Action principle for relativistic magnetohydrodynamics

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Collisionless Boltzmann (Vlasov) Equation and Modeling of Self-Gravitating Systems and Plasmas

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Overview

We move on from the covariant Poisson bracket theory of Marsden et al. [Ann. Phys. **169***, 29 (1986)], which uses a noncanonical bracket to perform constrained variations of an action functional.*

In this article this approach is used in order to obtain “Poisson brackets that are spacetime covariant [..] for a variety of relativistic field theories” including “electromagnetism, general relativity, and general relativistic fluids and plasmas in Eulerian representation”.

Here we follow a similar line of reasoning in order to construct a covariant action principle for ideal relativistic magnetohydrodynamics (MHD) in terms of natural Eulerian field variables.

Main reference

This presentation is based on the article

Action principle for relativistic magnetohydrodynamics

Eric D'Avignon, P. J. Morrison, F. Pegoraro
Physical Review D, **91**, 084050 (2015)

and on the PhD thesis

Aspects of Relativistic Hamiltonian Physics

by Eric D'Avignon,
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General framework

In the approach of Marsden *et al.* the field equations are shown to be equivalent to equations of the form

$$\{\mathscr{F}, \mathscr{S}\} = 0$$

with $\mathscr{F}$ an arbitrary function of the fields and $\mathscr{S}$ an action integral.

For the Relativistic Maxwell-Vlasov equations the particle Hamiltonian in an external electromagnetic field $F_{\mu\nu}$ con be written as

$$H = (m/2)\, u^{\mu} u_{\mu} = 1/(2m)\, (P^{\mu} - qA^{\mu}/c)\, (P_{\mu} - qA_{\mu}/c)$$

with P_{μ} the canonical momentum conjugate to x^{μ} and A_{μ} the 4-potential, leading to the Hamilton equations

$$dx^{\mu}/d\tau = \partial H/\partial P_{\mu}, \quad dP_{\mu}/d\tau = -\partial H/\partial x^{\mu} = (q/c)\, u^{\nu}\, \partial A_{\nu}/\partial x^{\mu}.$$

General framework, particles

The relativistic plasma distribution function in 8-D phase space $f(x,P)\, d^4x d^4P$ is constant along its particle world lines

$$df/d\tau + u^\mu\, \partial f/\partial x^\mu + (q/c)\,(u^\nu\, \partial A_\nu/\partial x^\mu)\, \partial f/\partial P_\mu \; = 0$$

or, in different notation, $\{f, H\}_{x,P} = 0$

with $\{f, g\}_{x,P} = (\partial f/\partial x^\mu)(\partial g/\partial P_\mu) \; - (\partial g/\partial x^\mu)\,(\partial f/\partial P_\mu)$

The basic field for the Vlasov theory is the plasma phase space distribution function f. Marsden *et al.* define the bracket of two functionals $\mathscr{F}, \mathscr{G}$ of f in the Lie-Poisson form

$$\{\mathscr{F}, \mathscr{G}\}(f) = \int f\, \{(\delta\mathscr{F}/\delta f)\,, (\delta\mathscr{G}/\delta f)\}_{x,P}\; d^4x d^4P$$

Define $\mathscr{S}[f] = \int f(x,P)\, H(x,P)\, d^4x d^4P,$ so that $\delta\mathscr{S}/\delta f = H$.
Then, the covariant bracket equation

$$\{\mathscr{F}, \mathscr{S}\}(f) = 0 \qquad \text{for all} \quad \mathscr{F}$$

is equivalent to the relativistic Vlasov equation.

