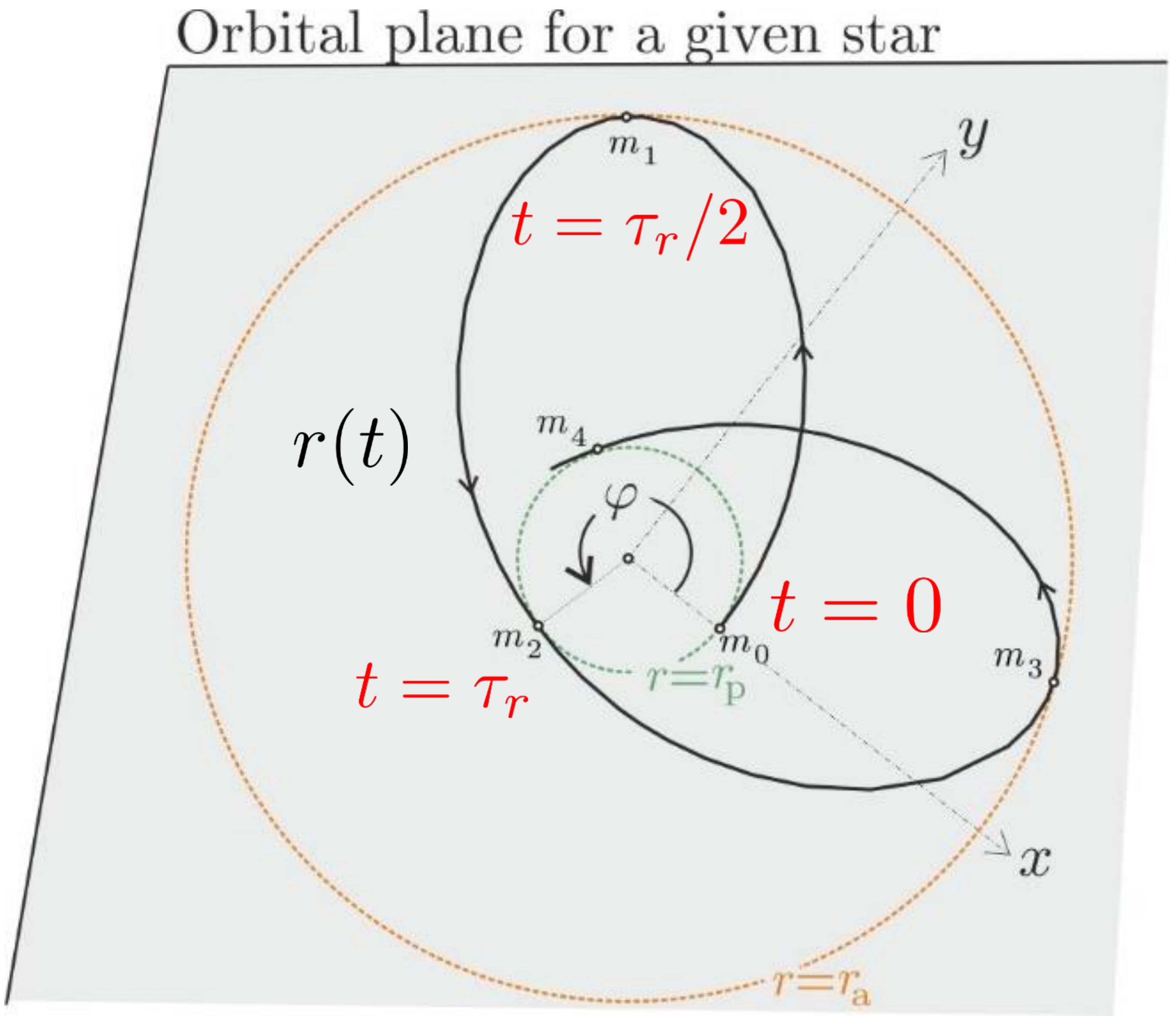
$$\forall t, \quad r(t) = r(t+\tau_r)$$

$$\tau_r = 2\int_{r_a}^{r_p} \frac{dr}{\sqrt{2[\xi - \psi(r)] - \frac{\Lambda^2}{r^2}}}$$

generally depends on ξ and Λ

*Def. An **orbit** is **isochrone** if $\tau_r = \tau_r(\xi)$*

*A **potential** is isochrone if **all** bounded & non radial orbits are isochrone.*



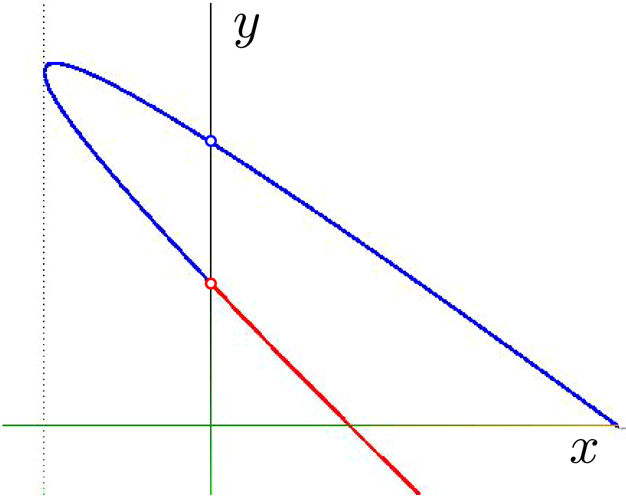
ISOCHRONE CLASSIFICATION : there are 4 classes of isochrone potentials …

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = \xi - \psi(r) - \frac{\Lambda^2}{2r^2} \xrightarrow[\left[x = 2r^2,\, y(x) = x\psi(x)\right]]{\text{Hénon variables}} \frac{1}{16}\left(\frac{dx}{dt}\right)^2 = \xi x - y(x) - \Lambda^2$$

