# Newtonian gravity in one dimension
## Vlasov simulations of a 1D expanding universe

**Giovanni MANFREDI**

*Institut de Physique et Chimie des Matériaux de Strasbourg*

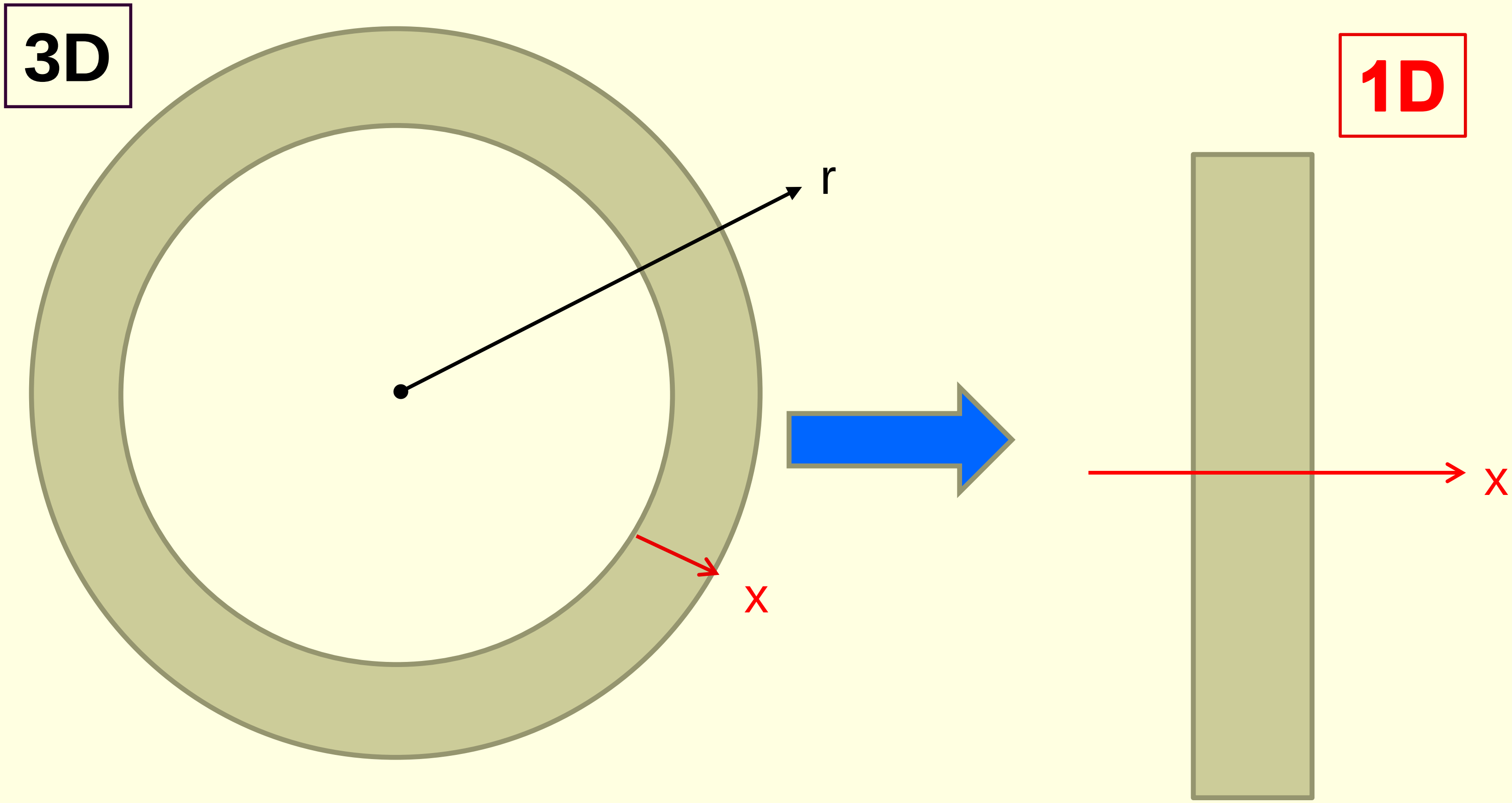
**Collaborators:**

J.-L. Rouet (Orléans), B. Miller and Y. Shiozawa (Texas)

# Motivations

- Expansion of the universe can be studied in a Newtonian framework
- Initial density fluctuations coupled to gravitational attraction lead to hierarchical (fractal?) structure
  - Confirmed by observations
- Simple model to investigate structure formation:
  - Newtonian gravity
  - Hubble-type expansion
  - One-dimensional geometry
  - **Mean-field Vlasov-Poisson model**

