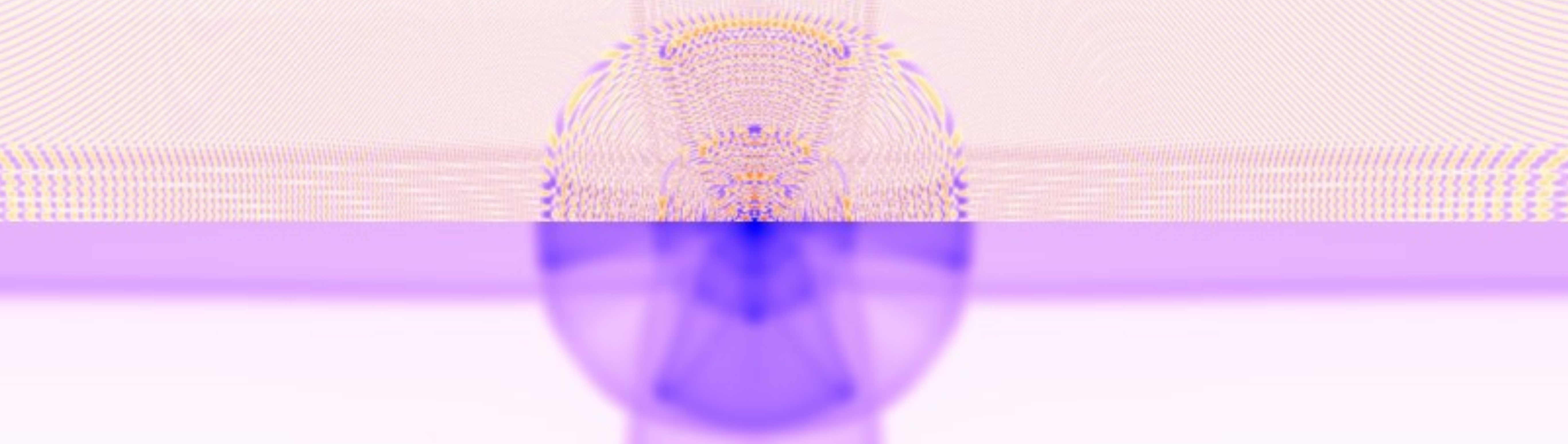
Solving the Vlasov equation in 2 spatial dimensions with the Schrödinger method



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arXiv: 1711.00140

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Fyzikální ústav
Akademie věd ČR, v. v. i.

1. **Introduction**

2. **Comparison** of solving
   coarse grained Vlasov with cold initial conditions (using `ColDICE`) and Schrödinger method (ScM) with
   a) 2D simulations: visually
   b) general formulas
   c) 2D simulations: quantitatively

3. **Application**: cosmological backreaction

4. **Summary & Outlook**

Vlasov equation: Recap and definitions

Continuous phase space distribution function $f(t, \boldsymbol{x}, \boldsymbol{u})$

- ensemble average of Klimontovich f_N, the N-body problem.
- dropping collision terms ~1/N
- moments

Gilbert (APJ 152, 1968)
Binney, Tremaine (1987)
Bertschinger astro-ph/9503125

$$M^{(n)}_{i_1\dots i_n}(\boldsymbol{x}) \equiv \int \mathrm{d}^3u \;\; u_{i_1}\cdots u_{i_n} f(\boldsymbol{x},\boldsymbol{u})$$

- density
- velocity
- velocity dispersion

$$n(\boldsymbol{x}) = M^{(0)} = e^{C^{(0)}}$$
$$u_i(\boldsymbol{x}) = C^{(1)}_i = M^{(1)}_i / n$$
$$C^{(2)}_{ij}(\boldsymbol{x}) = M^{(2)}_{ij}/n - u_i u_j$$

Vlasov (- Poisson) equation (collisionless Boltzmann)

$$\partial_t f(\boldsymbol{x},\boldsymbol{u}) = -\frac{\boldsymbol{u}}{a^2}\boldsymbol{\nabla}_x f + \boldsymbol{\nabla}_x \Phi \boldsymbol{\nabla}_u f$$

$$\Delta\Phi = \frac{4\pi G\rho_0}{a}\left(\int \mathrm{d}^3u \; f - 1\right)$$

nonlinearity

cosmological scale factor

$$\int_{\mathrm{vol}} \mathrm{d}^3x \int \mathrm{d}^3u \; f = \mathrm{vol}$$

Boltzmann hierarchy

$$\partial_t C^{(n)}_{i_1\cdots i_n} = -\frac{1}{a^2}\left\{\nabla_j C^{(n+1)}_{i_1\cdots i_n j} + \sum_{S\in\mathcal{P}(\mathcal{I}=\{i_1,\cdots,i_n\})} C^{(n+1-|S|)}_{\mathcal{I}\setminus S\cup\{j\}}\nabla_j C^{(|S|)}_S\right\} - \delta_{n1}\nabla_{i_1}\Phi$$

