Integer Lattice for Vlasov–Poisson

(Mocz & Succi, 2016)

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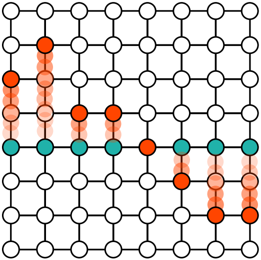

PRINCETON UNIVERSITY

Boltzmann Conference

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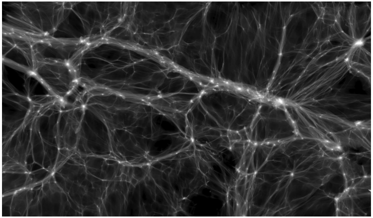
Outline

- Vlasov–Poisson equations
- Integer Lattice
- Other methods for VP: particle–mesh, finite volume, moving mesh, Schrödinger
- Test Problems (Jeans instability, Landau damping)
- Memory Saving ($N^6 \to N^4$)
- Extensions (weakly–collisional, Vlasov–Maxwell, ...)

Vlasov–Poisson



The dynamics of a collisionless, self-gravitating fluid is described by the evolution of the 6D distribution function $f(\mathbf{r},\mathbf{v},t)$:

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f - \nabla\Phi\cdot\frac{\partial f}{\partial \mathbf{v}} = 0 \quad (1)$$

Physical density:

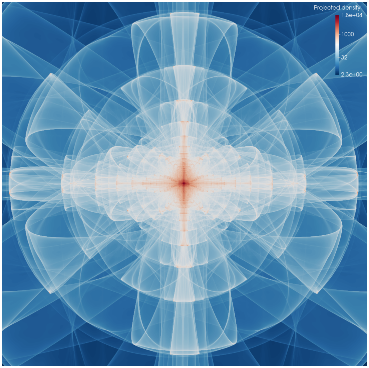
$$\rho(\mathbf{r},t) = \int(\mathbf{r},\mathbf{v},t)d^3\mathbf{v} \quad (2)$$

Poisson's equation:

$$\nabla^2\Phi = 4\pi G\rho \quad (3)$$

- a statement of conservation of phase-space density
- f constant along the trajectories of the system (Liouville's theorem)
- describes systems such as Cold Dark Matter (CDM)

Fine-grained phase-space structure



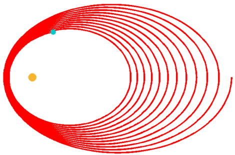
4D CDM phase-space

(Sousbi & Colombi, 2015)

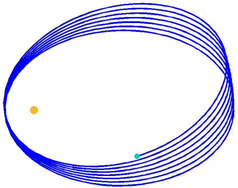
- understanding velocity field very important for modern cosmology
- structure formation
- modelling redshift-space distortions
- interpreting measurements of kinetic Sunyaev-Zeldovich effect
- shell-crossing
- N-body simulations only sample at a set of discrete locations, **poor** resolution in low-density regions

Designing Numerical Methods: Motivation

It is important to preserve physical symmetries and invariants at the discretized level

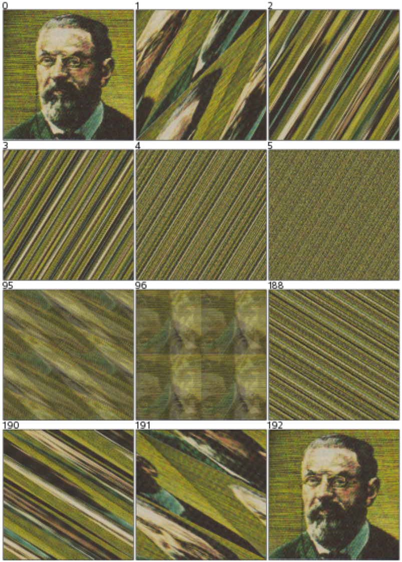


4^{th}-order Runge–Kutta



2^{nd}-order Leapfrog

Poincaré recurrence theorem



- systems will, after a sufficiently long but finite time, return to the initial state