# One-dimensional geometry



# Comoving coordinates

- Equation of motion:

$$\frac{d^2x}{dt^2} = E(x,t)$$

- Rescale space, time, and velocity:

$$x = C(t)\xi$$

$$dt = A^2(t)d\theta$$

$$\frac{dx}{dt} = \frac{C}{A^2}\frac{d\xi}{d\theta} + \dot{C}\xi$$

- We obtain the rescaled equation of motion:

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}\theta^2} + 2\,A^2\left(\frac{\dot{C}}{C} - \frac{\dot{A}}{A}\right)\frac{\mathrm{d}\xi}{\mathrm{d}\theta} + A^4\frac{\ddot{C}}{C}\,\xi = \frac{A^4}{C^3}\,\epsilon$$

# Comoving coordinates

- Rescale field and density to keep Poisson's equation invariant

$$\hat{\rho}(\xi,\theta) = C^3(t)\rho(r,t) \qquad \nabla_\xi \cdot \boldsymbol{\mathcal{E}} = -4\pi G\hat{\rho}$$

- Let us specify the scaling functions

$$A^2(t) = (t/t_0)^\beta \ ; \qquad C(t) = (t/t_0)^\gamma$$

- With $\gamma = 2/3$ (**Einstein-de Sitter universe**) and $\beta = 1$, all coefficients in the equation of motion become time-independent:

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}\theta^2} + \frac{1}{3\,t_0}\frac{\mathrm{d}\xi}{\mathrm{d}\theta} - \frac{2}{9\,t_0^2}\,\xi = \epsilon(\xi,\theta)$$

**Cancel at equilibrium** $\qquad \omega_J^2 t_0^2 = \frac{2D}{9}$

# N-body simulations

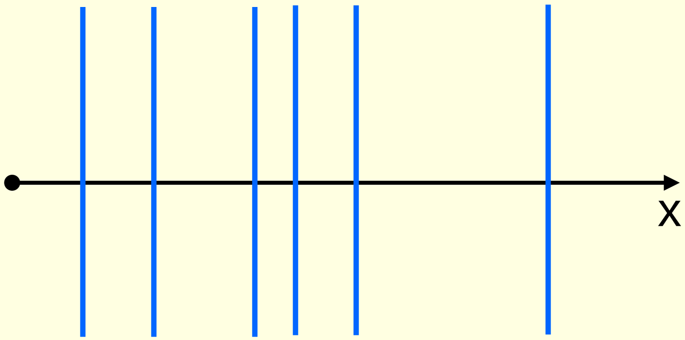
- Solve equations of motion for N>>1 “particles”:

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}\theta^2} + \frac{1}{\sqrt{2}}\frac{\mathrm{d}\xi}{\mathrm{d}\theta} - \xi = \epsilon(\xi, \theta)$$