Uhlemann, MK, Haugg 1403.5567

consistent truncation:
dust (pressureless perfect) **fluid**

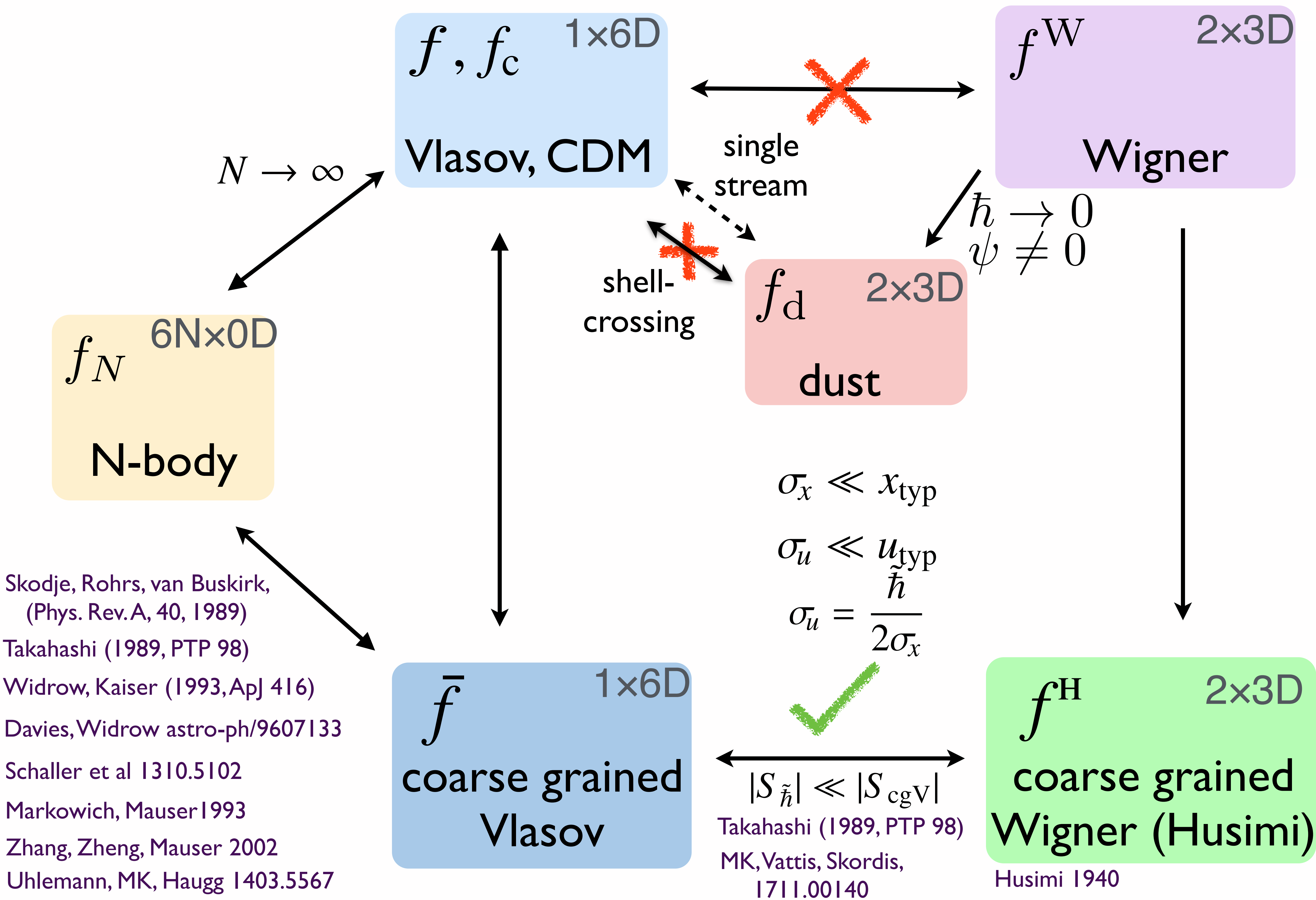
$$C^{(n\geq 2)}(\boldsymbol{x}) = 0$$

$$f_\mathrm{d}(t,\boldsymbol{x},\boldsymbol{u}) = n_\mathrm{d}(t,\boldsymbol{x})\,\delta_\mathrm{D}\big(\boldsymbol{u} - \boldsymbol{\nabla}\phi_\mathrm{d}(t,\boldsymbol{x})\big)$$

Definition of Cold Dark Matter (CDM)

$$\lim_{t\to 0} f_\mathrm{c}(t,\boldsymbol{x},\boldsymbol{u}) = f_\mathrm{d}(t,\boldsymbol{x},\boldsymbol{u})$$

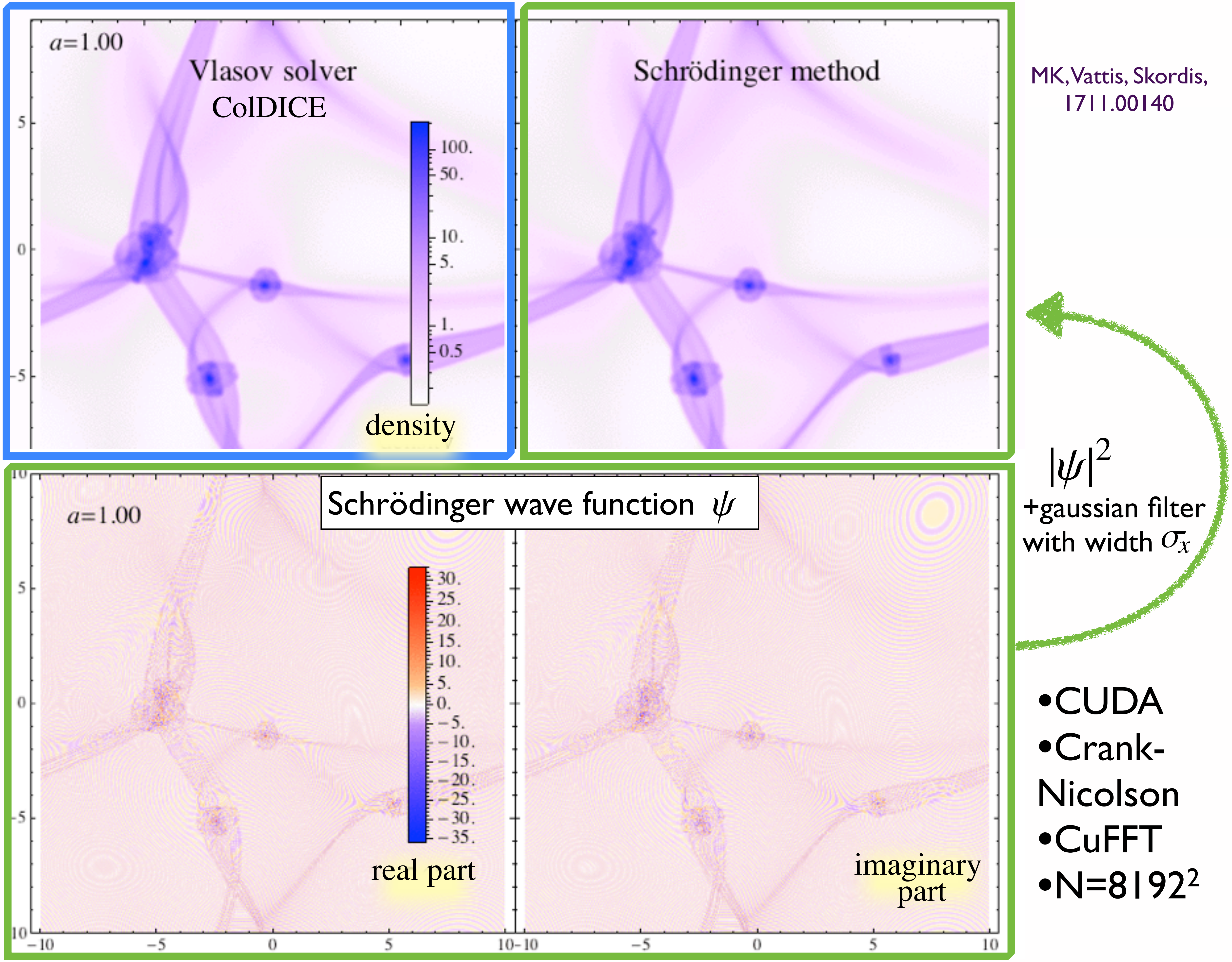
⇒Application in cosmology


$f\,, f_{\rm c}$ 1×6D
Vlasov, CDM
$f^{\rm W}$ 2×3D
Wigner
f_N 6N×0D
N-body
$f_{\rm d}$ 2×3D
dust
$\bar{f}$ 1×6D
coarse grained Vlasov
$f^{\rm H}$ 2×3D
coarse grained Wigner (Husimi)
$N \to \infty$
single stream
shell-crossing
$\hbar \to 0$
$\psi \neq 0$
$\sigma_x \ll x_{\rm typ}$
$\sigma_u \ll u_{\rm typ}$
$\sigma_u = \frac{\tilde{\hbar}}{2\sigma_x}$
$|S_{\tilde{\hbar}}| \ll |S_{\rm cgV}|$
Takahashi (1989, PTP 98)
MK, Vattis, Skordis, 1711.00140
Husimi 1940
Skodje, Rohrs, van Buskirk, (Phys. Rev. A, 40, 1989)
Takahashi (1989, PTP 98)
Widrow, Kaiser (1993, ApJ 416)
Davies, Widrow astro-ph/9607133
Schaller et al 1310.5102
Markowich, Mauser 1993
Zhang, Zheng, Mauser 2002
Uhlemann, MK, Haugg 1403.5567
Garny, Konstandin 1710.04846