General framework, particles + fields

The fields in the relativistic Maxwell-Vlasov equations are the triples $(A_\mu, \pi^{\mu\nu}, f)$ where $\pi^{\mu\nu} = F^{\mu\nu}$ are covariant momentum variables. Including the Maxwell fields dynamics in action variables (not illustrated in this presentation) the bracket of two functions of A, π, f is

$$\{\mathscr{F},\mathscr{G}\}_V(A,\pi,f) = \int f\,\{(\delta\mathscr{F}/\delta f),(\delta\mathscr{G}/\delta f)\}_{x,P}\ d^4x d^4P$$

$$+\int \big((\delta\mathscr{F}/\delta A_\mu)\,(\delta\mathscr{G}/\delta\pi^{\mu\nu}) - (\delta\mathscr{G}/\delta A_\mu)\,(\delta\mathscr{F}/\delta\pi^{\mu\nu})\big)V^\nu\ d^4x.$$

with V^μ an arbitrary vector field[1]. Setting the total action

$$\mathscr{S}(A,n,f) = \int \big(\pi^{\mu\nu}A_{\mu,\nu} - (1/4)\,\pi^{\mu\nu}\pi_{\mu\nu}\big)\,d^4x$$

$$+\int f(x,P)/(2m)\,(P_\mu - qA_\mu/c)(P^\mu - qA^\mu/c)\ d^4x d^4P,$$

the full field equations are $\quad \{\mathscr{F},\mathscr{S}\}_V(A,\pi,f) = 0 \quad \text{for all} \quad \mathscr{F}, V.$

[1] it corresponds to the arbitrariness in the choice of the direction of time and to the lapse in a 3+1 formulation

MHD equations

First we recall the equations of ideal nonrelativisitic ideal MHD

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\frac{\nabla p}{\rho} + \frac{1}{4\pi\rho}\left[(\nabla\times\mathbf{B})\times\mathbf{B}\right]$$

$$= -\frac{\nabla p}{\rho} + \frac{1}{4\pi\rho}\nabla\cdot\left(\mathbf{I}\,B^2/2 - \mathbf{B}\otimes\mathbf{B}\right)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) = -\mathbf{B}\,\nabla\cdot\mathbf{v} + \mathbf{B}\cdot\nabla\mathbf{v} - \mathbf{v}\cdot\nabla\mathbf{B}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0, \qquad \frac{\partial s}{\partial t} + \mathbf{v}\cdot\nabla s = 0\,.$$

Here ρ is the fluid density, p its pressure, s its specific entropy, $\mathbf{v}$ the velocity field, $\mathbf{B}$ the magnetic field, $\mathbf{I}$ the identity tensor[2].

Should one wish to add displacement current back into MHD, as is done in the most prevalent version of relativistic MHD, the momentum equation would have to be altered as follows:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\frac{\nabla p}{\rho} + \frac{1}{4\pi\rho}\left[\left(\nabla\times\mathbf{B} + \frac{\partial}{\partial t}\left(\frac{\mathbf{v}}{c^2}\times\mathbf{B}\right)\right)\times\mathbf{B}\right].$$

[2] The current $\mathbf{j}$ and electric field $\mathbf{E}$ have been eliminated from these equations, but they can be recovered from the ideal Ohm's Law, $\mathbf{E} + (\mathbf{v}/c)\times\mathbf{B} = 0$, and Ampére's Law, $\mathbf{j} = (c/4\pi)\nabla\times\mathbf{B}$

Relativistic MHD

The 4-vector field u^μ will denote the plasma 4-velocity[3] *at each point in spacetime; at each such point, this quantity will define a reference frame with locally vanishing 3-velocity*.

The fluid density is now $\rho = mn(1+\varepsilon)$, where n is the baryon number density, m is the fluid rest mass per baryon and ε is the internal energy per baryon, normalized to m.

Instead of the specific entropy s we will use the entropy density

$$\sigma = ns.$$

We will suppose that the energy can be written as $\varepsilon(n,\sigma)$, hence $\rho(n,\sigma)$, in which case the pressure is given in terms of n and σ by $p = n\partial\rho/\partial n + \sigma\partial\rho/\partial\sigma - \rho$, and $Tds = d(\rho/n) + pd(1/n)$.