Friction

Repulsive field

Gravitational interaction

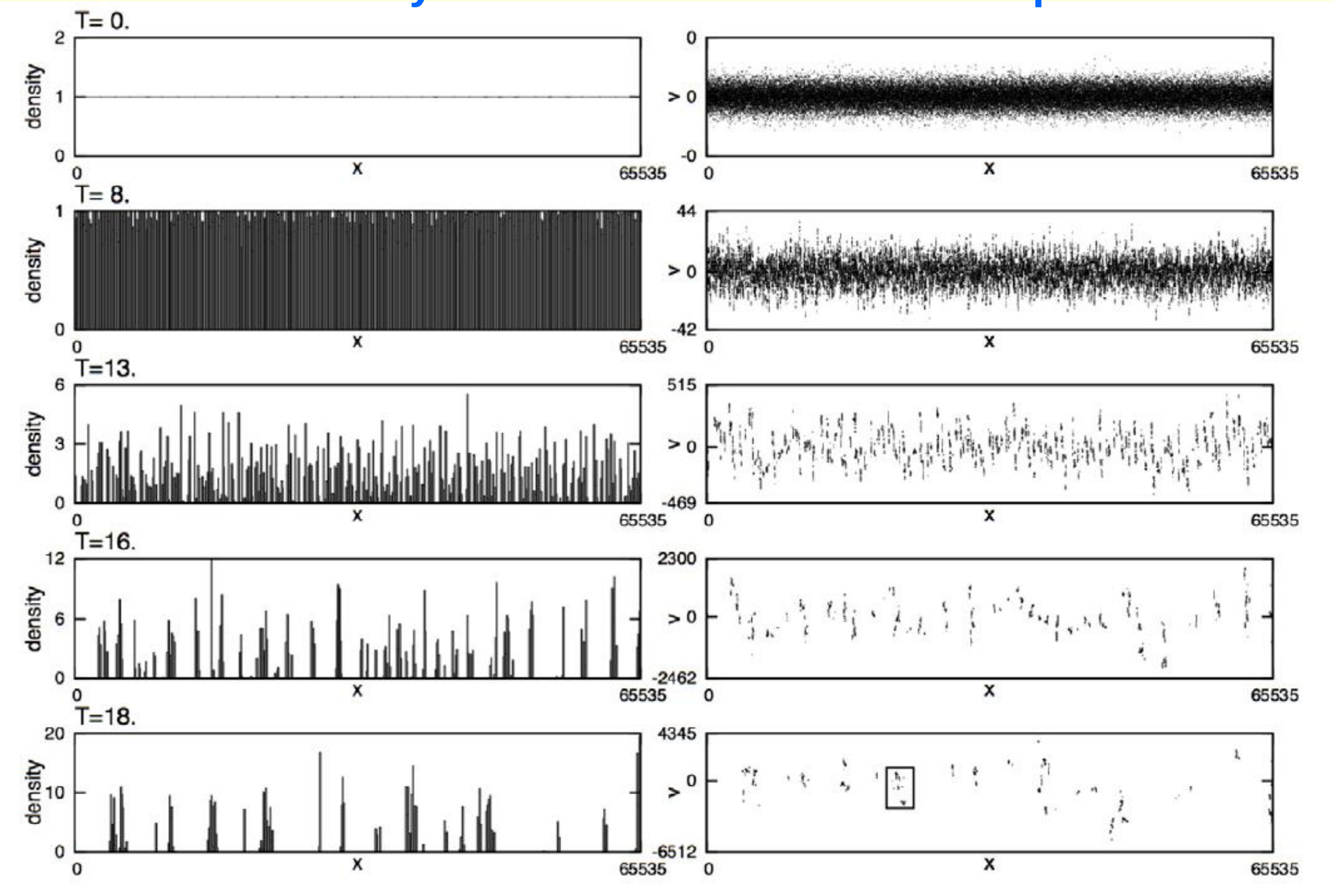



- In 1D, particles are actually **massive sheets**
- N ≈ $10^5$, but in principle we should approach the limit N → ∞
- **Continuous mean-field limit → Vlasov equation**

# Typical N-body results

(N = 66535)

