From Vlasov-Poisson to Euler in the gyrokinetic limit

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Collisionless Boltzmann equation and modeling of self-gravitating systems and plasmas, Nov 2017

Outline of the talk

- Introduction: from Vlasov-Poisson to Euler
- Main results
- Sketch of proofs
- From Vlasov-Poisson with point charge to the vortex-wave system

Aim: Vlasov-Poisson $\rightsquigarrow$ Euler equation in 2D

2D Vlasov-Poisson

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0$$

$f(t,x,v) \geq 0, \quad x, v \in \mathbb{R}^2$
[density of electric particles]

$\rho(t,x) = \int f(t,x,v)\,dv, \quad x \in \mathbb{R}^2$
[macroscopic density of particles]

$$E(t,x) = \left(\frac{x}{|x|^2} * \rho\right)(t,x)$$

2D Euler

$$\partial_t \omega + u \cdot \nabla \omega = 0$$

$$\mathrm{div}(u) = 0$$

$\omega(t,x), \quad x \in \mathbb{R}^2$
[vorticity]

$$u(t,x) = \left(\frac{x^\perp}{|x|^2} * \omega\right)(t,x)$$

The Vlasov-Poisson system with strong magnetic field

- **Gyrokinetic limit** for the Vlasov-Poisson system: the particles are submitted to a constant magnetic field, orthogonal to the plane, with strength tending to infinity.
- Corresponds to studying the asymptotics as $\varepsilon \to 0$ of

$$\partial_t f_\varepsilon + \frac{v}{\varepsilon} \cdot \nabla_x f_\varepsilon + \left(\frac{E_\varepsilon}{\varepsilon} + \frac{v^\perp}{\varepsilon^2} \right) \cdot \nabla_v f_\varepsilon = 0,$$
$$f_\varepsilon(0, x, v) = f_\varepsilon^0(x, v).$$

- The initial data f_ε^0 satisfy some suitable bounds for norms that are conserved by the flow of the Vlasov-Poisson equation.

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Assumptions on the initial data f_ε^0

- $f_\varepsilon^0 \in L^1 \cap L^\infty(\mathbb{R}^2)$, nonnegative and compactly supported
  $\rightsquigarrow$ unique global solution $f_\varepsilon \in L^\infty(L^1 \cap L^\infty)$ Okabe & Ukai 75.

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- Uniform bounds on physical quantities:

$$\|f_\varepsilon^0\|_{L^1} + \int_{\mathbb{R}^2} |x|^2 \rho_\varepsilon^0(x)\, dx + \mathcal{H}(f_\varepsilon^0) \leq C.$$

where the energy is defined by

$$\mathcal{H}(f) = \iint |v|^2 f(x,v)\, dx\, dv - \iint \ln|x-y| \rho(x)\rho(y)\, dx\, dy,$$

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- Additional boundedness assumption:
$\varepsilon^2 \|f_\varepsilon^0\|_{L^\infty} \ln\left(\|f_\varepsilon^0\|_{L^\infty} + 2\right) = o_\varepsilon(1)$.

Example: monokinetic-like data

The previous assumptions allow for initial data that converge to **monokinetic data:**

$$f_\varepsilon^0(x,v) \to \rho_0(x)\,\delta_{v=u_0(x)} \quad \text{as } \varepsilon \to 0, \quad \rho_0 \in L^\infty(\mathbb{R}^2).$$

Indeed, take

$$f_\varepsilon^0(x,v) = \rho_0(x)\frac{1}{\delta_\varepsilon^2}\Phi\left(\frac{v-u(x)}{\delta_\varepsilon}\right),$$

where $\varepsilon^2\delta_\varepsilon^{-2}|\ln\delta_\varepsilon|$ vanishes as $\varepsilon \to 0$ and Φ smooth cut-off function.