Integer Lattice Algorithm I

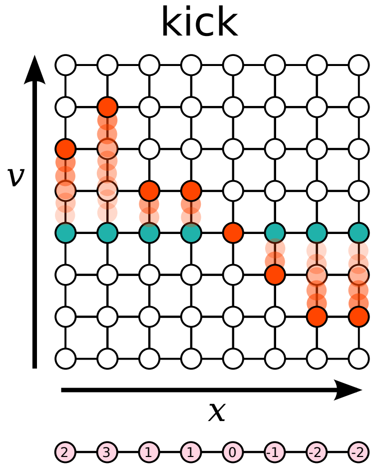
- define f on lattice sites (x, v)
- very **simple**, seemingly strange evolution equations:

$$v \leftarrow v + \lfloor \Delta t a \rceil \tag{4}$$

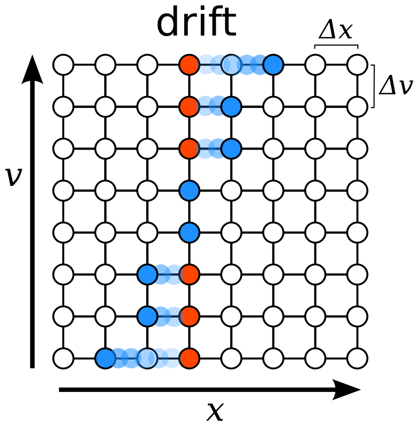
$$x \leftarrow v + \lfloor \Delta t v \rceil \tag{5}$$

- (units are $\Delta x = \Delta v = 1$)
- round-to-nearest-int operator $\lfloor \cdot \rceil$ restricts motion to the lattice
- acceleration a is obtained from a Poisson solver
- based on idea by (Syer & Tremaine 1995) which used an integer lattice to search for equilibria solutions

Integer Lattice Algorithm II

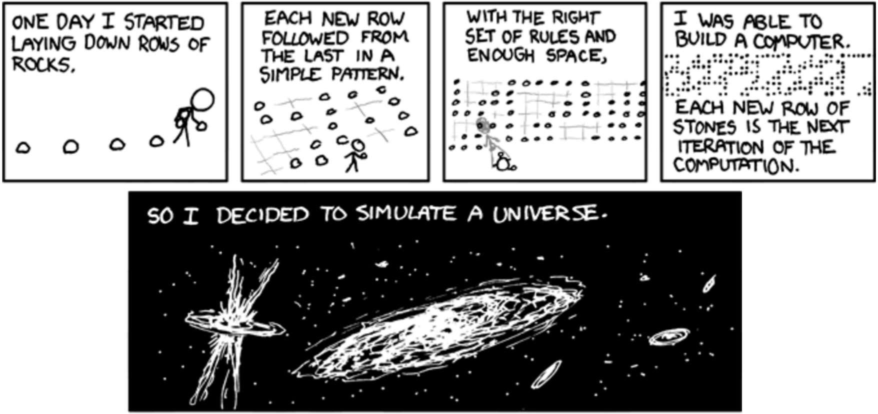


$a\Delta t$ from Poisson solver



$v\Delta t$ from grid location

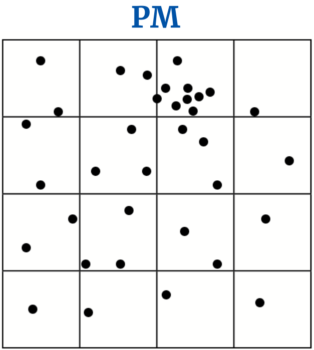
A bunch of rocks



Algorithm Properties

- each step is a simple permutation of lattice sites!
- no roundoff error!
- reversible
- conservative
- Lagrangian
- discrete system has a Hamiltonian, approaches continuous system in the limit of increased resolution
- all orbits are periodic
- analogy to chaos is long-period orbits

Other Methods: N-body Particle-Mesh (PM)



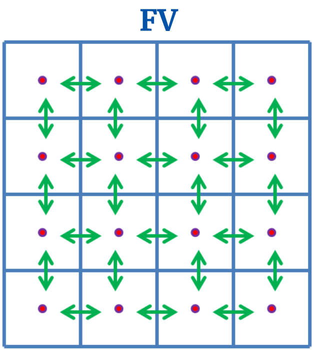
- N particles
- 2^{nd}-order symplectic leap-frog scheme ('kick'-'drift'-'kick')
- cloud-in-cell (CIC) binning

$$v \leftarrow v + 0.5\Delta t a \tag{6}$$

$$x \leftarrow x + \Delta t v \tag{7}$$

$$v \leftarrow v + 0.5\Delta t a \tag{8}$$

Other Methods: Finite Volume (FV)