2a) Comparison
of
2D cosmological
simulations by eye

for `ColDICE` and ScM

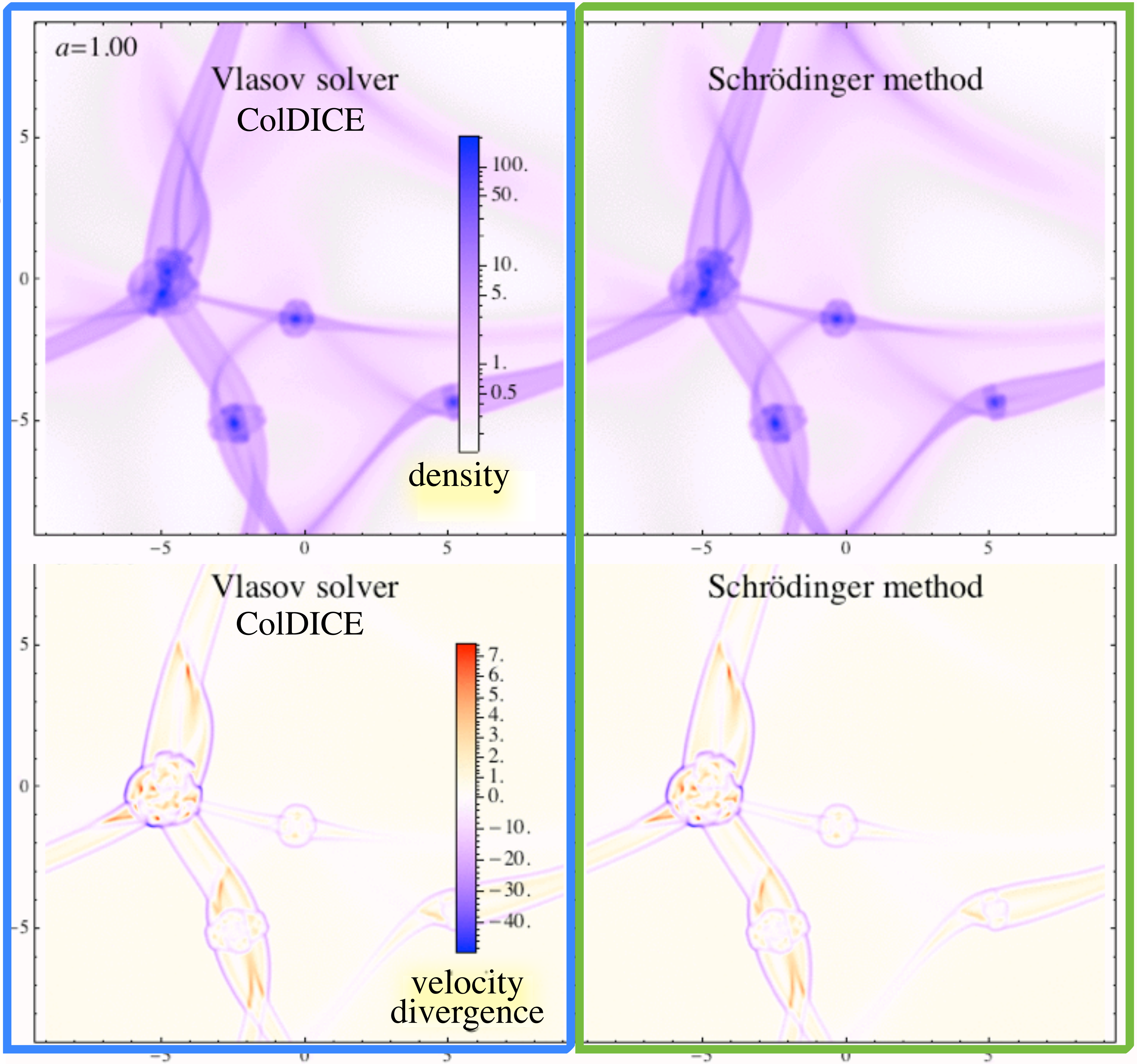
Sousbie, Colombi, (JCPH,321, 644, 2016) 1509.07720



MK, Vattis, Skordis, 1711.00140

- CUDA
- Crank-Nicolson
- CuFFT
- $N=8192^2$

Sousbie, Colombi, (JCPH,321, 644, 2016) 1509.07720



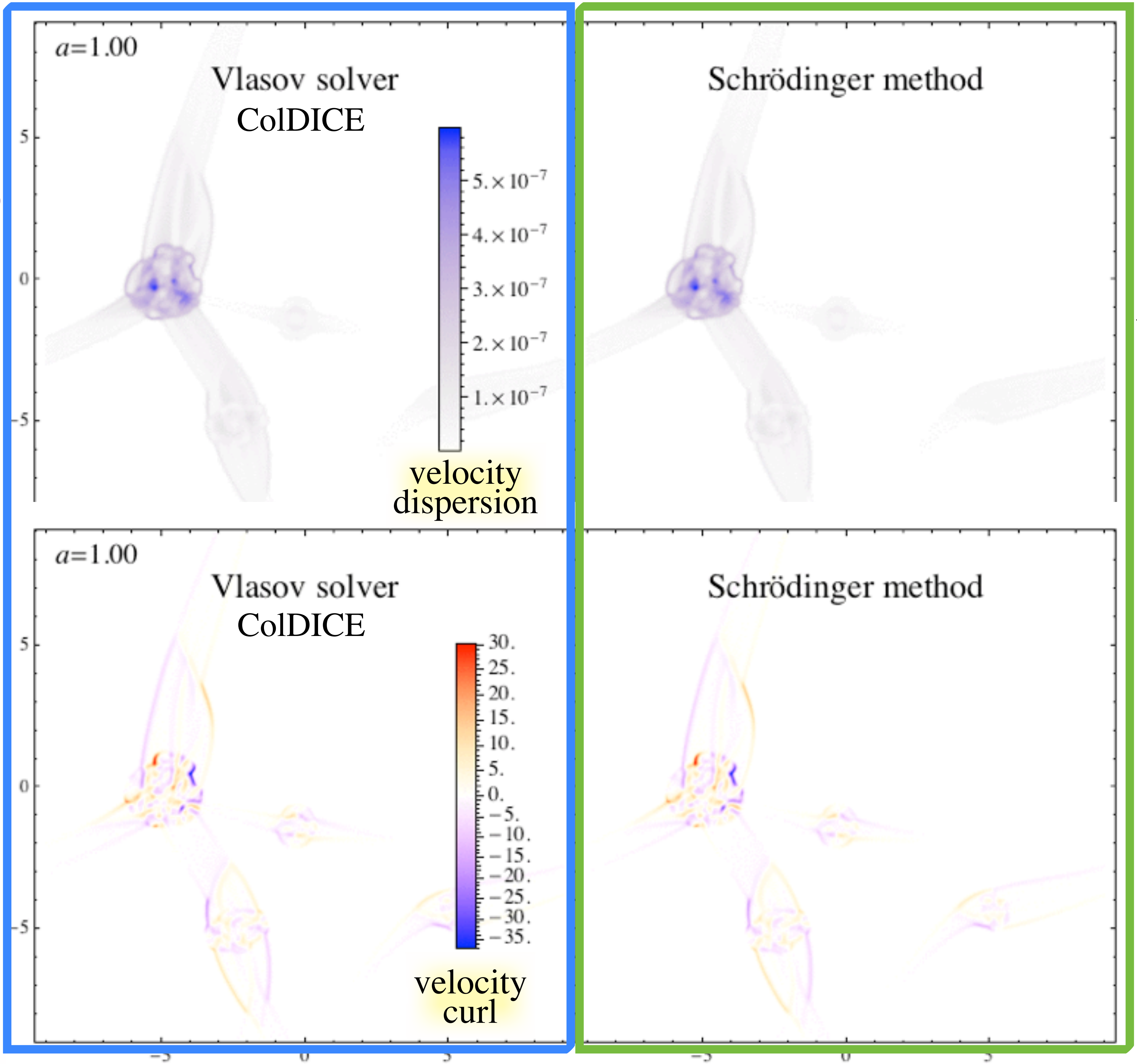
MK, Vattis, Skordis, 1711.00140

$M^{\mathrm{w}(0)} = |\psi|^2$

+gaussian filter with width σ_x, σ_u

$M_i^{\mathrm{w}(1)} = \tilde{\hbar}\, \Im \{\psi_{,i}\bar{\psi}\}$

Sousbie, Colombi, (JCPH,321, 644, 2016) 1509.07720



MK, Vattis, Skordis, 1711.00140

$$M^{\mathrm{w}(2)}_{ij} = \frac{\tilde{\hbar}^2}{2}\Re\left\{\psi_{,i}\bar{\psi}_{,j} - \psi_{,ij}\bar{\psi}\right\}$$