Main result

Theorem 1

Let f_ε^0 satisfy the previous assumptions and f_ε denote the corresponding global solution. There exists a subsequence $\varepsilon_n \to 0$ as $n \to +\infty$ such that

- *ρ_{ε_n} converges to ρ in $C(\mathbb{R}_+, \mathcal{M}^+(\mathbb{R}^2) - w^*)$;*
- *ρ belongs moreover to $L^\infty(\mathbb{R}_+, H^{-1}(\mathbb{R}^2))$;*
- *ρ is a global* **generalized "vortex sheet" solution** *of the 2D Euler equation.*

Notion of generalized solution to 2D Euler

Notion of "vortex sheet" solution: for $\rho \in \mathcal{M}^+ \cap H^{-1}(\mathbb{R}^2))$ need to define the product $u \cdot \nabla \rho$ in the sense of distributions, where $u = x^\perp/|x|^2 * \rho$.

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Observed by Delort 91, Schochet 95.
If ρ is sufficiently smooth we have by symmetrization:

$$\langle \mathrm{div}(u\rho), \Phi\rangle = -\langle \left(\frac{x^\perp}{|x|^2} * \rho\right) \rho, \nabla\Phi\rangle = \iint H_\Phi(x,y)\,\rho(x)\,\rho(y)\,dx\,dy,$$

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where

$$H_\Phi(x,y) = \frac{1}{2}\frac{(x-y)^\perp}{|x-y|^2}\cdot(\nabla\Phi(x) - \nabla\Phi(y)).$$

H_Φ bounded on $\mathbb{R}^2 \times \mathbb{R}^2$ and continuous off the diagonal $\{(x,x), x \in \mathbb{R}^2\}$.

Notion of generalized solution to 2D Euler

We set for ρ positive bounded Radon measure belonging to H^{-1}:

$$\mathcal{H}_\Phi[\rho,\rho] = \iint H_\Phi(x,y)\, d\rho(x)\, d\rho(y).$$

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Definition

We say that $\rho \in L^\infty(\mathcal{M}^+ \cap H^{-1}(\mathbb{R}^2))$ is a vortex sheet solution of the Euler equation with initial datum ρ_0 if for all $\Phi \in C_c^\infty(\mathbb{R}^2)$

$$\int \Phi\, d\rho(t,x) = \int \Phi\, d\rho_0(x) + \int_0^t \mathcal{H}_\Phi[\rho(s),\rho(s)]\, ds.$$

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Delort 91, Schochet 95: global existence of such solutions.

Previous results on the asymptotics for Vlasov-Poisson

- Golse & Saint-Raymond 99, Saint-Raymond 02: compactness method, same assumptions except that $\varepsilon\|f_\varepsilon^0\|_{L^\infty} = o_\varepsilon(1)$.
- Brenier 00: different time scaling, modulated energy method.
- Bostan, Finot & Hauray 15 different scaling, effective dynamics for the asymptotics of the shifted density $f_\varepsilon(t, x - R(-t/\varepsilon v)^\perp, R(-t/\varepsilon)v)$.
- Other regimes leading to various equations: Brenier 00 (quasineutral limit), Frénod & Sonnendrücker 98, 99, 01 , Han-Kwan 10, Ghendrih, Hauray & Nouri 09, Hauray and Nouri 11, Barré, Chiron, Goudon & Masmoudi 15.

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- Introduction: from Vlasov-Poisson to Euler in the gyrokinetic limit
- Main results
- Sketch of proofs
  - Uniform estimates $\rightsquigarrow$ compactness.
  - New lagrangian coordinates $\rightsquigarrow$ new weak formulation
  - Passing to the limit
- From Vlasov-Poisson with point charge to the vortex-wave system

Conserved quantities, uniform bounds & quantitative estimates

Let f_ε be a solution as in the Theorem 1.

- The quantities $\|f_\varepsilon(t)\|_{L^p}$ and $\mathcal{H}(f_\varepsilon(t))$ are conserved.
- In particular: $\|f_\varepsilon(t)\|_{L^1} + \int |x|^2 \rho_\varepsilon(t) + \mathcal{H}(f_\varepsilon(t)) \leq C$.
- Already known: this implies $\|\rho_\varepsilon(t)\|_{H^{-1}} \leq C$.