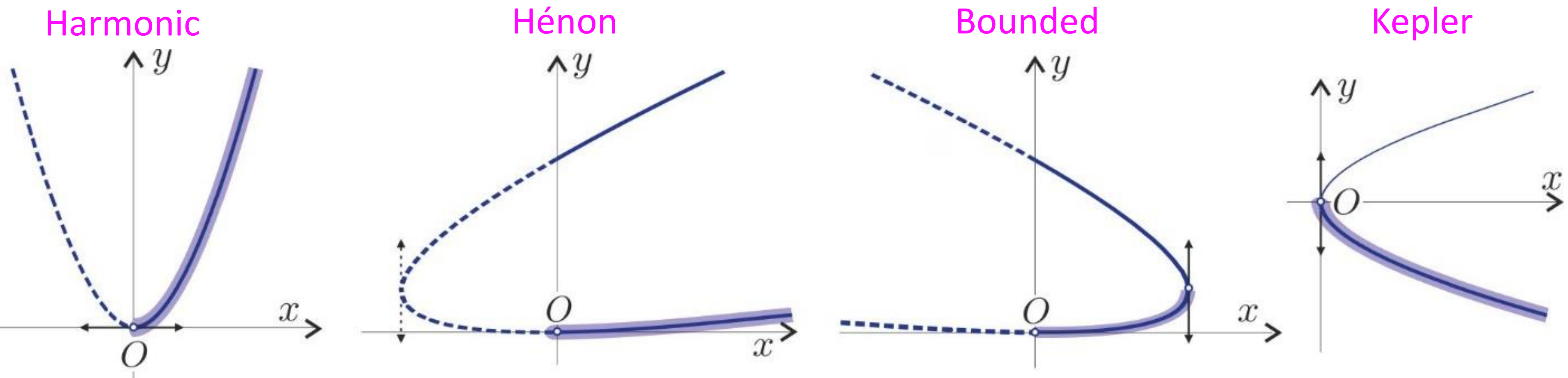
*Thm. A potential is **isochrone** iff the graph of $y(x)$ is a **parabola**.*

2 parameters for the orbital equation, i.e. ξ and Λ :
- If $\xi \to \xi + \epsilon$ then $(x, y) \to (x, y + \epsilon x)$ (Transvection)
- If $\Lambda^2 \to \Lambda^2 + \lambda^2$ then $(x, y) \to (x, y + \lambda^2)$ (y–Translation)

Fundamental isochrone group : $\mathbb{A} = \left\{(\epsilon, \lambda) \in \mathbb{R}^2,\ (x, y) \to (x, y + \epsilon x + \lambda^2)\right\} \simeq (\mathbb{R}^2, +)$



Action of a transvection on a parabola



$\psi_{\rm ha}(r) = \frac{1}{2}\omega^2 r^2$

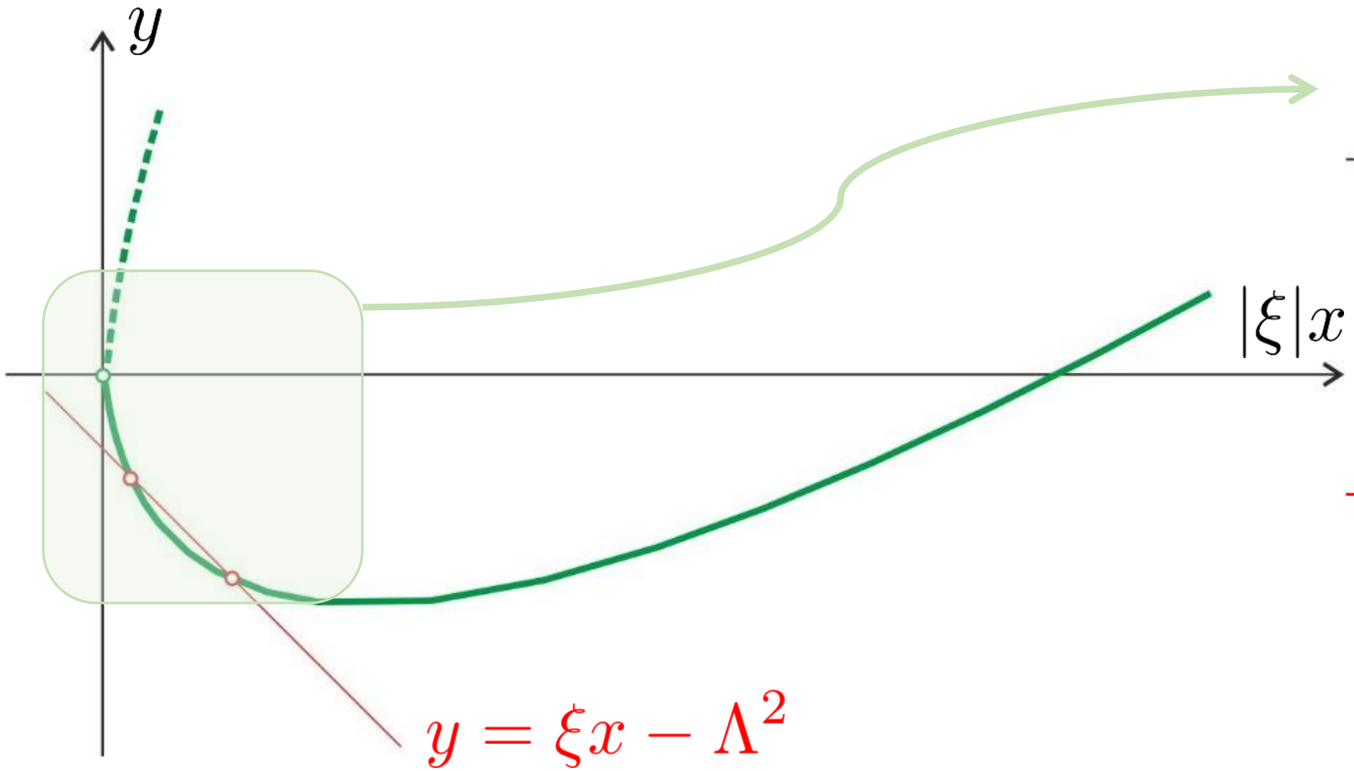
$\psi_{\rm he}(r) = -\frac{\mu}{b + \sqrt{b^2 + r^2}}$