+gaussian filter with width σ_x, σ_u

$$M^{\mathrm{w}(1)}_{i} = \tilde{\hbar}\,\Im\left\{\psi_{,i}\bar{\psi}\right\}$$

2b) Comparison
of
mathematical formulations

of CDM and ScM

Classical (CDM)	Schrödinger–Poisson
Degrees of freedom: 2×d	**2**
$\mathbb{R}^d \to \mathbb{R}^{2\times d}: \quad \boldsymbol{X}(\boldsymbol{q}), \boldsymbol{U}(\boldsymbol{q})$	$\mathbb{R}^d \to \mathbb{R}^2: \quad \mathfrak{R}\{\psi(\boldsymbol{x})\}, \mathfrak{I}\{\psi(\boldsymbol{x})\}$
Dynamics: 2 non-localities	**1 non-locality**
Hamiltonian equations: $a^2\partial_t \boldsymbol{X}(\boldsymbol{q}) = \boldsymbol{U}(\boldsymbol{q})$, $\partial_t \boldsymbol{U}(\boldsymbol{q}) = -\boldsymbol{\nabla}_x \Phi_c(\boldsymbol{x})\vert_{\boldsymbol{x}=\boldsymbol{X}(\boldsymbol{q})}$	Hamiltonian equations (Arriola, Soler, (JSP, 103, 2001)): $i\tilde{\hbar}\partial_t\psi(\boldsymbol{x}) = -\frac{\tilde{\hbar}^2}{2a^2}\Delta\psi(\boldsymbol{x}) + \Phi_\psi(\boldsymbol{x})\psi(\boldsymbol{x})$
Poisson equation: $\Delta\Phi_c(\boldsymbol{x}) = \frac{4\pi G\rho_0}{a}\left(\sum_{\boldsymbol{q} \text{ with } \boldsymbol{x}=\boldsymbol{X}(\boldsymbol{q})} \frac{1}{\vert\det \partial_{q^i} X^j(\boldsymbol{q})\vert} - 1\right)$ — sum over streams	Poisson equation: $\Delta\Phi_\psi(\boldsymbol{x}) = \frac{4\pi G\rho_0}{a}\left(\vert\psi(\boldsymbol{x})\vert^2 - 1\right)$
Phase space distr.: non-local	**quasi-local**
$f_c(\boldsymbol{x},\boldsymbol{u}) = \sum_{\boldsymbol{q} \text{ with } \boldsymbol{x}=\boldsymbol{X}(\boldsymbol{q})} \frac{\delta_D(\boldsymbol{u}-\boldsymbol{U}(\boldsymbol{q}))}{\vert\det \partial_{q^i} X^j(\boldsymbol{q})\vert}$ — sum over streams	$f_H(\boldsymbol{x},\boldsymbol{u}) := \left\vert \int d^dx' \frac{e^{-\frac{(\boldsymbol{x}-\boldsymbol{x}')^2}{4\sigma_x^2} - \frac{i}{\tilde{\hbar}}\boldsymbol{u}\cdot\boldsymbol{x}'}}{(2\pi\tilde{\hbar})^{d/2}(2\pi\sigma_x^2)^{d/4}} \psi(\boldsymbol{x}')\right\vert^2$ — Gaussian has effective of a few σ_x
Moments: non-local	**quasi-local**
$M^{c(n)}_{i_1,\dots,i_n}(\boldsymbol{x}) = \sum_{\boldsymbol{q} \text{ with } \boldsymbol{x}=\boldsymbol{X}(t,\boldsymbol{q})} \frac{U_{i_1}(\boldsymbol{q})\dots U_{i_n}(\boldsymbol{q})}{\vert\det \partial_{q^i} X^j(\boldsymbol{q})\vert}$ — sum over streams	$M^{H(n)}_{i_1,\dots,i_n}(\boldsymbol{x}) = e^{\frac{\sigma_x^2}{2}\Delta}\left\{\frac{(-i)^n\partial^n}{\partial J_{i_1}\dots\partial J_{i_n}} e^{-\frac{1}{2}\sigma_u^2\boldsymbol{J}^2} \psi\left(\boldsymbol{x}+\frac{\tilde{\hbar}}{2}\boldsymbol{J}\right)\bar{\psi}\left(\boldsymbol{x}-\frac{\tilde{\hbar}}{2}\boldsymbol{J}\right)\Big\vert_{\boldsymbol{J}=0}\right\}$ — n derivatives of ψ

Convergence of ScM to coarse grained Vlasov

Coarse grained Vlasov

$$\partial_t \bar{f} = -\frac{\boldsymbol{u}}{a^2}\boldsymbol{\nabla}_x \bar{f} - \frac{{\sigma_u}^2}{a^2}\boldsymbol{\nabla}_x\boldsymbol{\nabla}_u \bar{f} + \boldsymbol{\nabla}_x \bar{\Phi}\exp({\sigma_x}^2 \overleftarrow{\boldsymbol{\nabla}}_x \overrightarrow{\boldsymbol{\nabla}}_x)\boldsymbol{\nabla}_u \bar{f}$$

$$\bar{f}(\boldsymbol{x},\boldsymbol{u}) = \int \frac{d^3x'\, d^3u'}{(2\pi\sigma_x\sigma_u)^3} e^{-\frac{(\boldsymbol{x}-\boldsymbol{x}')^2}{2{\sigma_x}^2} - \frac{(\boldsymbol{u}-\boldsymbol{u}')^2}{2{\sigma_u}^2}} f(\boldsymbol{x}',\boldsymbol{u}') = e^{\frac{{\sigma_x}^2}{2}\Delta_x + \frac{{\sigma_u}^2}{2}\Delta_u}\{f\} \qquad \sigma_u = \frac{\tilde{\hbar}}{2\sigma_x}$$

Husimi equation (automatically satisfied if Schrödinger-Poisson is solved)

$$\partial_t f_{\mathrm{H}} = -\frac{\boldsymbol{u}}{a^2}\boldsymbol{\nabla}_x f_{\mathrm{H}} - \frac{{\sigma_u}^2}{a^2}\boldsymbol{\nabla}_x\boldsymbol{\nabla}_u f_{\mathrm{H}} + \Phi_{\mathrm{H}}\exp({\sigma_x}^2 \overleftarrow{\boldsymbol{\nabla}}_x \overrightarrow{\boldsymbol{\nabla}}_x)\frac{2}{\tilde{\hbar}}\sin\left(\frac{\tilde{\hbar}}{2}\overleftarrow{\boldsymbol{\nabla}}_x \overrightarrow{\boldsymbol{\nabla}}_u\right) f_{\mathrm{H}}$$

$$\partial_t f_{\mathrm{H}} = S_{\mathrm{V}} + S_{\mathrm{cgV}} + S_{\tilde{\hbar}} + O(\tilde{\hbar}^2{\sigma_x}^2, \tilde{\hbar}^4)$$

$$S_{\mathrm{V}} \equiv -\frac{\boldsymbol{u}}{a^2}\cdot\boldsymbol{\nabla}_x f_{\mathrm{H}} + \boldsymbol{\nabla}_x\Phi_{\mathrm{H}}\cdot\boldsymbol{\nabla}_u f_{\mathrm{H}}$$ Vlasov source

$$S_{\mathrm{cgV}} \equiv -\frac{{\sigma_u}^2}{a^2}\boldsymbol{\nabla}_x\cdot\boldsymbol{\nabla}_u f_{\mathrm{H}} + {\sigma_x}^2(\partial_{x_i}\partial_{x_j}\Phi_{\mathrm{H}})(\partial_{x_i}\partial_{u_j} f_{\mathrm{H}}),$$

coarse graining source

$$S_{\tilde{\hbar}} \equiv -\frac{\tilde{\hbar}^2}{24}(\partial_{x_i}\partial_{x_j}\partial_{x_k}\Phi_{\mathrm{H}})(\partial_{u_i}\partial_{u_j}\partial_{u_k} f_{\mathrm{H}})$$