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- Already known: this implies $\|\rho_\varepsilon(t)\|_{H^{-1}} \leq C$.
- Quantitative estimates:

$$\|\rho_\varepsilon(t)\|_{L^2} \leq C\|f_\varepsilon^0\|_{L^\infty}^{1/2}, \quad \|E_\varepsilon(t)\|_{H^1_{\text{loc}}} \leq C(1 + \|f_\varepsilon^0\|_{L^\infty}^{1/2})$$

$$\rightsquigarrow \|E_\varepsilon(t)\|_{L^q_{\text{loc}}} \leq C\sqrt{q}(1 + \|f_\varepsilon^0\|_{L^\infty}^{1/2}), \quad \forall q \geq 2.$$

New lagrangian coordinates for Vlasov-Poisson

DiPerna & Lions 89: theory on transport equations.

Vlasov-Poisson

$$f_\varepsilon\left(X_\varepsilon(t,x,v), V_\varepsilon(t,x,v)\right) = f_\varepsilon^0(x,v)$$

$$\begin{cases} \dot{X}_\varepsilon = \dfrac{V_\varepsilon}{\varepsilon}, \quad \dot{V}_\varepsilon = \dfrac{V_\varepsilon^\perp}{\varepsilon^2} + \dfrac{E_\varepsilon(X_\varepsilon)}{\varepsilon} \\ (X_\varepsilon, V_\varepsilon)(0,x,v) = (x,v) \end{cases}$$

$$E_\varepsilon = \frac{x}{|x|^2} * \rho_\varepsilon$$

Euler (only in 2D)

$$\omega\left(X(t,x)\right) = \omega^0(x)$$

$$\begin{cases} \dot{X} = u(X) \\ X(0,x) = x \end{cases}$$

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We set $Z_\varepsilon = X_\varepsilon + \varepsilon V_\varepsilon^\perp$,

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$$E_\varepsilon^\perp = \frac{x^\perp}{|x|^2} * \rho_\varepsilon$$

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Then: $\dot{Z}_\varepsilon = E_\varepsilon^\perp(X_\varepsilon)$

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Bostan, Hauray & Finot: similar combination of coordinates.

Weak formulation for the spatial density

Proposition

For all $\Phi \in C_c^\infty(\mathbb{R}^2)$,

$$\int \Phi \rho_\varepsilon(t,x)\,dx - \int \Phi \rho_\varepsilon^0(x)\,dx = \int_0^t \mathcal{H}_\Phi[\rho_\varepsilon(s), \rho_\varepsilon(s)]\,ds + o_\varepsilon(1).$$

Proof of Theorem 1 with the proposition

Delort, Schochet:

$$\rho_{\varepsilon_n} \rightharpoonup \rho \quad \text{in } C(\mathcal{M}^+) \cap L^\infty(H^{-1})$$

implies the convergence of the nonlinear term:

$$\int \mathcal{H}_\Phi[\rho_{\varepsilon_n}(s), \rho_{\varepsilon_n}(s)]\, ds \to \int \mathcal{H}_\Phi[\rho(s), \rho(s)]\, ds.$$

Therefore one can pass to the limit in the previous weak formulation.

Proof of the proposition

By using $f_\varepsilon(t) = (X_\varepsilon(t), V_\varepsilon(t))_\# f_\varepsilon^0$ and changing variables:

$$\iint f_\varepsilon(t,x,v)\Phi(x)\,dx\,dv = \iint f_\varepsilon^0(x,v)\Phi(X_\varepsilon(t,x,v))\,dx\,dv$$

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where

$$R_1 = \iint f_\varepsilon^0(x,v)\left(\Phi(X_\varepsilon) - \Phi(Z_\varepsilon)\right)$$

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where

$$R_1 = \iint f_\varepsilon^0(x,v)\left(\Phi(X_\varepsilon) - \Phi(Z_\varepsilon)\right) \le \|D\Phi\|_{L^\infty}\varepsilon \iint |v| f_\varepsilon \le C\varepsilon.$$