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[3]We use signature and units such that $u_\mu u^\mu = g_{\mu\nu}u^\mu u^\nu = 1$, where the Minkowski metric $g_{\mu\nu}$ is given by $\mathrm{dia}(1,-1,-1,-1)$

4-vector electromagnetic field representation

Given u^μ, one can also define the two 4-vectors

$$B^\mu \equiv \mathscr{F}^{\mu\nu} u_\nu = \gamma(\mathbf{v}\cdot\mathbf{B}, \mathbf{B} - \mathbf{v}\times\mathbf{E})$$

$$E^\mu \equiv F^{\mu\nu} u_\nu = \gamma(\mathbf{v}\cdot\mathbf{E}, \mathbf{E} + \mathbf{v}\times\mathbf{B})$$

where $\mathscr{F}_{\mu\nu} = \varepsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}/2$ is the dual of $F^{\mu\nu}$. In terms of the 4-vectors B^μ and E^μ the field tensor has the decomposition[4]

$$F^{\mu\nu} = \varepsilon^{\mu\nu\lambda\sigma} B_\lambda u_\sigma + (u^\mu E^\nu - u^\nu E^\mu)\ ,$$

a form valid for any timelike 4-vector u^μ. Different values of B^μ and E^μ can correspond to the same field tensor, for one can add any quantity proportional to u^μ to either 4-vector while leaving the field tensor unchanged; however the representation is unique if the constraints $E^\lambda u_\lambda = B^\lambda u_\lambda = 0$ are imposed.

[4] $\mathbf{B}^i = B^i$ and $\mathbf{E}^i = E^i$ in the reference frame defined by u^μ

Relativistic ideal MHD magnetic 4-vector

In MHD one eliminates the electric field from the theory. In a relativistic context, this is done by setting $E^\mu = F^{\mu\lambda} u_\lambda = 0$, which gives $\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$ and, in a given reference frame,

$$B^\mu = \gamma \left(\mathbf{v} \cdot \mathbf{B}, \mathbf{B}/\gamma^2 + \mathbf{v} \left(\mathbf{v} \cdot \mathbf{B} \right) \right), \, B_\mu B^\mu = -(\mathbf{B} \cdot \mathbf{B})/\gamma^2 + (\mathbf{v} \cdot \mathbf{B})^2) = -B^2_{\mathrm{rest}}.$$

For convenience $b^\mu \equiv B^\mu / \sqrt{4\pi}$ will be used, in which case the MHD field tensor and its dual have the forms

$$\begin{aligned} F^{\mu\nu} &= \sqrt{4\pi}\, \varepsilon^{\mu\nu\lambda\sigma} b_\lambda u_\sigma \\ \mathscr{F}^{\mu\nu} &= \sqrt{4\pi}\, (b^\mu u^\nu - u^\mu b^\nu) \,. \end{aligned}$$

The restriction $b^\lambda u_\lambda = 0$, can be lifted by defining a family of vectors

$$h^\mu = b^\mu + \alpha\, u^\mu$$

where α is an arbitrary scalar field and now, in general, $h^\mu u_\mu = \alpha \neq 0$. The field tensor $F^{\mu\nu}$ and its dual $\mathscr{F}^{\mu\nu}$ are unchanged when written in terms of h^μ.

Relativistic ideal MHD equations

Each equation of relativistic MHD can be written as the vanishing of a divergence:

$$\partial_\mu (n u^\mu) = 0, \qquad \partial_\mu (\sigma u^\mu) = 0, \qquad \partial_\mu T^{\mu\nu} = 0\,,$$

$$\partial_\mu \mathscr{F}^{\mu\nu} = 0, \quad \Rightarrow \quad \partial_\nu (b^\mu u^\nu - u^\mu b^\nu) = \partial_\nu (h^\mu u^\nu - u^\mu h^\nu) = 0$$