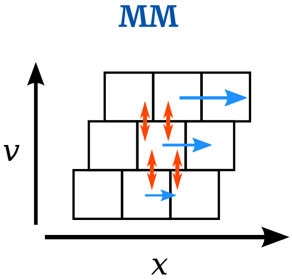
- standard FV for collisionless Boltzmann equations (Yoshikawa, Yoshida, Umemura, 2013)
- 6D, conservative, diffusive, Eulerian

$$v \leftarrow v - 0.5\frac{\Delta t}{\Delta v}\sum_{n\in\text{neighbours}}\mathcal{F}_{v,n} \tag{9}$$

$$x \leftarrow x - \frac{\Delta t}{\Delta x}\sum_{n\in\text{neighbours}}\mathcal{F}_{x,n} \tag{10}$$

$$v \leftarrow v - 0.5\frac{\Delta t}{\Delta v}\sum_{n\in\text{neighbours}}\mathcal{F}_{v,n} \tag{11}$$

Other Methods: Finite Volume Moving Mesh (MM)



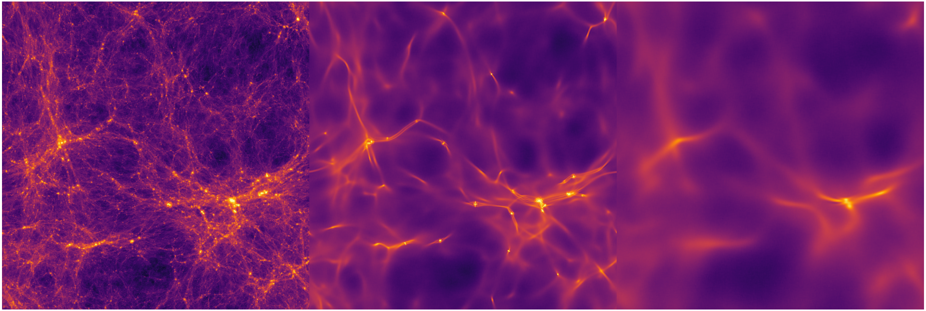
- Shear cells by their velocity
- reduces numerical diffusion, makes the scheme Galilean-invariant
- improves timestep condition

$$\Delta t = C \cdot \min\left(\frac{\Delta x}{v_{\max}}, \frac{\Delta v}{a_{\max}}\right) \quad (12)$$

$$\Delta t = C \cdot \min\left(\frac{\Delta x}{\Delta v}, \frac{\Delta v}{a_{\max}}\right) \quad (13)$$

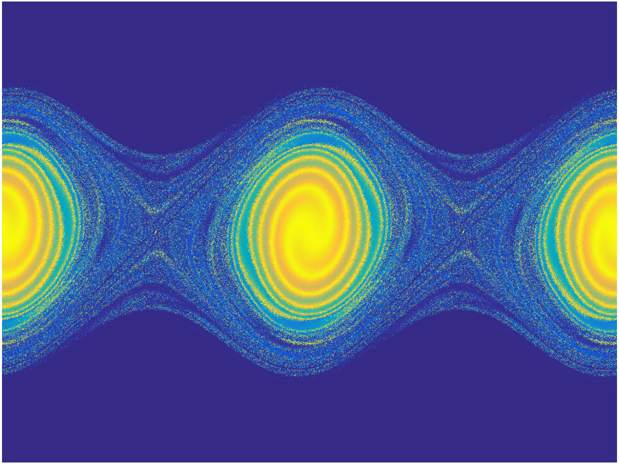
Other Methods: Schrödinger limit $\hbar \to 0$

- Approximate 6D phase-space with 3D wave-function. Below are some of my simulations for axion dark matter