quantum artifact source

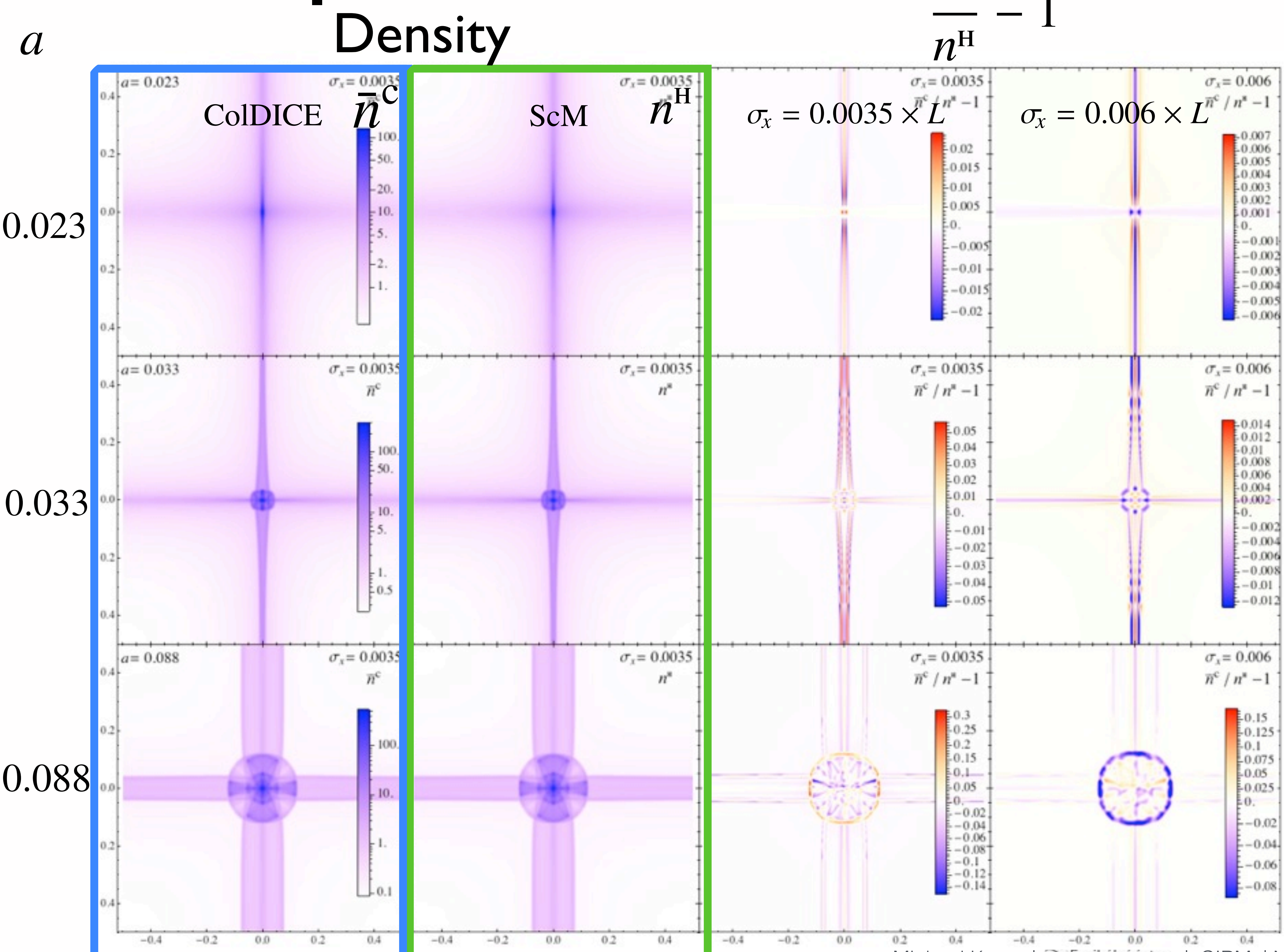
Necessary to approximate coarse grained Vlasov:

$$|S_{\tilde{\hbar}}| \ll |S_{\mathrm{cgV}}|$$

2c) Quantitative **Comparison** of **2D** cosmological **simulations**

for ColDICE and ScM

Sine collapse

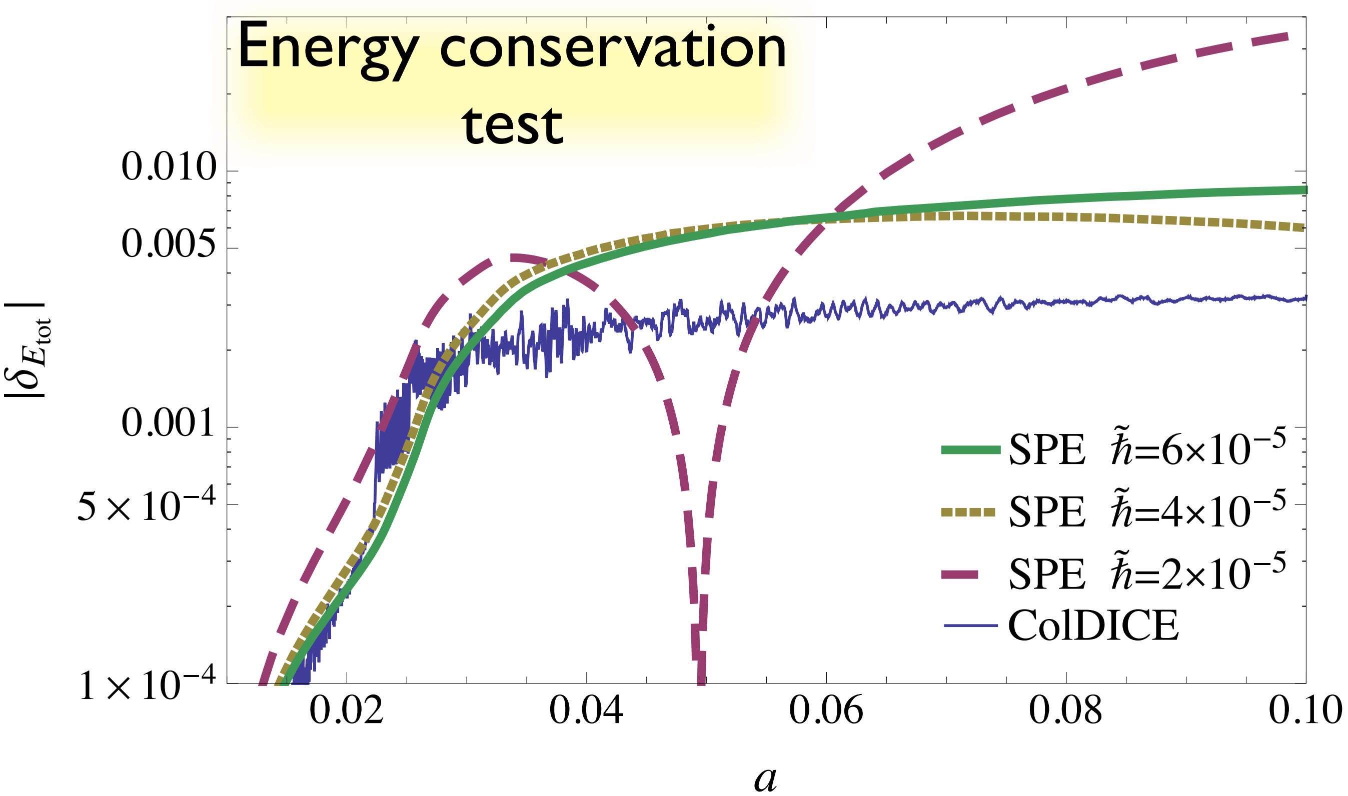


Sine collapse

Numerical convergence

- the larger $\tilde{\hbar}$, the better
- similar to `ColDICE`

$$\delta_{E_{\mathrm{tot}}} \equiv \frac{E_{\mathrm{tot}}(a)}{E(a_{\mathrm{ini}})} - 1 \overset{?}{=} 0$$