The stress-energy tensor $T^{\mu\nu} = T^{\mu\nu}_{fl} + T^{\mu\nu}_{EM}$ is considerably more complex when written in terms of h^μ rather than b^μ

$$T^{\mu\nu}_{fl} = (\rho + p)\, u^\mu u^\nu - p\, g^{\mu\nu}\,,$$

$$T^{\mu\nu}_{EM} = \frac{1}{4\pi}\left(F^{\mu\lambda} F_\lambda{}^\nu + \frac{1}{4} g^{\mu\nu} F_{\lambda\sigma} F^{\lambda\sigma}\right) = -b^\mu b^\nu - \left(b_\lambda b^\lambda\right) u^\mu u^\nu + \frac{1}{2} g^{\mu\nu} b_\lambda b^\lambda$$

$$= -h^\mu h^\nu - \left(h_\lambda h^\lambda\right) u^\mu u^\nu + \left(h_\lambda u^\lambda\right) \left(h^\mu u^\nu + u^\mu h^\nu\right) + \frac{1}{2} g^{\mu\nu} \left(h_\lambda h^\lambda - \left(h_\lambda u^\lambda\right)^2\right).$$

Note that $T^{\mu\nu}_{EM}$ does not depend on the choice of α. The field part $T^{\mu\nu}_{EM}$ depends on b^μ or h^μ only through the tensor $F^{\mu\nu}$ in which α cancels out. This system preserves $b^\mu u_\mu = 0$ and $u^\mu u_\mu = 1$.

Covariant Poisson bracket formulation

The covariant Poisson bracket formalism requires two parts:
i) an action $\mathscr{S}$ that is a covariant functional of the field variables and
ii) a covariant Poisson bracket $\{\,,\}$ defined on functionals of the fields.
Instead of the usual extremization $\delta S = 0$, the theory arises from setting $\{F, S\} = 0$ for all functionals F.

A general Poisson bracket for fields Ψ has the form

$$\{F, G\} = \int dz\, (\delta F/\delta \Psi)\, \mathscr{J}\, (\delta G/\delta \Psi),$$

where $\delta F/\delta \Psi$ is the functional derivative, dz is an appropriate spacetime measure, and $\mathscr{J}$ is a cosymplectic operator that provides $\{F, G\}$ with the properties of antisymmetry and the Jacobi identity.

Thus $\{F, S\} = 0 \quad \forall\, F \quad \Rightarrow \quad \mathscr{J}\, \delta S/\delta \Psi = 0\,.$

If $\mathscr{J}$ is nondegenerate the covariant Poisson bracket formalism reproduces the conventional variational principle. *MHD when written in terms of Eulerian variables possesses noncanonical Poisson brackets for which $\mathscr{J}$ possess degeneracy that is reflected in the existence of "Casimirs".*

In such a case the Poisson bracket naturally enforces constraints.

Covariant Action $S[n,\sigma,u,F]$

$$
\begin{aligned}
S[n,\sigma,u,F] &= \int d^4x \left(\frac{1}{2}(p+\rho)u_\lambda u^\lambda + \frac{1}{2}(p-\rho) - \frac{1}{16\pi} F_{\lambda\sigma}F^{\lambda\sigma} \right) \\
\Rightarrow S[n,\sigma,u,b] &= \frac{1}{2}\int d^4x \left((p+\rho - b_\lambda b^\lambda) u_\lambda u^\lambda + p - \rho \right) \\
\Rightarrow S[n,\sigma,u,h] &= \frac{1}{2}\int d^4x \left((p+\rho - h_\sigma h^\sigma) u_\lambda u^\lambda + (h_\lambda u^\lambda)^2 + p - \rho \right) .
\end{aligned}
$$

As in Marsden *et al.* the action is the sum of the fluid action together with the standard expression for the electromagnetic action[5].