ΛCDM | 2.5e−21 eV | 2.5e−22 eV

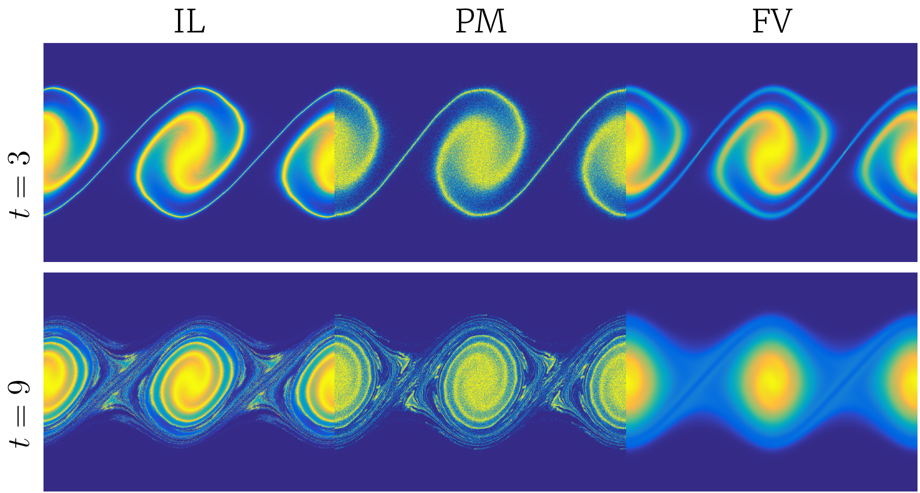
Jeans Instability Test



x–v plane

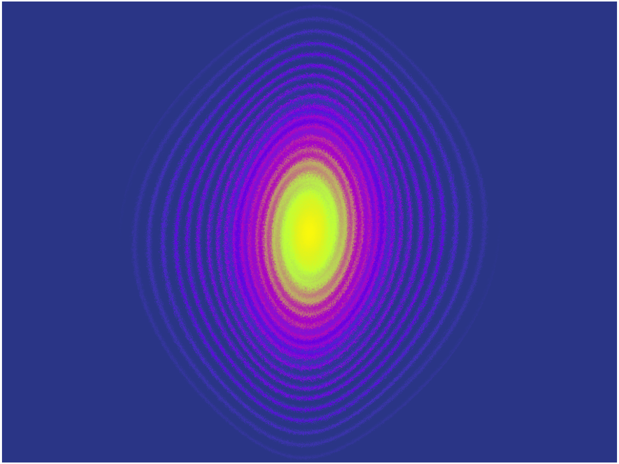
- Gaussian–like profile in velocity direction
- seed Jeans unstable large–scale mode
- gravitational collapse

Jeans Instability Comparison



Lattice-noise vs Monte-Carlo vs Diffusion

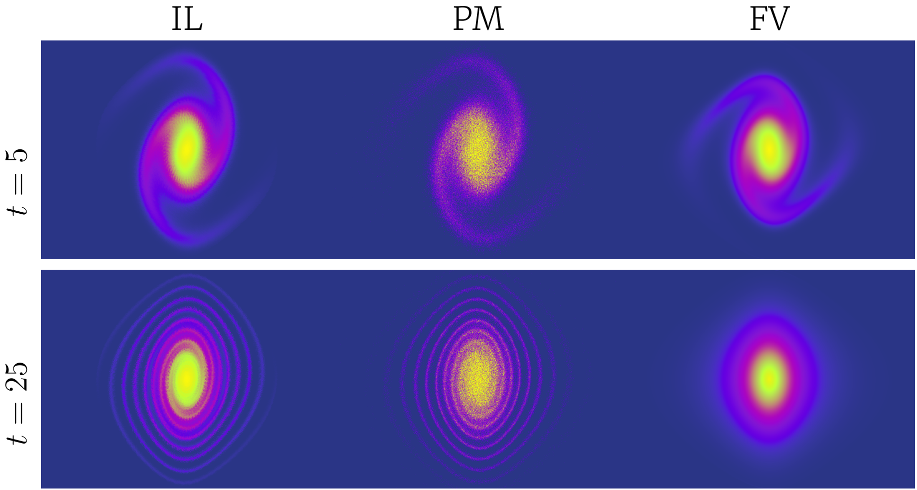
Landau Damping Test



x–v plane

- Gaussian IC close to smooth equilibrium
- will wind up, evolve thin spiral arms
- complex structure approximates smooth equilibrium but never reaches it

Landau Damping Comparison



Lattice-noise vs Monte-Carlo vs Diffusion

Comparison Summary

- IL recovers fine phase-space features. Lattice-noise
- PM has Monte-Carlo noise
- FV, MM has diffusion (erases phase-space structure)
- MM less diffusive than FV

Comparison Table

method	conservative?	reversible?	Galilean-invariant?
IL	✓	✓	✓
PM	✓	×	✓
FV	✓	×	×
MM	✓	×	✓

Comparison Table

method	error	computing time for Jeans test [s]
IL	‘lattice noise’	3
PM	Monte–Carlo	10
FV	2^{nd}–order (diffusive)	800
MM	2^{nd}–order (diffusive)	1000