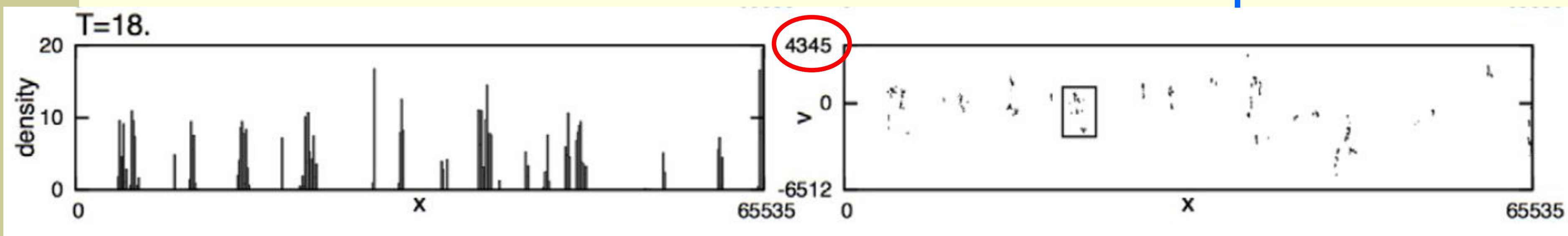
**Density** **Phase space**



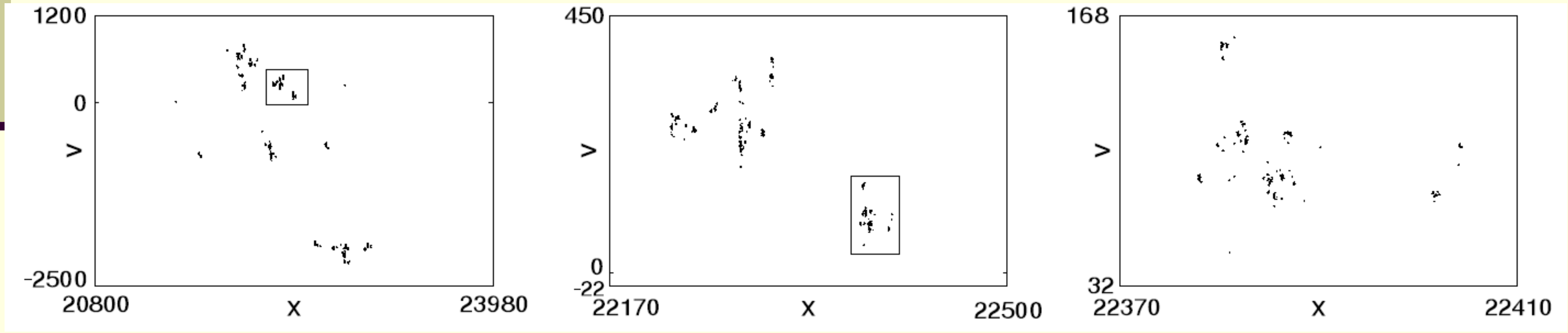

Bruce N. Miller and Jean-Louis Rouet, J. Stat. Mech. P12028 (2010).

# Typical N-body results, contd

**Density** **Phase space**



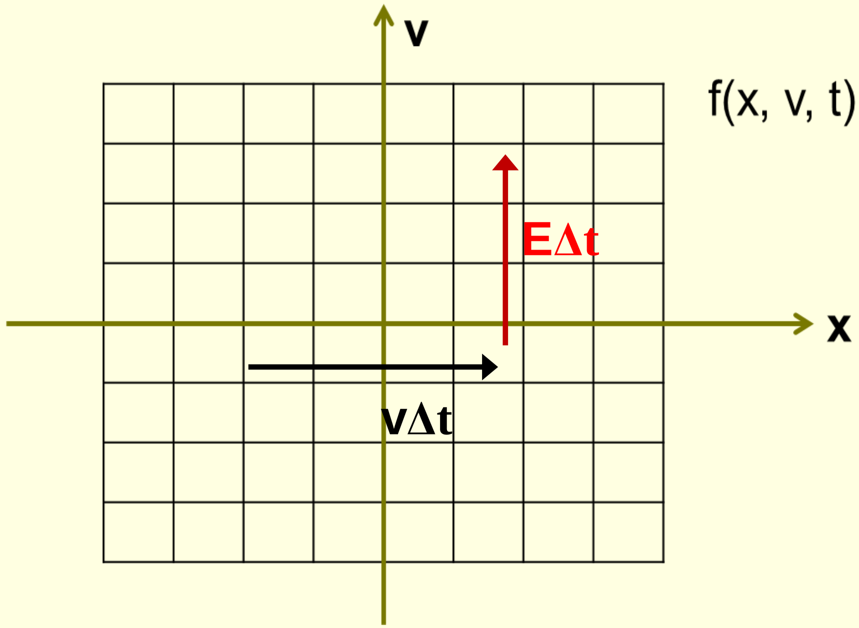

**Phase-space consecutive zooms**




Bruce N. Miller and Jean-Louis Rouet, J. Stat. Mech. P12028 (2010).

# Vlasov approach

- **Solve directly Vlasov equation in the phase space, instead of the N-body eqs. of motion:**
  - Mean field approach ($N \to \infty$)
  - Better statistical properties than N-body codes; lower numerical noise; better resolution at low densities.



# Vlasov approach: new scaling

- **Problem**: with the previous scaling ($\gamma = 2/3$, $\beta = 1$) the rescaled velocity $\eta$ grows indefinitely: **unsuited to Vlasov approach**
- One can show that, with the alternative scaling:

$$A^2(t) = (t/t_0)^\beta \qquad \beta = \frac{3\alpha - 1}{3\alpha} = \frac{9 - \sqrt{2}}{9} \approx 0.84$$

the new velocity space $\eta$ is bounded.

- **Still same expansion factor (EdS): $C(t) = (t / t_0)^{2/3}$**
- The resulting equation of motion is

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}\hat{\theta}^2} + \frac{3+\sqrt{2}}{3\sqrt{2}} \frac{1}{\hat{\theta}/3+1} \frac{\mathrm{d}\xi}{\mathrm{d}\hat{\theta}} - \frac{1}{[\hat{\theta}/3+1]^2}\xi = \frac{1}{[\hat{\theta}/3+1]^2}\hat{\epsilon}$$