Note that the EM term has an opposite sign than usual, and the coupling term is missing, because they have been combined via an integration by parts:

$$
\int d^4x \left(j^\mu A_\mu + \frac{1}{16\pi} F_{\lambda\sigma}F^{\lambda\sigma} \right) = \int d^4x \left(\frac{1}{4\pi} A_\mu \partial_\nu F^{\mu\nu} + \frac{1}{16\pi} F_{\lambda\sigma}F^{\lambda\sigma} \right) = \int d^4x \left(-\frac{1}{4\pi} F^{\mu\nu}\partial_\nu A_\mu + \frac{1}{16\pi} F_{\lambda\sigma}F^{\lambda\sigma} \right)
$$

$$
= \int d^4x \left(-\frac{1}{16\pi} F^{\mu\nu}F_{\mu\nu} \right), \qquad \text{using} \quad \partial_\nu A_\mu = (\partial_\nu A_\mu - \partial_\mu A_\nu)/2 + (\partial_\nu A_\mu + \partial_\mu A_\nu)/2.
$$

[5] The integrand when evaluated on the constraint $u_\lambda u^\lambda = 1$ is the total pressure $p + |b_\lambda b^\lambda|/2$.

Momentum variable. $u^\mu, b^\mu \Leftrightarrow m^\mu, h^\mu$

From the action one derives a momentum m_μ by functional differentiation,

$$m_\mu = \frac{\delta S}{\delta u^\mu} = (p + \rho - h_\sigma h^\sigma)\, u_\mu + \left(h_\lambda u^\lambda\right) h_\mu \equiv \mu u_\mu + \alpha h_\mu\,.$$

The quantity $\mu = p + \rho - h_\lambda h^\lambda$ is a modified enthalpy density.

$\alpha = h_\lambda u^\lambda$ and $u^\mu = (m^\mu - \alpha h^\mu)/\mu$ imply $\alpha = h_\lambda m^\lambda / (\mu + h_\sigma h^\sigma)$. Then

$$u^\mu = \frac{m^\mu}{\mu} - \frac{h_\lambda m^\lambda}{\mu(\mu + h_\sigma h^\sigma)} h^\mu$$

$$b^\mu = h^\mu \left(1 + \frac{(h_\lambda m^\lambda)^2}{\mu(\mu + h_\sigma h^\sigma)^2}\right) - \frac{h_\lambda m^\lambda}{\mu(\mu + h_\sigma h^\sigma)} m^\mu\,.$$

Action in momentum variables

Express the action in terms of the variables m^μ and h^μ which will turn out to be the appropriate variables for the covariant Poisson bracket:

$$S[n,\sigma,m,h] = \frac{1}{2}\int d^4x \left(\frac{m_\lambda m^\lambda}{\mu} - \frac{\left(h_\lambda m^\lambda\right)^2}{\mu(\mu + h_\sigma h^\sigma)} + p - \rho \right) .$$

After taking variations of the action, one may impose the constraint $u_\lambda u^\lambda = 1$. In terms of the momentum m^μ, this constraint becomes

$$1 = u_\lambda u^\lambda = \frac{1}{\mu^2}\left(m_\lambda m^\lambda - 2\frac{\left(h_\lambda m^\lambda\right)^2}{\mu + h_\sigma h^\sigma} + \frac{\left(h_\lambda m^\lambda\right)^2}{(\mu + h_\sigma h^\sigma)^2}\left(h_\tau h^\tau\right) \right) . \qquad (1)$$

All functional derivatives of the action can be reduced to simple expressions, provided $1 = u_\lambda u^\lambda$ is applied only after functional differentiation.

$$\frac{\delta S}{\delta n} = \left(-\frac{m_\lambda m^\lambda}{2\mu^2} + \frac{\left(h_\lambda m^\lambda\right)^2}{2\mu^2(\mu + h_\sigma h^\sigma)} + \frac{\left(h_\lambda m^\lambda\right)^2}{2\mu\,(\mu + h_\sigma h^\sigma)^2} \right)\frac{\partial \mu}{\partial n} + \frac{1}{2}\frac{\partial p}{\partial n} - \frac{1}{2}\frac{\partial \rho}{\partial n} = -\frac{\partial \rho}{\partial n} .$$