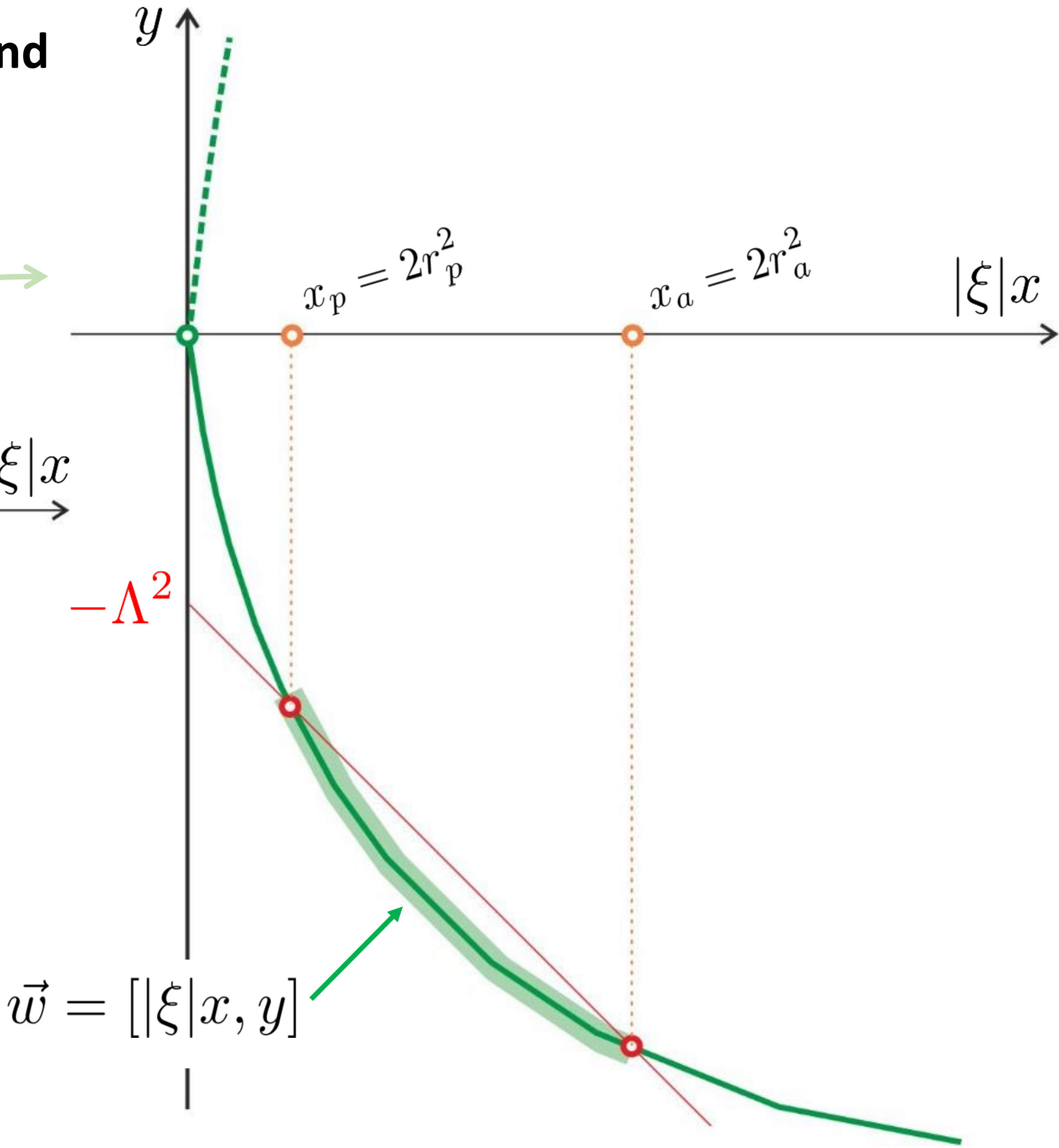
$\psi_{\rm bo}(r) = \frac{\mu}{b + \sqrt{b^2 - r^2}}$

$\psi_{\rm ke}(r) = -\frac{\mu}{r}$

… because there are 4 group orbits under $\mathbb{A}-$action on parabolas !

Draw your own isochrone orbit of energy ξ and momentum Λ in the potential ψ





$$\frac{1}{16}\left(\frac{dx}{dt}\right)^2 = \xi x - y(x) - \Lambda^2$$

Orbital equation

A reference frame for each isochrone parabola

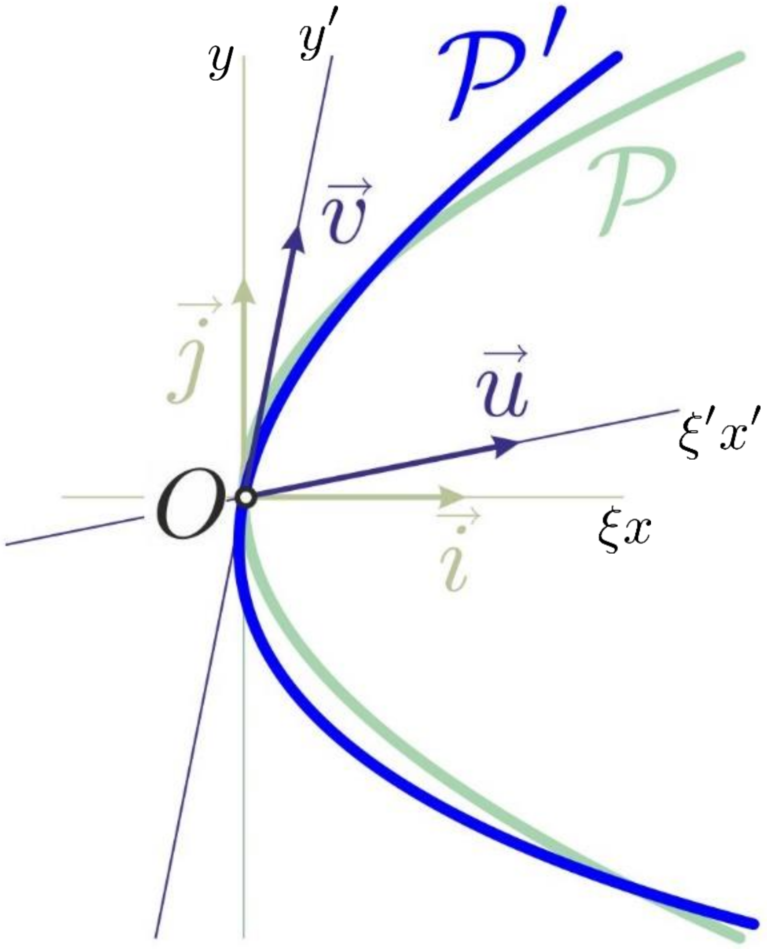
Def. $\mathscr{R} = \left(O, \vec{i}, \vec{j}\right)$ is the reference frame of the parabola $\mathcal{P}$

Using the affine coordinate system $\vec{w} = [\xi x, y]$ the orbital equation writes

$$\frac{1}{16}\left[\frac{d}{d\tau}\left(\vec{w}|\vec{i}\right)\right]^2 = \left(\vec{w}|\vec{i}-\vec{j}\right) + \left(\vec{w}_\Lambda|\vec{j}\right)$$

$d\tau = \xi dt$
Proper time

$$\frac{1}{16}\left(\frac{dx}{dt}\right)^2 = \xi x - y(x) - \Lambda^2$$



Look for the more general ***linear*** transformation B such that *(D. Lynden-Bell idea)*

if $\vec{w}' = B(\vec{w})$ then $\xi' x' - y'(x') = \xi x - y(x)$

B generalizes the Bohlin-Levi-Civita Transformation to all isochrones :
if $\vec{w}$ is an isochrone orbit in the isochrone ψ
then $\vec{w}'$ is an isochrone orbit in the isochrone ψ'