On the other hand,

$$\frac{d}{dt}\iint f_\varepsilon^0(x,v)\Phi(Z_\varepsilon(t,x,v))$$

$$= \iint f_\varepsilon^0(x,v)E_\varepsilon^\perp(X_\varepsilon(x,v))\cdot\nabla\Phi(Z_\varepsilon(t,x,v)) \quad [\dot{Z}_\varepsilon = E_\varepsilon^\perp(X_\varepsilon))]$$

On the other hand,

$$
\begin{aligned}
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&= \iint f_\varepsilon(t,x,v)E_\varepsilon^\perp(x)\cdot\nabla\Phi(x+\varepsilon v^\perp) \quad [f_\varepsilon(t) = (X_\varepsilon(t), V_\varepsilon(t))_\# f_\varepsilon^0]
\end{aligned}
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where

$$R_2 = \iint \rho_\varepsilon E_\varepsilon^\perp(x)\cdot\Big(\nabla\Phi(x+\varepsilon v^\perp) - \Phi(x)\Big)\,dx.$$

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$$R_2 = \iint \rho_\varepsilon E_\varepsilon^\perp(x)\cdot\Big(\nabla\Phi(x+\varepsilon v^\perp) - \Phi(x)\Big)\, dx.$$

So

$$R_2 \le \varepsilon\|D^2\Phi\|_{L^\infty}\|\int |v|f_\varepsilon\|_{L^{q'}}\|E_\varepsilon\|_{L^q}.$$

We conclude with the quantitative estimates.

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Vlasov-Poisson with point charge $\rightsquigarrow$ Vortex-wave system

Consider the **interaction** of bounded density with a point charge located at ξ with intensity $q > 0$:

$$f(t,x,v) \rightsquigarrow f(t,x,v) + q\delta_{x=\xi(t)} \otimes \delta_{v=\eta(t)}.$$

Question: in the gyrokinetic limit do we get the **interaction** of bounded vorticity with point vortex of circulation q ?

Vlasov-Poisson with point charge $\rightsquigarrow$ Vortex-wave system

Vlasov-Poisson with charge

$$\partial_t f + \frac{v}{\varepsilon^2} \cdot \nabla_x f + \frac{E_{\text{total}}}{\varepsilon} \cdot \nabla_v f = 0$$

$$E_{\text{total}} = E + q\frac{x-\xi}{|x-\xi|^2}$$

$$E = \frac{x}{|x|^2} * \rho$$

$$\dot{\xi} = \eta, \quad \dot{\eta} = qE(\xi)$$

Euler: vortex-wave system

$$\partial_t \omega + u_{\text{total}} \cdot \nabla \omega = 0$$

$$u_{\text{total}} = u + q\frac{(x-\xi)^\perp}{|x-\xi|^2}$$

$$u = \frac{x^\perp}{|x|^2} * \omega$$

$$\dot{\xi} = u(\xi)$$

Vlasov-Poisson with charge

$$\partial_t f + v \cdot \nabla_x f$$
$$+(E + q\frac{x-\xi}{|x-\xi|^2}) \cdot \nabla_v f = 0$$

$$\dot{\xi} = \eta, \quad \dot{\eta} = qE(\xi)$$

Caprino, Marchioro & Pulvirenti 10
Desvillettes, Miot & Saffirio 14
Crippa, Ligabue & Saffirio 17, 3D case
existence, uniqueness for
$f \in L^\infty_c, \xi \notin \mathrm{supp}(\rho)$

Vortex-wave system

$$\partial_t \omega + (u + q\frac{(x-\xi)^\perp}{|x-\xi|^2}) \cdot \nabla \omega = 0$$

$$\dot{\xi} = u(\xi)$$

Marchioro & Pulvirenti 91
Lacave & Miot 09

: existence, uniqueness for
$\omega \in L^\infty_c, \xi \notin \mathrm{supp}(\omega)$