Action in momentum variables

$$\frac{\delta S}{\delta \sigma} = -\frac{\partial \rho}{\partial \sigma} \, . \qquad \frac{\delta S}{\delta m^{\nu}} = \frac{m_{\nu}}{\mu} - \frac{(h_{\lambda} m^{\lambda})}{\mu(\mu + h_{\tau} h^{\tau})} h_{\nu} = u_{\nu} \, ,$$

$$\frac{\delta S}{\delta h^{\nu}} = \frac{m_{\lambda} m^{\lambda}}{\mu^2} h_{\nu} - \frac{\left(h_{\lambda} m^{\lambda}\right)^2}{\mu^2(\mu + h_{\sigma} h^{\sigma})} h_{\nu} - \frac{(h_{\lambda} m^{\lambda})}{\mu(\mu + h_{\sigma} h^{\sigma})} m_{\nu} = \left(1 + \frac{\left(h_{\lambda} m^{\lambda}\right)^2}{\mu(\mu + h_{\sigma} h^{\sigma})^2}\right) h_{\nu} - \frac{(h_{\lambda} m^{\lambda})}{\mu(\mu + h_{\sigma} h^{\sigma})} m_{\nu} = b_{\nu} \, .$$

The relationship $\delta S/\delta h^{\nu} = b_{\nu}$ gives a meaning to h^{ν}: it is a conjugate to b^{ν}, just as m^{ν} is to u^{ν}.

The covariant Poisson bracket for relativistic MHD is obtained by extending the nonrelativistic bracket of Ref. P. J. Morrison and J. M. Greene, Phys. Rev. Lett. **45***, 790 (1980) to spacetime.*

A difficulty arises in choosing the equivalent of the nonrelativistic momentum and field, because the 4-vectorial equivalents of $\mathbf{M} = \rho \mathbf{v}$ and $\mathbf{B}$ will no longer produce the correct equations.

Relativistic MHD bracket

The 4-vectors m^ν and h^ν provide the appropriate replacements, giving the relativistic MHD bracket

$$\{F,G\} = \int d^4x \left(n \left(\frac{\delta F}{\delta m_\mu} \partial_\mu \frac{\delta G}{\delta n} - \frac{\delta G}{\delta m_\mu} \partial_\mu \frac{\delta F}{\delta n} \right) + \sigma \left(\frac{\delta F}{\delta m_\mu} \partial_\mu \frac{\delta G}{\delta \sigma} - \frac{\delta G}{\delta m_\mu} \partial_\mu \frac{\delta F}{\delta \sigma} \right) \right.$$
$$+ m_\nu \left(\frac{\delta F}{\delta m_\mu} \partial_\mu \frac{\delta G}{\delta m_\nu} - \frac{\delta G}{\delta m_\mu} \partial_\mu \frac{\delta F}{\delta m_\nu} \right) + h^\nu \left(\frac{\delta F}{\delta m_\mu} \partial_\mu \frac{\delta G}{\delta h^\nu} - \frac{\delta G}{\delta m_\mu} \partial_\mu \frac{\delta F}{\delta h^\nu} \right)$$
$$\left. + h^\mu \left[\left(\partial_\mu \frac{\delta F}{\delta m_\nu} \right) \frac{\delta G}{\delta h^\nu} - \left(\partial_\mu \frac{\delta G}{\delta m_\nu} \right) \frac{\delta F}{\delta h^\nu} \right] \right).$$

The bracket is complicated, but one can derive the equations of motion fairly quickly thanks to the simple functional derivatives

$$\frac{\delta S}{\delta n} = -\frac{\partial \rho}{\partial n}; \qquad \frac{\delta S}{\delta \sigma} = -\frac{\partial \rho}{\partial \sigma}; \qquad \frac{\delta S}{\delta m_\nu} = u^\nu; \qquad \frac{\delta S}{\delta h_\nu} = b^\nu,$$

where u^μ and b^μ here are shorthands for their expressions in terms of the fields m^μ and h^μ.