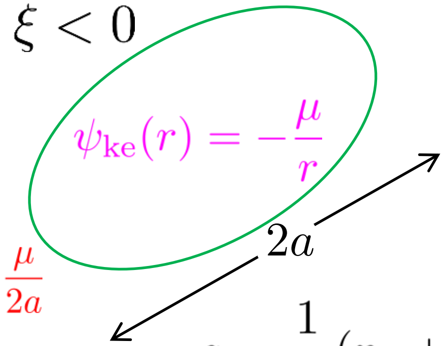
$$\frac{1}{16}\left[\frac{a}{d\tau'}\left(\vec{w}'|\vec{u}\right)\right] = \left(\vec{w}'|\vec{u}-\vec{v}\right) + \left(\vec{w}'_\Lambda|\vec{v}\right)$$

if $\mathcal{P}$ is the keplerian parabola, ψ is the keplerian potential

ψ' is generally not keplerian in $\mathscr{R}$ but it is always keplerian in $\mathscr{R}'$

The isochrony property is relative to a frame
Isochrone theory of relativity

Consequence of the isochrone relativity



$\xi < 0$

$\psi_{\mathrm{ke}}(r) = -\frac{\mu}{r}$

$2a$

$\xi = -\frac{\mu}{2a}$

$$a = \frac{1}{2}\left(r_a + r_p\right)$$

Kepler Law 3 ***(1618)***

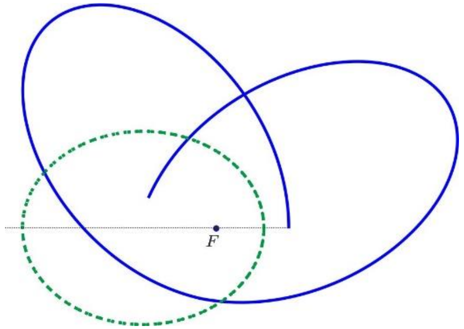
$$\tau_r^2 = \frac{4\pi^2}{\mu} a^3$$

Kepler is Isochrone

3 other fundamental isochrones

Hénon

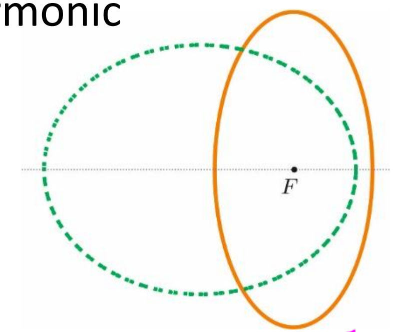
$$\psi_{\mathrm{he}}(r) = -\frac{\mu}{b + \sqrt{b^2 + r^2}}$$



$$a = \frac{1}{2}\left(\sqrt{b^2 + r_a^2} + \sqrt{b^2 + r_p^2}\right)$$

For all isochrones

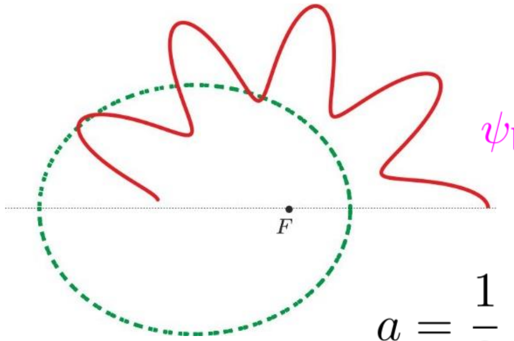
Harmonic



$$a = 2^{-2/3} R$$

$$\psi_{\mathrm{ha}}(r) = \frac{1}{2}\omega^2 r^2$$

Bounded



$$\psi_{\mathrm{bo}}(r) = \frac{\mu}{b + \sqrt{b^2 - r^2}}$$

$$a = \frac{1}{2}\left(\sqrt{b^2 - r_a^2} + \sqrt{b^2 - r_p^2}\right)$$

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Hénon's proposition 1958

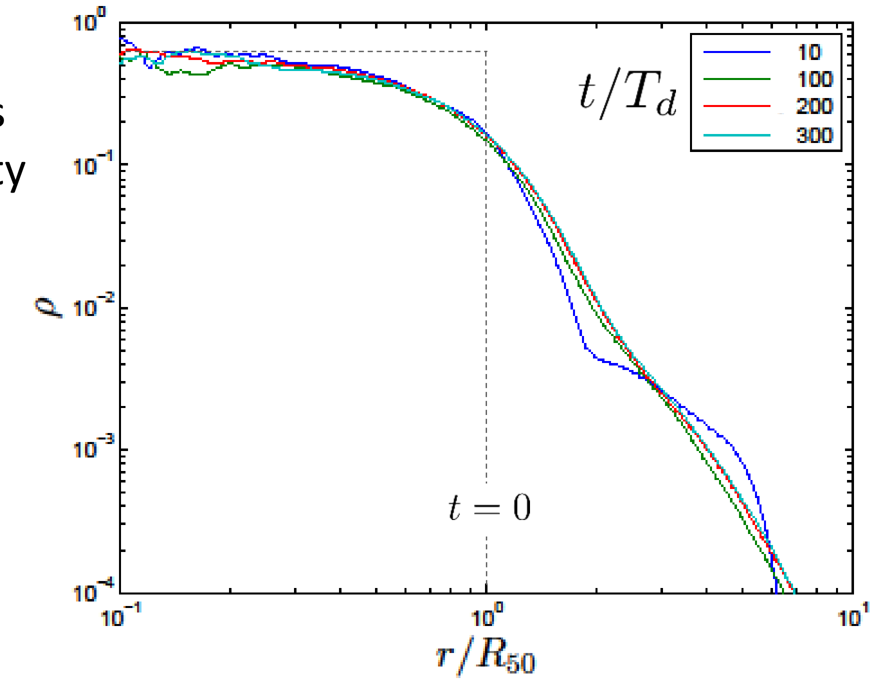
The isochrone model (Hénon) is the fundamental state in which an isolated self-gravitating system settles down after a violent relaxation (which was not yet described in 1958…).

Mechanism invoked : resonance (Physical proof needed…)