Comparison Table

method	resolution	computational cost $[\mathcal{O}(\cdot)]$	memory scaling
IL	$N_x^3 N_v^3$	$N_t N_x^3 N_v^3$ $N_t^2 N_x^3 N_v^3$ *	$N_x^3 N_v^3$ $N_t N_x^3$ *
PM	N	$N_t N$	N
FV	$N_x^3 N_v^3$	$N_t N_x^3 N_v^3$	$N_x^3 N_v^3$
MM	$N_x^3 N_v^3$	$N_t N_x^3 N_v^3$	$N_x^3 N_v^3$

* memory efficient version

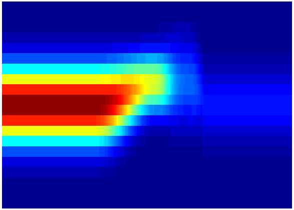
Memory saving

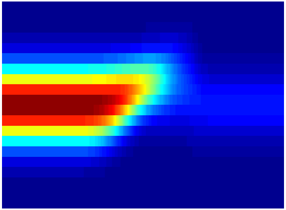
- assume $N_x \sim N_v \sim N_t :\sim N$
- 6D phase-space solvers are memory limited ($\mathcal{O}(N^6)$)
- But IL is *exactly* reversible
- suffices to store N^3 accelerations at each timestep and 'recover' the distribution function
- memory scaling is now $\mathcal{O}(N^4)$. (Computation increases by factor of N – but that's OK!)

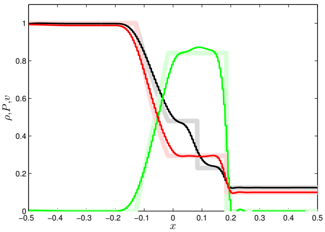
Extension to collisions (add BGK source term)

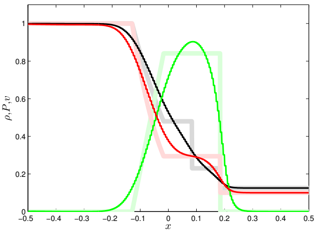
Strongly collisional shock

Weakly collisional shock



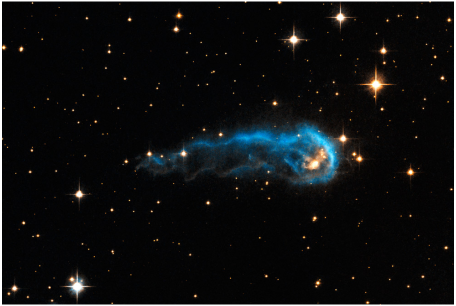






Weakly collisional fluid dynamics

- galaxy moving through intracluster medium experiences drag force. Gas is stripped
- Ram-pressure stripping is always solved in the fluid regime. $P = \frac{1}{2}\rho v^2$
- But it is actually weakly-collisional!
- $\lambda_{\rm mfp} \sim L_{\rm galaxy}$



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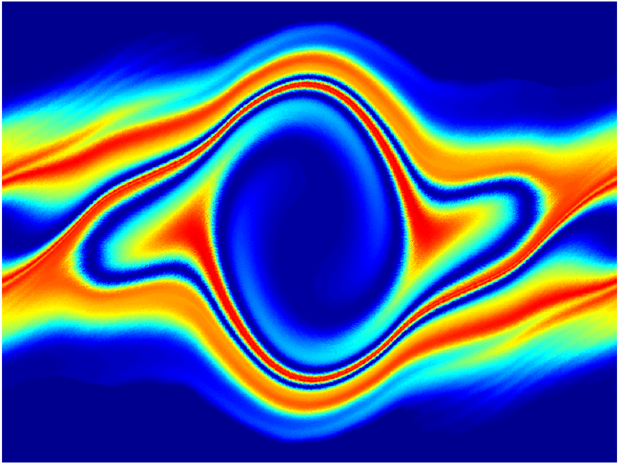
Extension to plasma: Two-Stream Instability



x–v plane

- Vlasov–Maxwell equations (collisionless)
- two beams in opposite directions develop instability
- alternative to particle-in-cell (PIC)

Summary

- Integer lattice is a very simple, powerful tool to study the dynamics of Vlasov–Poisson systems (e.g. CDM)
- numerical method has many good and unique theoretical properties (such as reversibility and no roundoff errors)
- memory-efficient implementation overcomes memory limitations of phase-space methods