Assumptions on the initial data f_ε^0

- $f_\varepsilon^0 \in L^1 \cap L^\infty$ compactly supported, vanishes near ξ_ε^0
  $\rightsquigarrow$ unique global solution $f_\varepsilon \in L^\infty(L^1 \cap L^\infty)$ for each fixed $\varepsilon > 0$.
- Uniform bounds on physical quantities:

$$\|f_\varepsilon^0\|_{L^1} + \int |x|^2 \rho_\varepsilon^0(x)\,dx + |\xi_\varepsilon^0| + \mathcal{H}(f_\varepsilon^0, \xi_\varepsilon^0, \eta_\varepsilon^0) \leq C$$

where

$$\mathcal{H}(f, \xi, \eta) = \mathcal{H}(f) + |\eta|^2 - q \int \ln|x - \xi| \rho(x)\,dx.$$

- $\varepsilon^2 \|f_\varepsilon^0\|_{L^\infty} = o_\varepsilon(1)$.
- $\int \rho_\varepsilon^0 < 1$.

Convergence to a nonlinear equation

Theorem 1

Up to a subsequence:

- $\rho_{\varepsilon_n} \rightharpoonup \rho$ *in* $C_w(\mathbb{R}_+, \mathcal{M}_+(\mathbb{R}^2))$ *and* $\xi_{\varepsilon_n} \to \xi$ *locally uniformly;*
- $\rho \in L^\infty(\mathbb{R}_+, H^{-1}(\mathbb{R}^2))$*;*

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- $\rho \in L^\infty(\mathbb{R}_+, H^{-1}(\mathbb{R}^2))$;
- *There exists a defect measure* $\nu \in [L^\infty(\mathbb{R}_+, \mathcal{M}(\mathbb{R}^2)]^4$ *such that* (ρ, ξ) *satisfies for all test function:*

$$\int_{\mathbb{R}^2} \Phi d(\rho(t) + \delta_{\xi(t)}) = \int_{\mathbb{R}^2} \Phi d(\rho_0 + \delta_{\xi_0}) + \int_0^t \mathcal{H}_\Phi[\rho + q\delta_\xi, \rho + q\delta_\xi]\, ds + \int_O^t \int_{\mathbb{R}^2} D\nabla^\perp\Phi : d\nu\, ds. \quad \text{(NLE)}$$

The special cases $q = 0$ and $q = 1$

Case $q = 0$ $\rightsquigarrow$ first part of the talk: convergence to a generalized solution of the Euler equation.

Case $q = 1$ (NLE) reduces to the generalized formulation of the Euler equation for the total measure-valued vorticity $\omega = \rho + \delta_\xi$.

Decoupling the equation into a system of PDE/ODE

Theorem 2

Let (ρ, ξ) be an accumulation point given by Theorem 1 and such that ν vanishes. If moreover $\rho \in L^\infty_{loc}(\mathbb{R}_+, L^p(\mathbb{R}^2))$ for some $p > 2$ and $\xi \in C^1(\mathbb{R}_+, \mathbb{R}^2)$ then (ρ, ξ) satisfies the system

$$\begin{cases} \partial_t \rho + \left(E^\perp + q\dfrac{(x-\xi)^\perp}{|x-\xi|^2} \right) \cdot \nabla \rho = 0 \\ \dot{\xi}(t) = qE^\perp(t, \xi(t)), \end{cases}$$

where $E = \frac{x}{|x|^2} * \rho$.

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where $E = \frac{x}{|x|^2} * \rho$.

Remark

- *If $q = 0$ retrieve 2D Euler,*
- *If $q = 1$ retrieve vortex-wave system.*

Basic properties of the new system of ODE

Theorem 3

- *Global existence of a solution with* $\rho \in L^\infty(\mathbb{R}_+, L^\infty(\mathbb{R}^2))$*, compactly supported ;*
- *Uniqueness holds if moreover* $\xi(0) \notin suppp(\rho(0))$*. In this case, the solution satisfies* $\xi(t) \notin supp(\rho(t))$*, for all* $t > 0$*. This means:* **no collision occurs** *between the plasma particles and the point charge.*

Thank you for your attention.