Forgotten because …

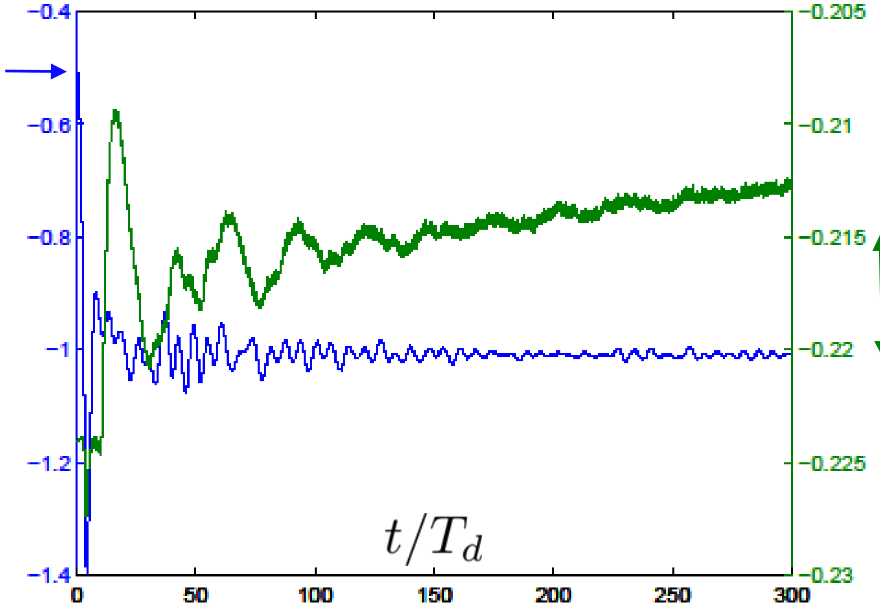
- This proposition was published at the very end of the paper in french…
- Because globular clusters evolve and donnot stay in this initial fundamental state;
- The mechanism of this evolution was poorly known at this epoch;
- The « hand-made » King(1966) model (with 3 free parameters !) fits very well the data (mass density profile);
- LSB galaxies were theorized to exist in 1976 and studied in detail since 2000 …
- This proposition doesn't apply to HSB galaxies, which are the most studied and simulated : non isolated systems
- … ?