# Vlasov-Poisson equations

- The resulting Vlasov-Poisson equations are

$$\frac{\partial f}{\partial \hat{\theta}} + \hat{\eta}\frac{\partial f}{\partial \xi} + \frac{\partial}{\partial \hat{\eta}}\left(\frac{\hat{\epsilon}}{\mu^2(\hat{\theta})}\, f - \frac{K}{\mu(\hat{\theta})}\,\hat{\eta} f\right) = 0,$$

$$\frac{\partial \hat{\epsilon}}{\partial \xi} = 1 - \int f(\xi, \hat{\eta}, \hat{\theta})\, d\hat{\eta},$$

where $K = (3+\sqrt{2})/3\sqrt{2}$ and $\mu(\hat{\theta}) = \hat{\theta}/3 + 1.$

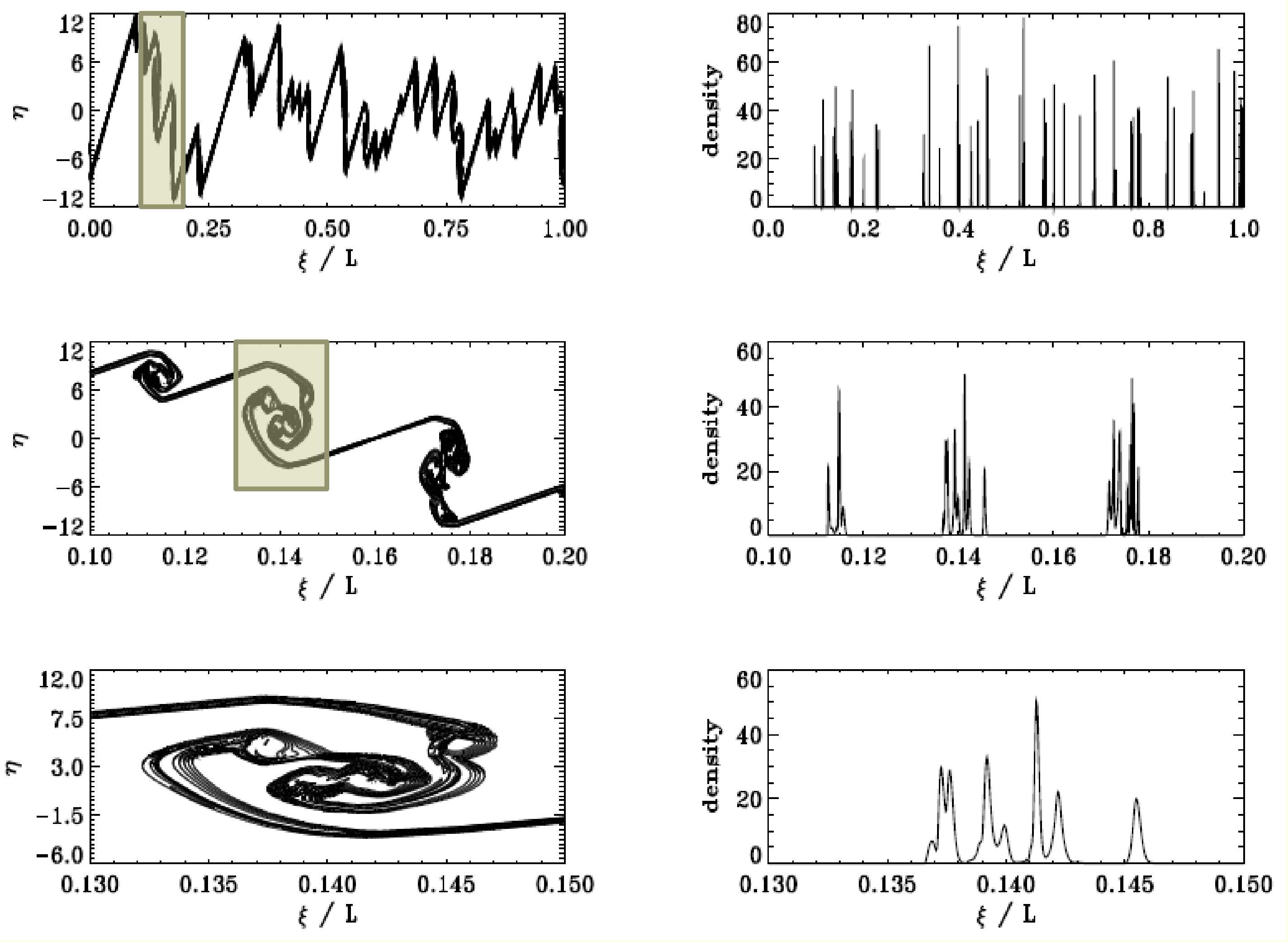
- Note that some terms are time-dependent
- We shall use these equations for the Vlasov simulations