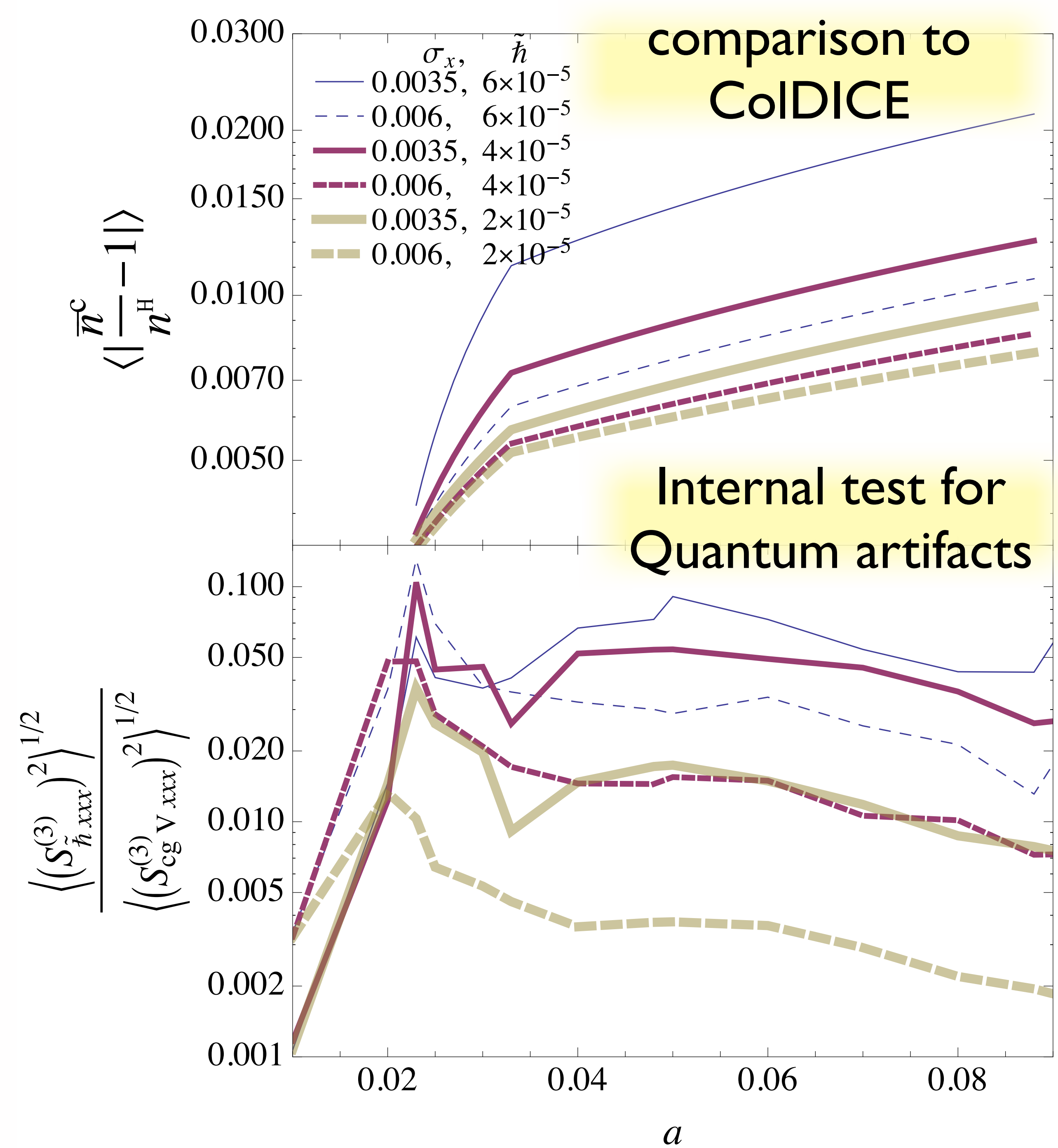
$$K(t) = \frac{\tilde{\hbar}^2}{2a^2}\int d^3x \; |\nabla_x \psi|^2 \qquad E(t) := K(t) + W(t)$$

$$E_{\mathrm{tot}} := E(t) + E_{\mathrm{exp}}(t)$$

$$W(t) = \frac{1}{2}\int d^3x \;\; \Phi_\psi \, |\psi|^2 \qquad E_{\mathrm{exp}} = \int_{a_{\mathrm{ini}}}^{a} \frac{2K(a') + W(a')}{a'} da'$$

Convergence to Vlasov

- the smaller $\tilde{\hbar}$, the better
- $\langle \bar{n}^{\mathrm{C}} / n^{\mathrm{H}} - 1 \rangle \lesssim 1\%$



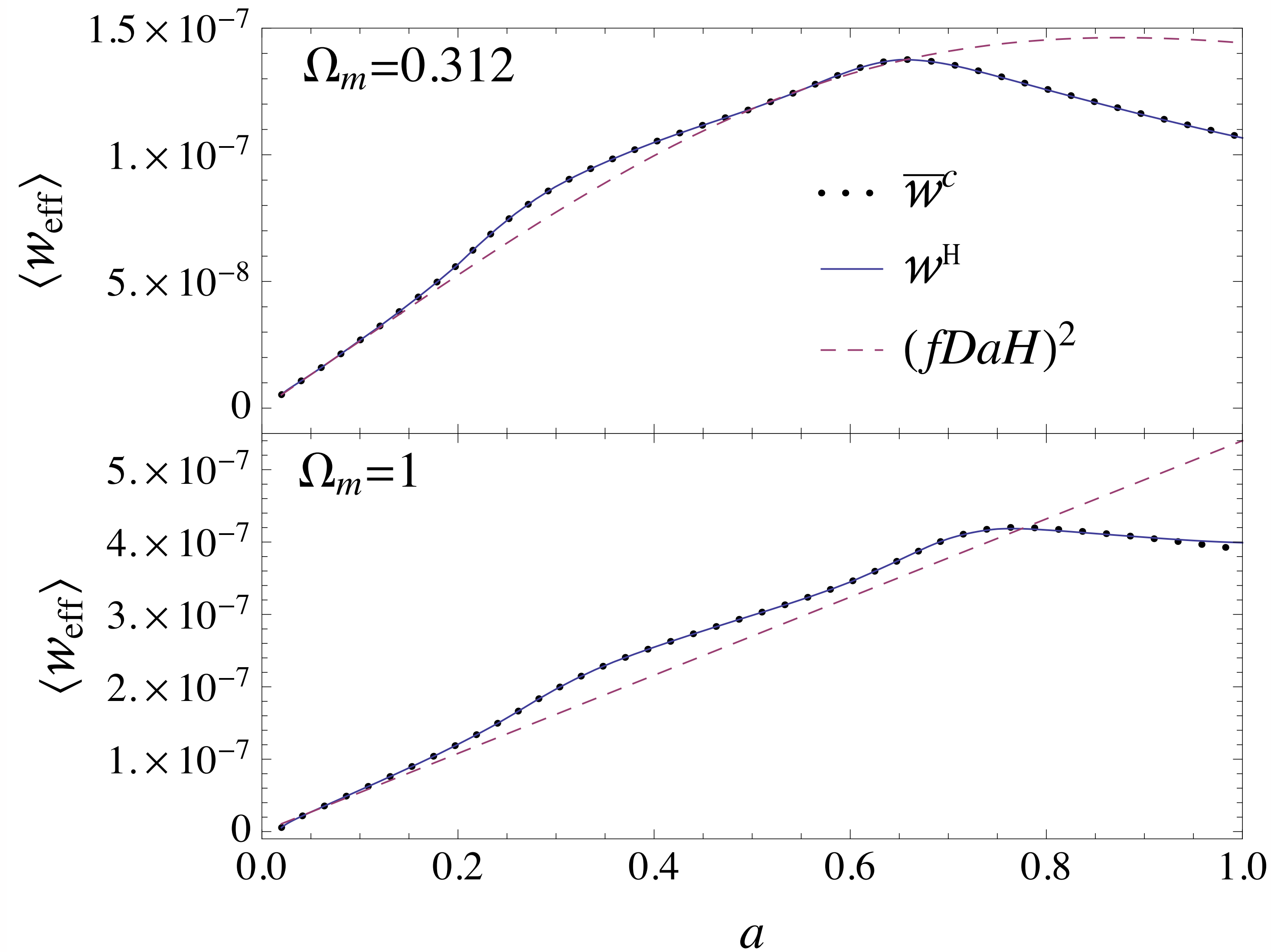
3) Cosmological backreaction estimate from ColDICE and ScM