Mass density



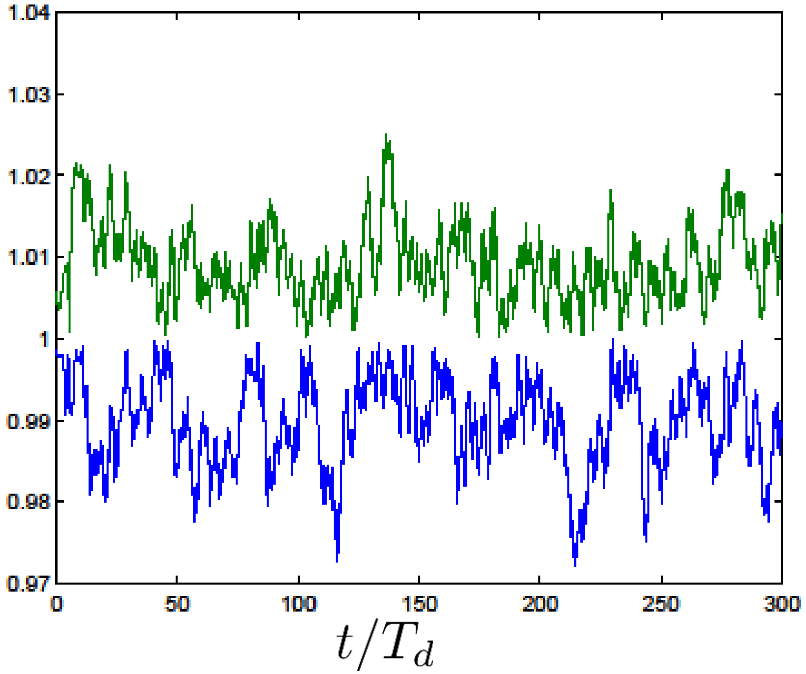
Initial value

Virial ratio



Total Energy

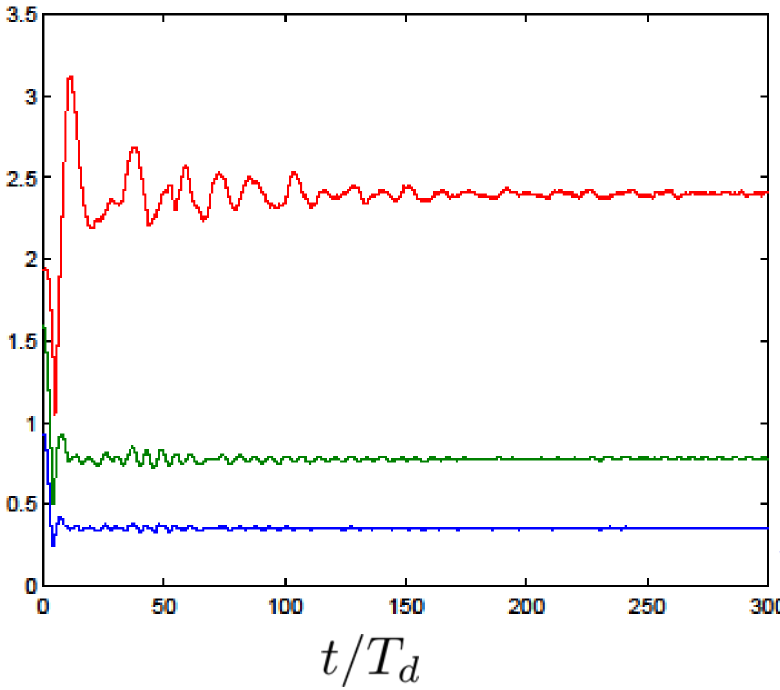
Axial ratio



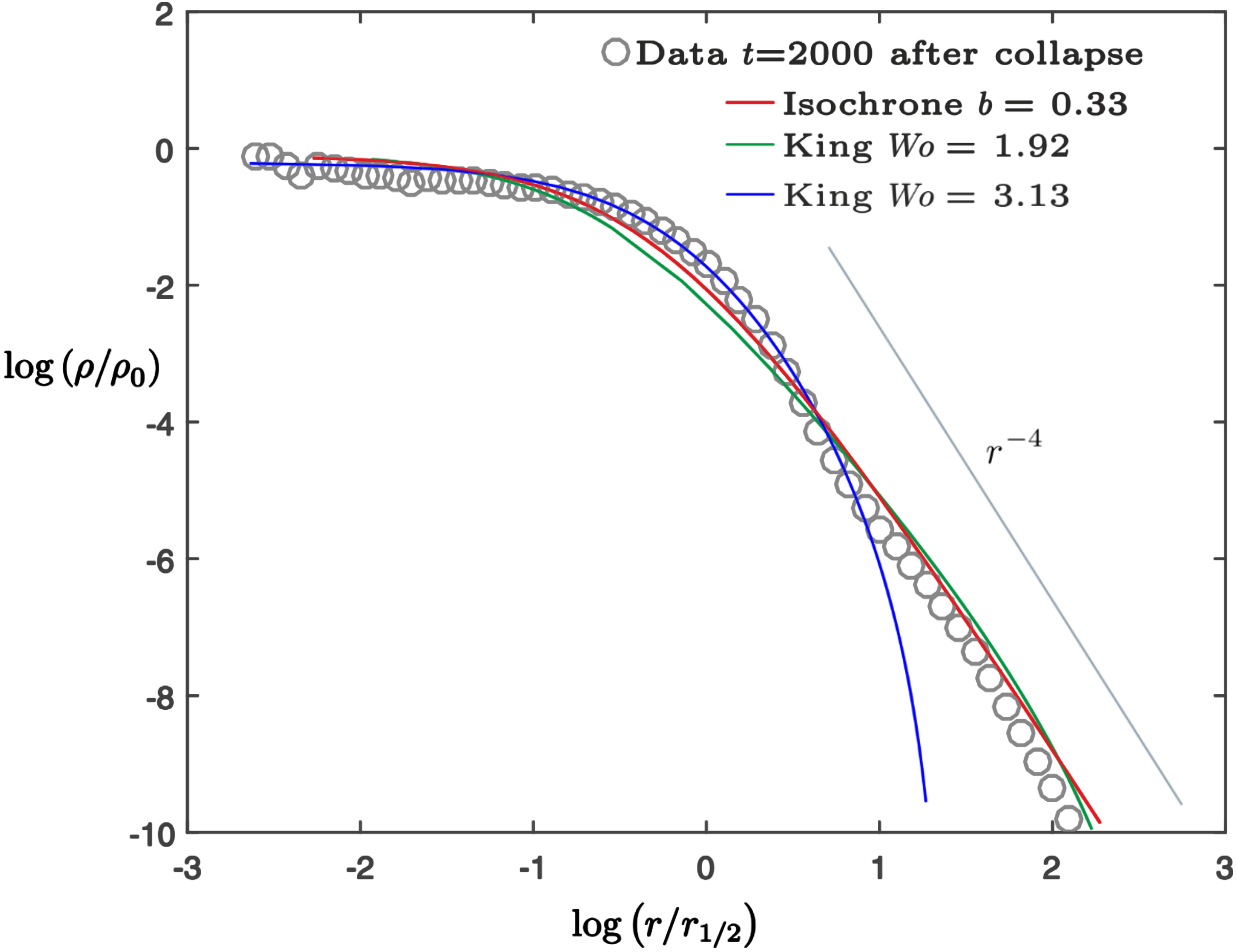
Collapse of 30K particles
Equal mass

Uniform in phase space
R=1 at t=0

Tree Code (Gadget 2)
$\varepsilon \sim R_{10}/10$



Mass density analysis

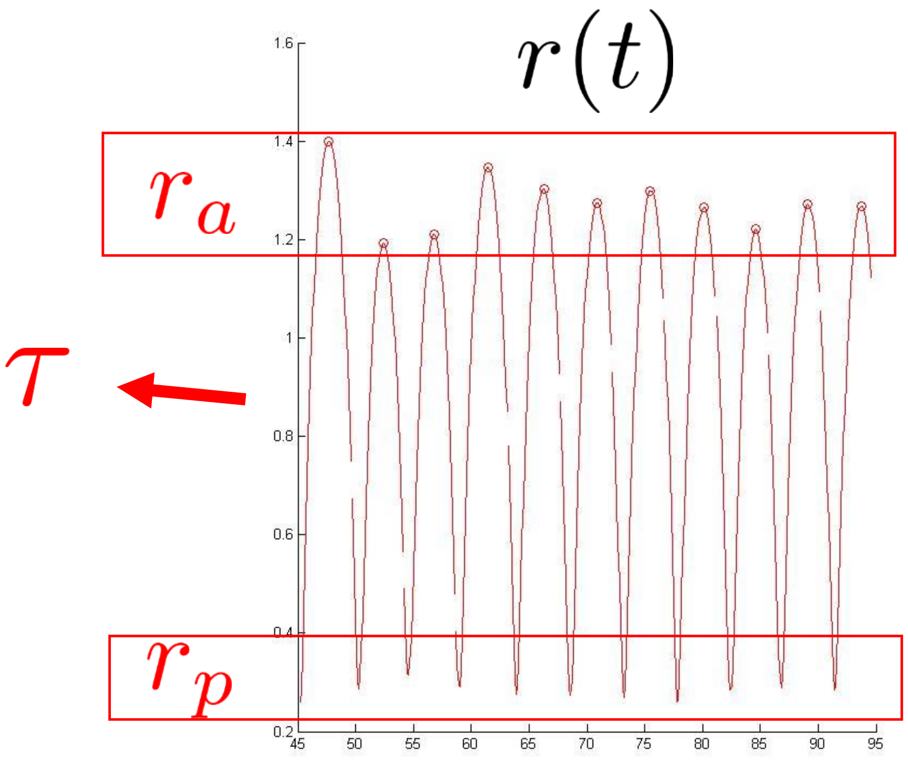


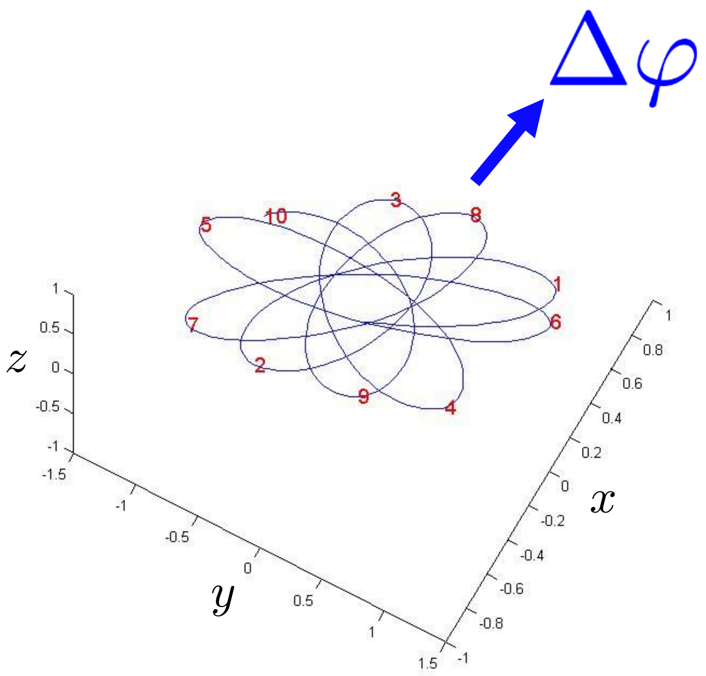
$T_d \simeq 10$

Ambiguous !