# Typical Vlasov-Poisson simulation

- **Initial condition:**
  - Density: $n = 1 + \varepsilon \sum \tilde{\rho}(k)\, e^{-ikx}$
  - Fluctuation spectrum: $\tilde{\rho}(k) \sim k^n,\ n = 1 - 3,\ \varepsilon \ll 1$
  - Maxwellian velocity distribution with “thermal” width $\eta_{th}$ = 0.2
- **Computational parameters:**
  - L = 31416
  - Max. velocity = 20
  - dt = 0.001, Nx = 16383, Nv = 1500
- **Linear analysis:**
  - Neglecting the friction term in the Vlasov eq., one obtains the dispersion relation: $\omega^2 = -\omega_J^2 + 3k^2\eta_{th}^2$
  - **Instability** for small $k$ (large wave lengths)

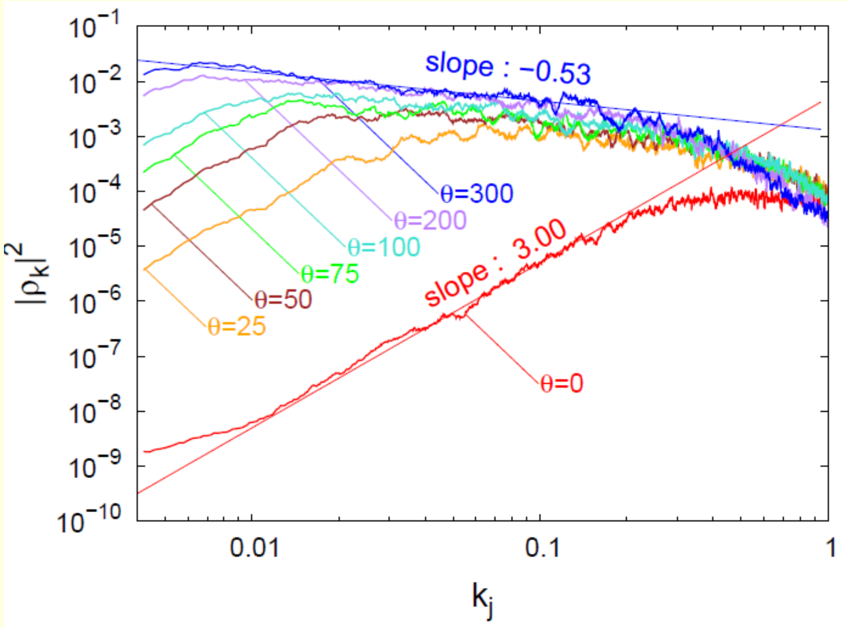
# Results – Phase space and matter density