Effective pressure

$$T^i{}_i = \rho_0 M^{(2)}_{ii}/a^5 = 2P_{\rm eff}$$

$$w_{\rm eff} \equiv P_{\rm eff}a^3/\rho_0$$

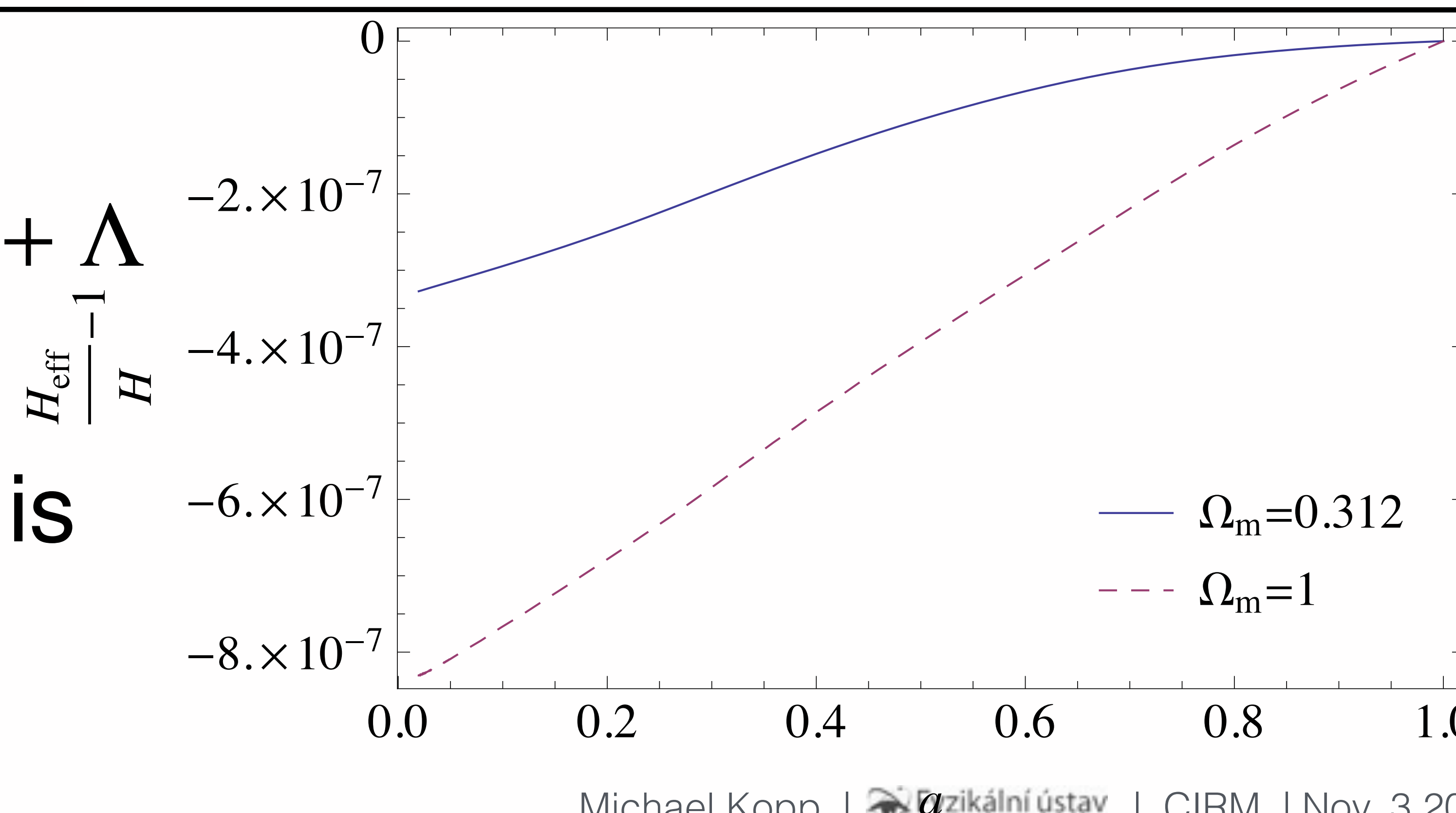
- Excellent agreement between ColDICE and ScM
- ScM can be used as basis for EFTofLSS



Backreaction estimate

$$3H^2_{\rm eff} + 2\dot{H}_{\rm eff} = -8\pi G\langle P_{\rm eff}\rangle + \Lambda$$

- effect on expansion rate is negligibly small



4) Summary

MK, Vattis, Skordis, 1711.00140

1. Convergence of ScM to coarse grained Vlasov for $\hbar \to 0$
2. Excellent agreement with ColDICE
3. Many advantages:
   a) only 2 degrees of freedom, UV complete
   b) phase space can be avoided
   c) quasi-local in eulerian space
   d) *f* sampled uniformly, but minimal resolution $\tilde{\hbar}$

Future plans

Schive et al (ApJS, 186, 2010)
Schive Chiueh, Broadhurst (Nature 2014)

1. 3D implementation with AMR, using GAMER
2. Warm initial conditions and test against non-cold Vlasov solvers, like VlaPoly

Colombi, Touma (MNRAS 441, 2014)