Isochrony analysis

(n = 200 orbits analyzed)

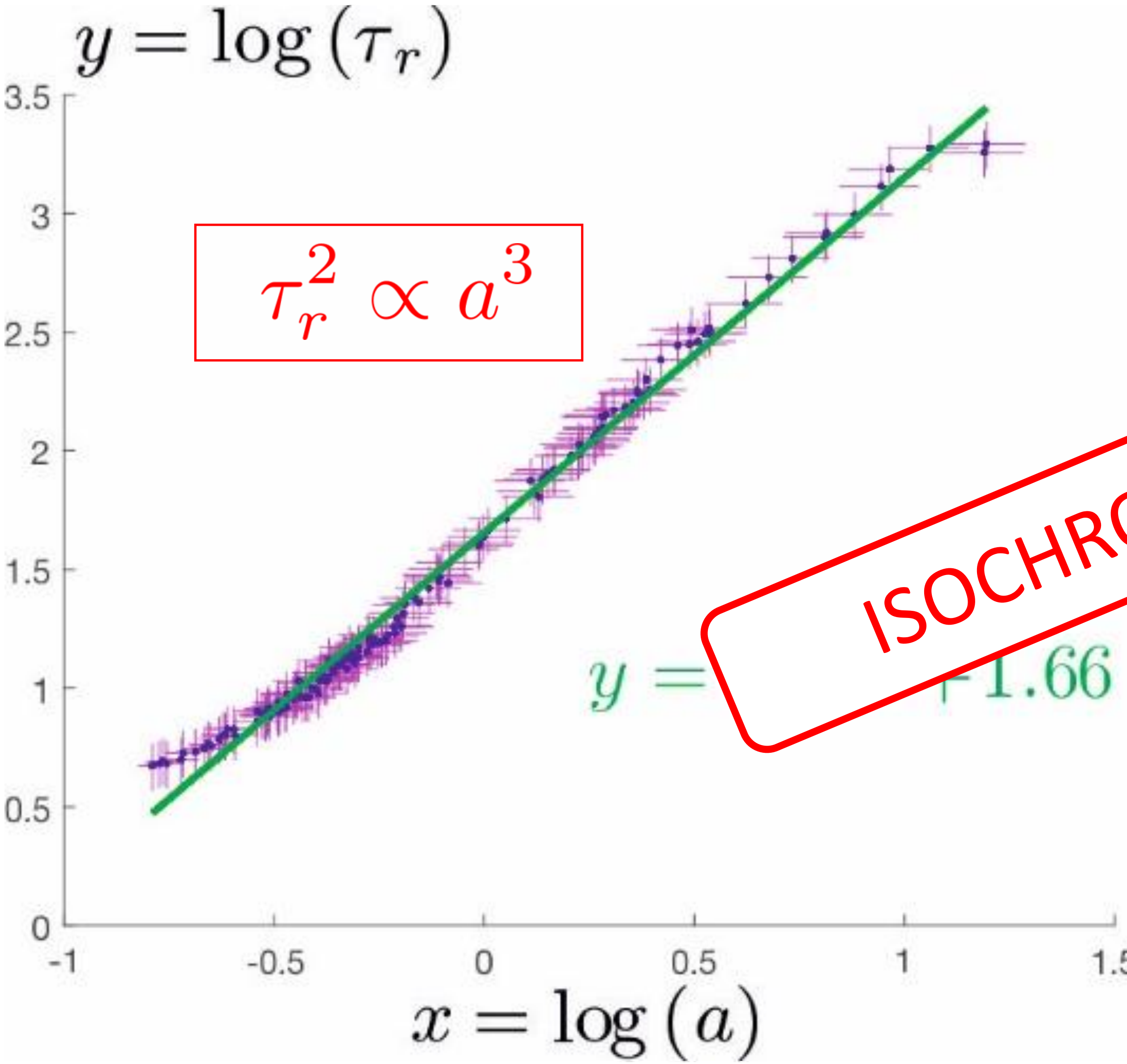


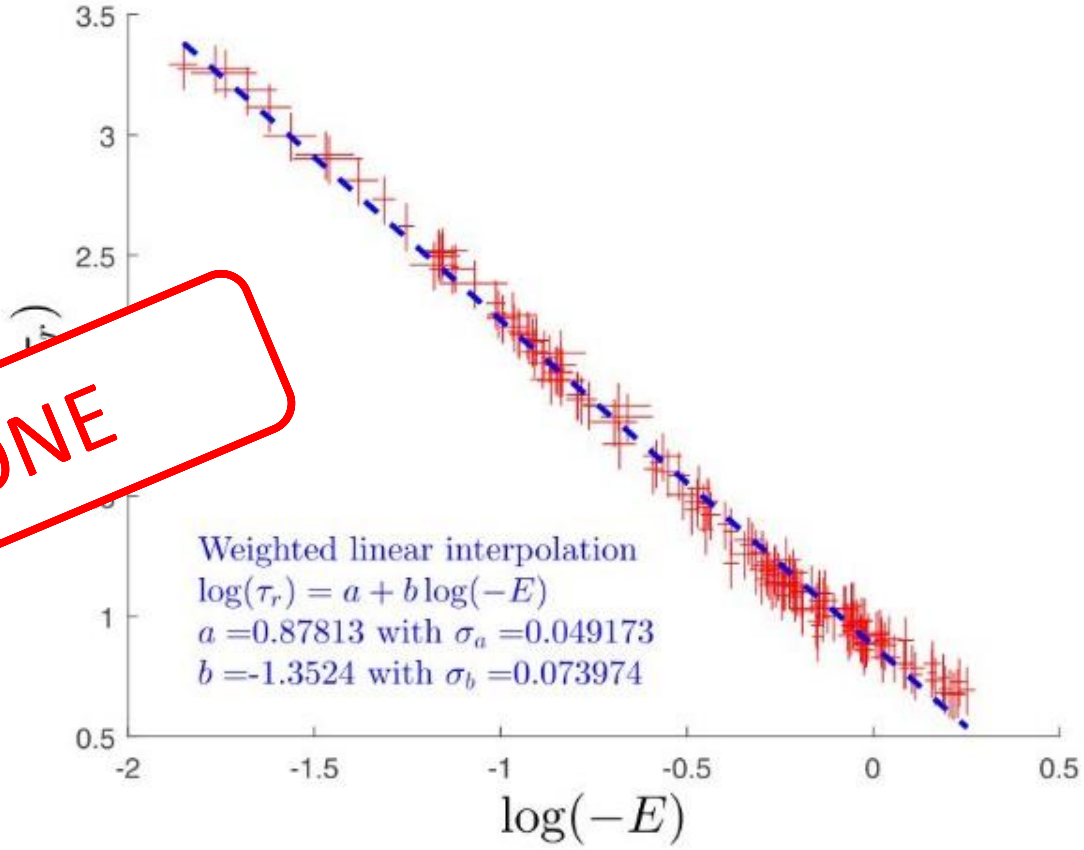


$$E = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left[\frac{1}{2} m \left(\frac{d\vec{r}}{dt} \right)^2 + m\psi(\vec{r}) \right] dt$$

$$L^2 = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left(m\vec{r} \wedge \frac{d\vec{r}}{dt} \right)^2 dt$$