$\theta = 300$

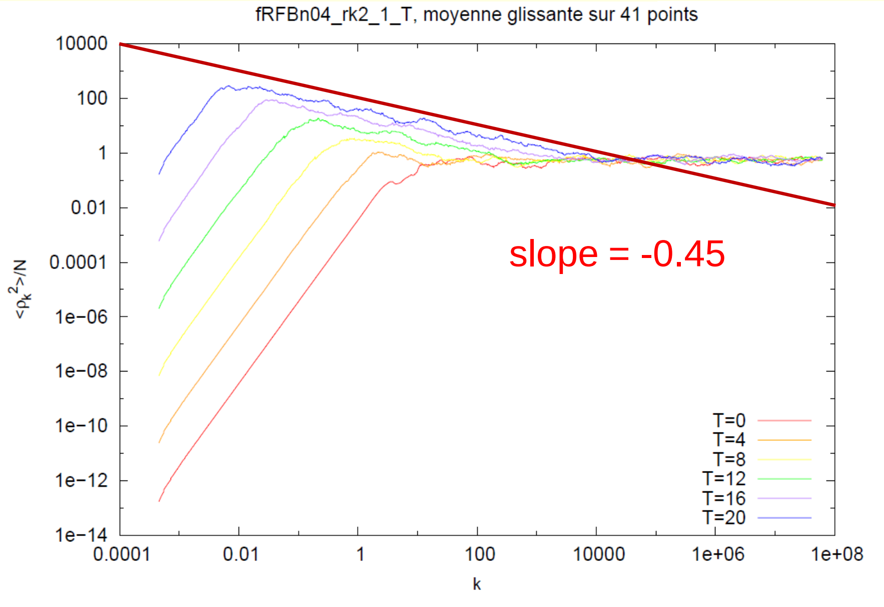


# Density wavenumber spectrum

**Vlasov code**



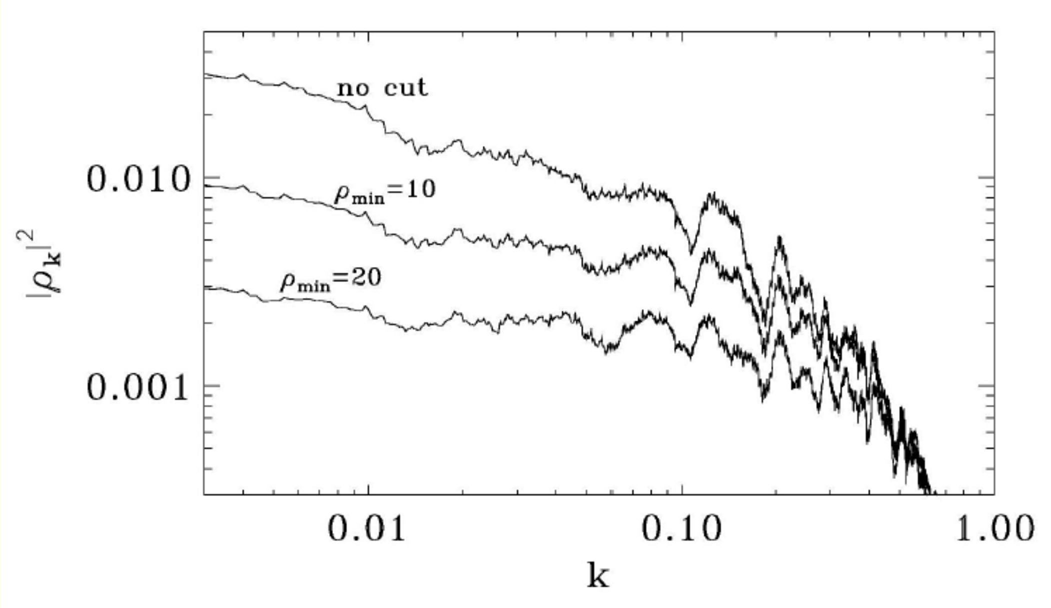

**N-body code**

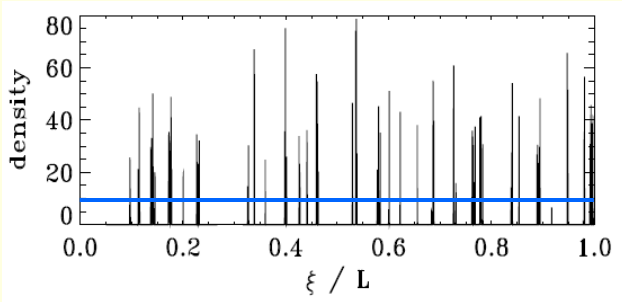



**Analytical prediction: slope = −0.57**

D. Benhaiem, M. Joyce, and F. Sicard, Mon. Not. R. Astron. Soc. **429**, 3423 (2013).

# High-density power spectrum




- Density values below $\rho_{min}$ have been removed
- The spectrum becomes shallower with increasing cut-off
- For $\rho_{min}$=4, one finds the N-body spectrum slope (–0.45)