Using $F = \int d^4x\, n(x)\, \delta^4(x - x_0)$ in $\{F, S\} = 0$ gives, after an integration by parts, $\partial_\mu (n u^\mu) = 0$, which is the continuity equation[6]. In the same manner one finds the adiabaticity equation from a σ variation.
The h^μ variation gives

$$\partial_\nu (h^\mu u^\nu) - h^\nu \partial_\nu u^\mu = 0\,.$$

The above equation does not coincide with Maxwell's equations[7] since they correspond to $\pounds_u h^\mu = 0$, the Lie-dragging of the four-dimensional vector density h^μ by u^μ. The theory obtained from the variational principle can be viewed as a family of theories, only some of which correspond to physical systems. However, if $\partial_\mu h^\mu = 0$, then one obtains the usual form of relativistic MHD.

[6] evaluated implicitly at x_0; however, since that point is arbitrary, the result holds for the entire spacetime

[7] although they are analogous to the nonrelativistic equation $\partial \mathbf{B}/\partial t = -\mathbf{B}\,\nabla\cdot\mathbf{v} + \mathbf{B}\cdot\nabla\mathbf{v} - \mathbf{v}\cdot\nabla\mathbf{B}$

Resulting field equations. Setting of $\partial_\mu h^\mu = 0$

The situation is analogous to that in nonrelativistic Hamiltonian MHD formalism which can describe systems with $\nabla \cdot \mathbf{B} \neq 0$: in both cases, the physical systems are a subset of the full class of systems described by the formalism. In the nonrelativistic case the condition $\nabla \cdot \mathbf{B} = 0$ is maintained by the dynamics and the similar situation[8] arises for h^μ.

With h^μ thus specified, we recover the usual form of Maxwell's equations (for $E^\mu = 0$) $\quad \partial_\mu (h^\mu u^\nu - u^\mu h^\nu) = 0.$

The m^λ variation gives momentum equation

$$0 = n\partial^\mu \left(\frac{\partial p}{\partial n}\right) + \sigma \partial^\mu \left(\frac{\partial p}{\partial \sigma}\right) + m_\nu \partial^\mu (u^\nu) + \partial_\nu (m^\mu u^\nu) + h_\nu \partial^\mu (b^\nu) - \partial_\nu (h^\nu b^\mu)$$

$$= \partial_\nu \left(\left(\rho + p - \left(h_\lambda h^\lambda\right)\right) u^\mu u^\nu + g^{\mu\nu}\left[-p + \frac{1}{2}\left(h_\lambda h^\lambda - \left(h_\lambda u^\lambda\right)^2\right)\right] - h^\mu h^\nu + \left(h_\lambda u^\lambda\right)\left(h^\mu u^\nu + u^\mu h^\nu\right)\right),$$

Once derived it can be replaced with the simpler, equivalent version involving b^μ.

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[8] There also exists an alternative bracket that builds in $\partial_\mu h^\mu = 0$ where the constraint is enforced by the bracket's Jacobi identity. An extensive discussion how to impose $\partial_\mu h^\mu = 0$ as an "initial condition" is given in D'Avignon *et al.* Phys. Rev. D, 91, 084050 (2015)

Conclusions

The relativistic ideal MHD equations have been cast into a covariant action formalism using a noncanonical bracket.
Given a relativistic MHD problem posed in terms of (u^μ, b^μ), we must determine the associated problem in terms of (m^μ, h^μ), this requires the determination[9] of α such that $\partial_\mu h^\mu = 0$.

Features that remain to be covered:
3+1 reductions,
additional Casimirs (magnetic helicity),
the relation to Lagrangian action principles, brackets in systems possessing extra symmetry (e.g. spherical or toroidal).

Thank you for your attention

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[9] See procedure in Eric D'Avignon, P. J. Morrison, F. Pegoraro *Physical Review D*, **91**, 084050 (2015)