Isochrony analysis

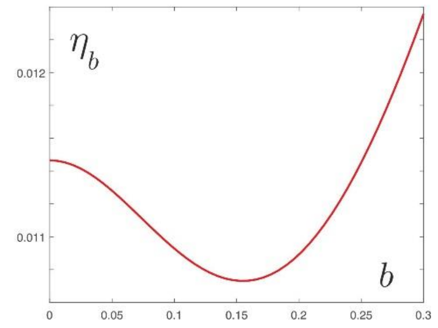




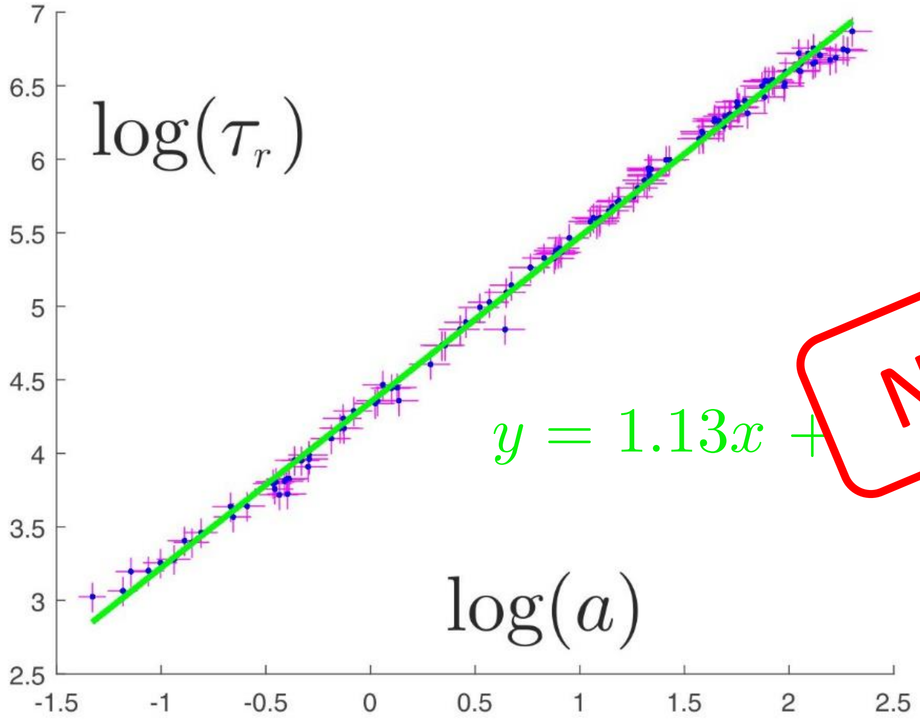
$$\tau_r = \tau_r(E) \simeq cst.E^{-3/2}$$

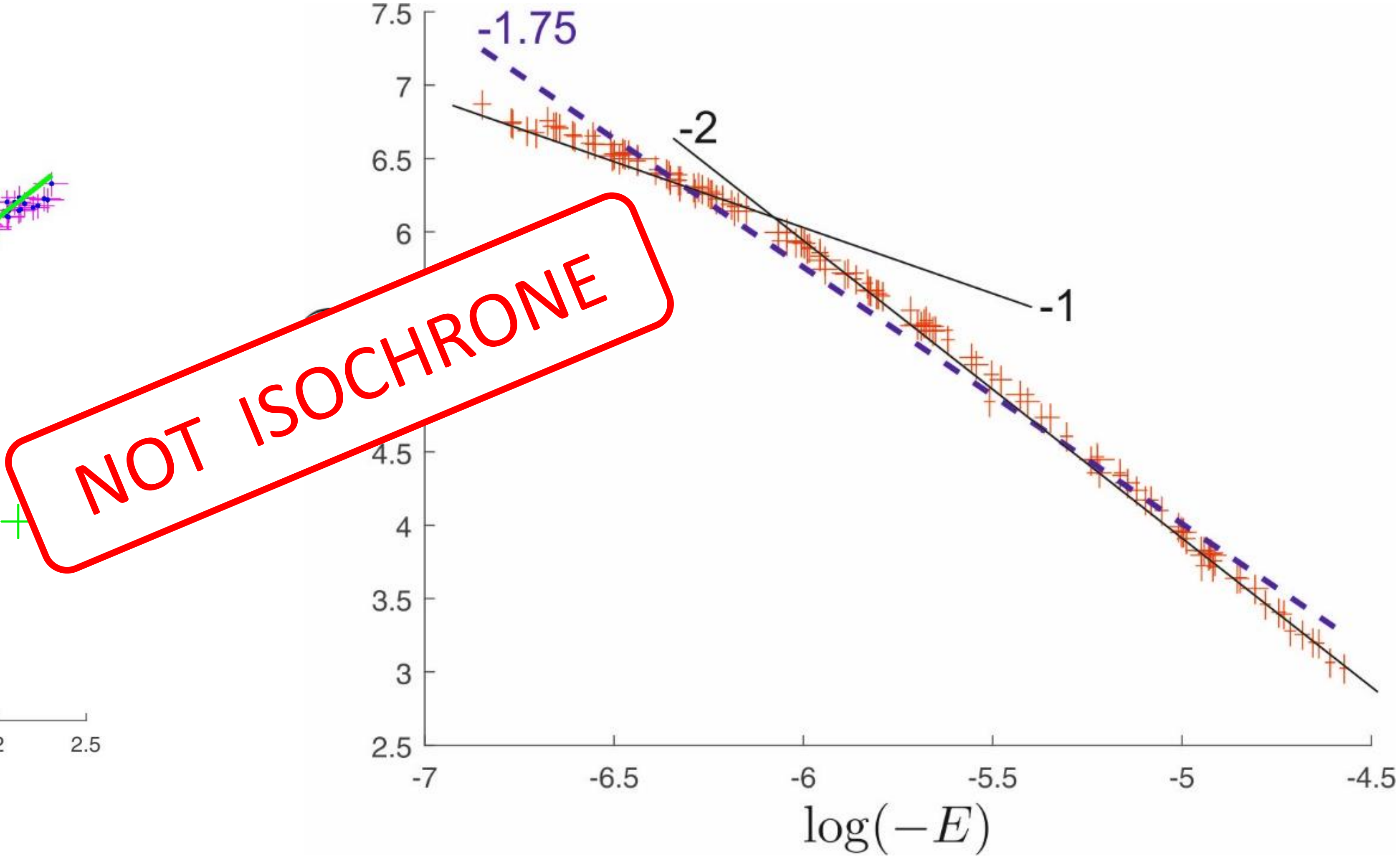
Same analysis for a King Model …

$Wo = 1.8$: isochrone-like density



Best b : 0.15





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Isochrony appears to be fundamental for dynamics in self-gravitating systems
(For physicists as well as for mathematicians)

Understand the resonance Hénon mecanism seems crucial !

Pay more attention to LSB galaxies to understand HSB ones…

Thank you for your attention