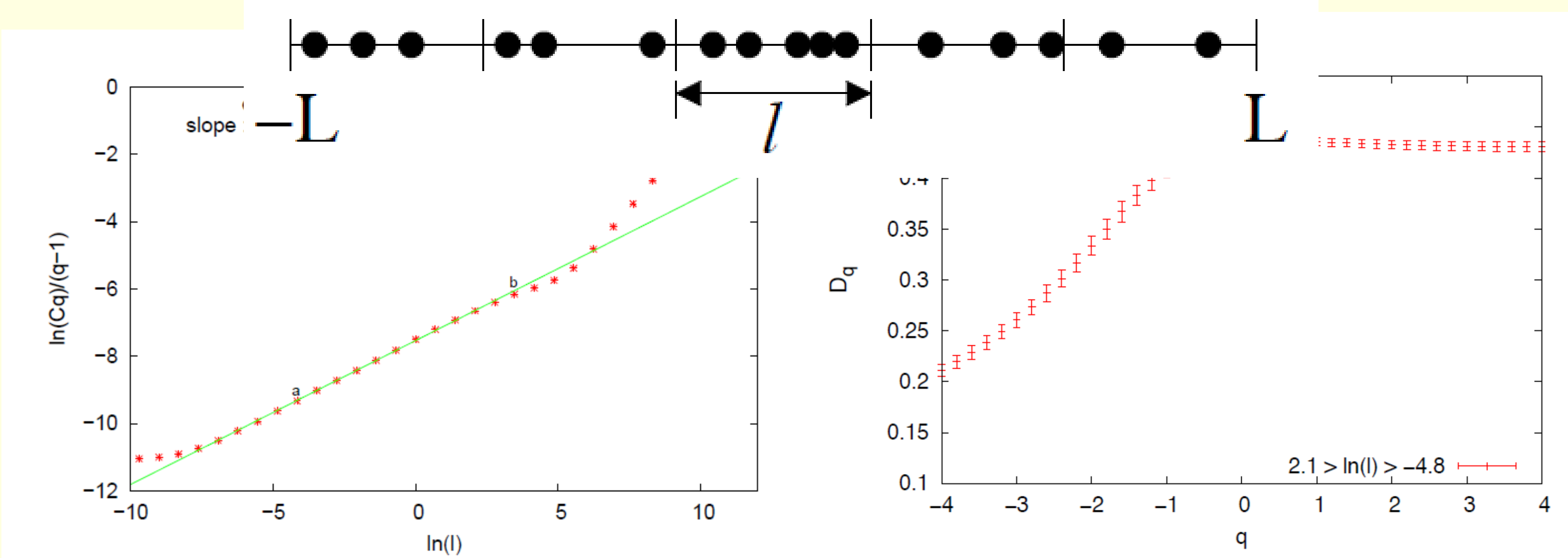
**N-body spectrum is shallower because low density regions are not well represented**

# Fractal dimension (box counting)

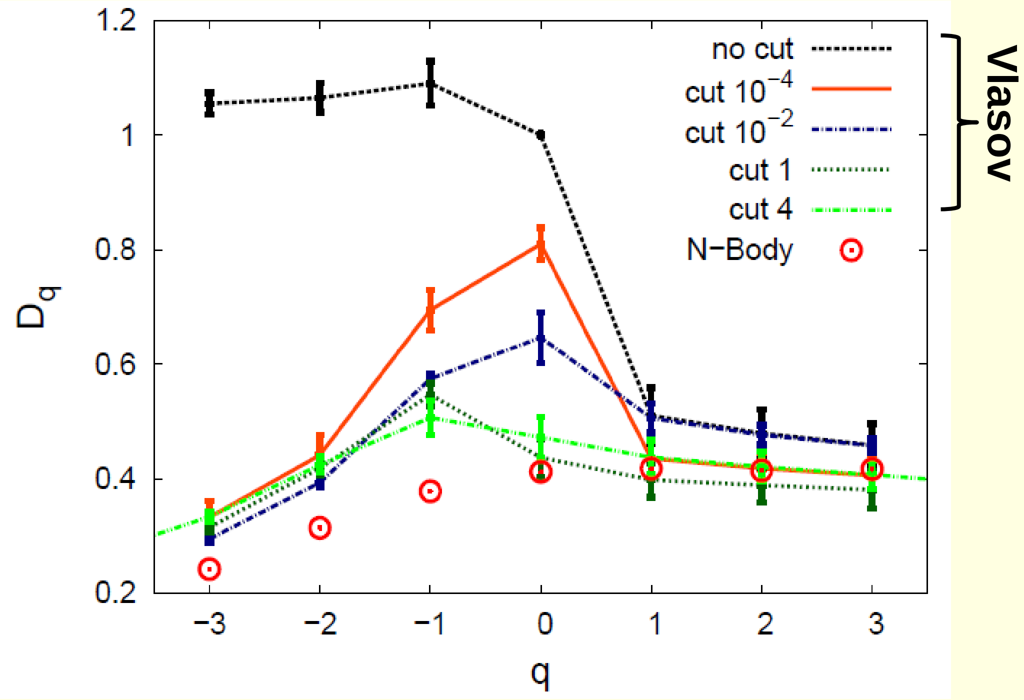
$$D_q = \frac{1}{q-1} \lim_{l \to 0} \frac{\ln C_q}{\ln l}, \quad C_q = \Sigma \mu_i^q,$$

$$\mu_i = \int_{Cell\ "i"} \varrho(x)\, dx$$

box $i$ $\mu_i = N_i/N$



# Fractal dimension – Results




$$D_q = \frac{1}{q-1} \lim_{l \to 0} \frac{\ln C_q}{\ln l}, \quad C_q = \Sigma \mu_i^q,$$

# Conclusions

- **All technical details here:**

**G. Manfredi, J.-L. Rouet, B. Miller, Y. Shiozawa, Phys. Rev. E 93, 042211 (2016).**