3. Non-perturbative field theory methods applied to ScM

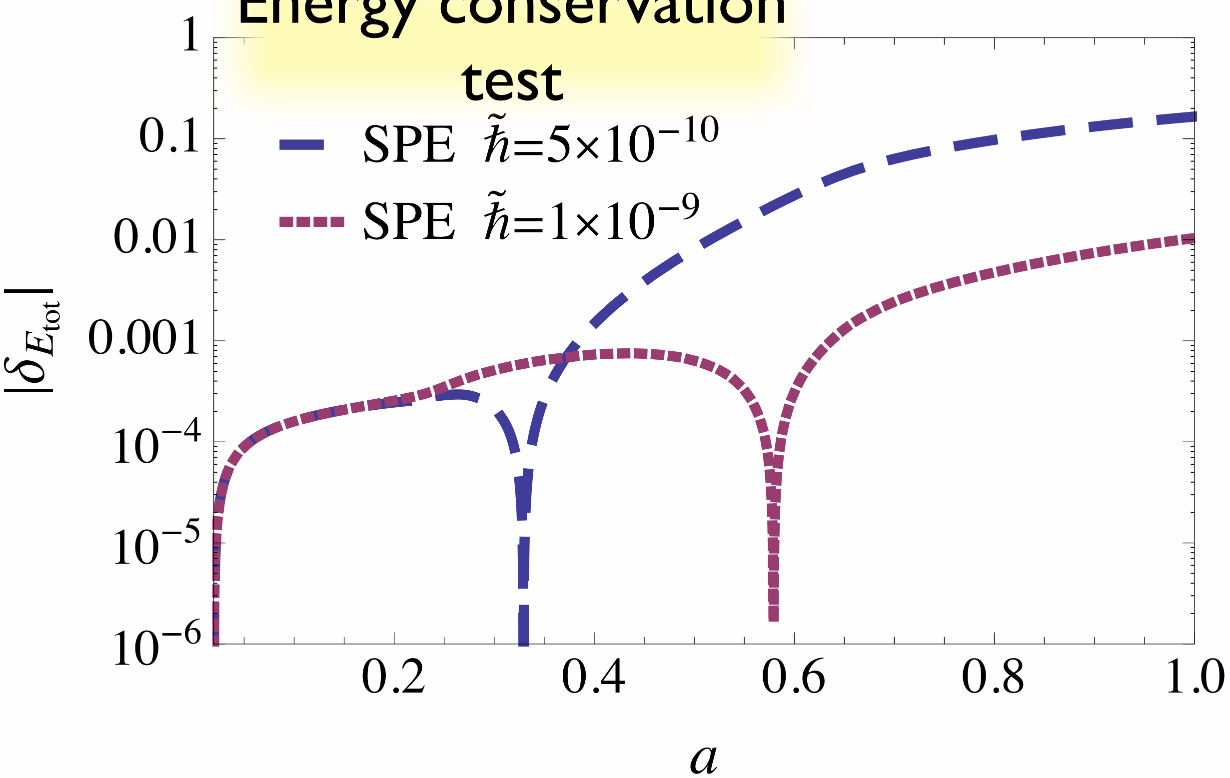
Backup slides

Gaussian random field collapse

Numerical convergence

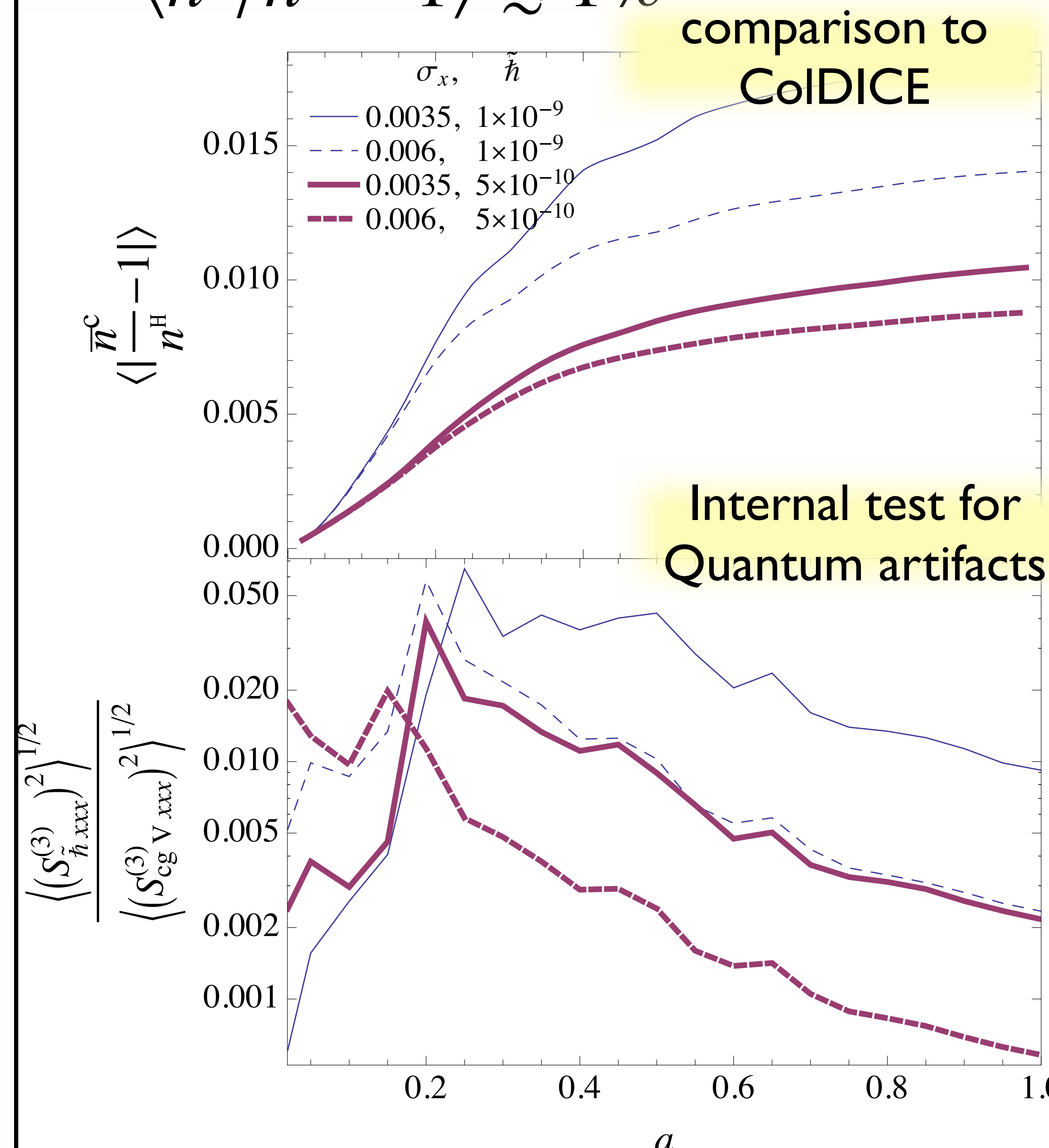
- is easier for larger
- less than `ColDICE` (has AMR)

$$\delta_{E_{\rm tot}} \equiv \frac{E_{\rm tot}(a)}{E(a_{\rm ini})} - 1$$



convergence to Vlasov

- is better for smaller $\tilde{\hbar}$
- $\langle \bar{n}^{\rm C}/n^{\rm H} - 1\rangle \lesssim 1\%$



Discretization and solution of Schrödinger Equation

$$i\tilde{\hbar}\partial_a\psi = -\frac{\tilde{\hbar}^2}{2a^3H}\Delta\psi + \frac{\Phi_\psi}{aH}\psi.$$

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operator splitting via alternating direction implicit (ADI) method

Peaceman, Rachford (1955)
Guenther, (1995).

$\lambda = da/\epsilon$

$$e^{-\frac{i\lambda\tilde{\hbar}\epsilon}{4Ha^3}\frac{\partial^2}{\partial x^2}}S(x,y) = e^{\frac{i\lambda\tilde{\hbar}\epsilon}{4Ha^3}\frac{\partial^2}{\partial x^2}}\psi(a,x,y)$$

$$e^{-\frac{i\lambda\tilde{\hbar}\epsilon}{4Ha^3}\frac{\partial^2}{\partial y^2}}T(x,y) = e^{\frac{i\lambda\tilde{\hbar}\epsilon}{4Ha^3}\frac{\partial^2}{\partial y^2}}S(x,y)$$

$$e^{\frac{i\lambda\epsilon}{2\tilde{\hbar}Ha}\Phi_\psi}\psi(a+\lambda\epsilon,x,y) = e^{-\frac{i\lambda\epsilon}{2\tilde{\hbar}Ha}\Phi_\psi}T(x,y)$$

---

5 point stencil finite differences

$$\frac{\partial^2 f(x_0)}{\partial x^2} \approx \frac{1}{12\epsilon^2}\hat{\delta}_x f(x_0)$$

$$\hat{\delta}_x f(x_0) \equiv -f(x_0-2\epsilon) + 16f(x_0-\epsilon) - 30f(x_0) + 16f(x_0+\epsilon) - f(x_0+2\epsilon)$$

Crank-Nicolson form.
Pentadiagonal matrix inversion

Goldberg, Schey, Schwartz (1967)
Jia, Jiang (2013)

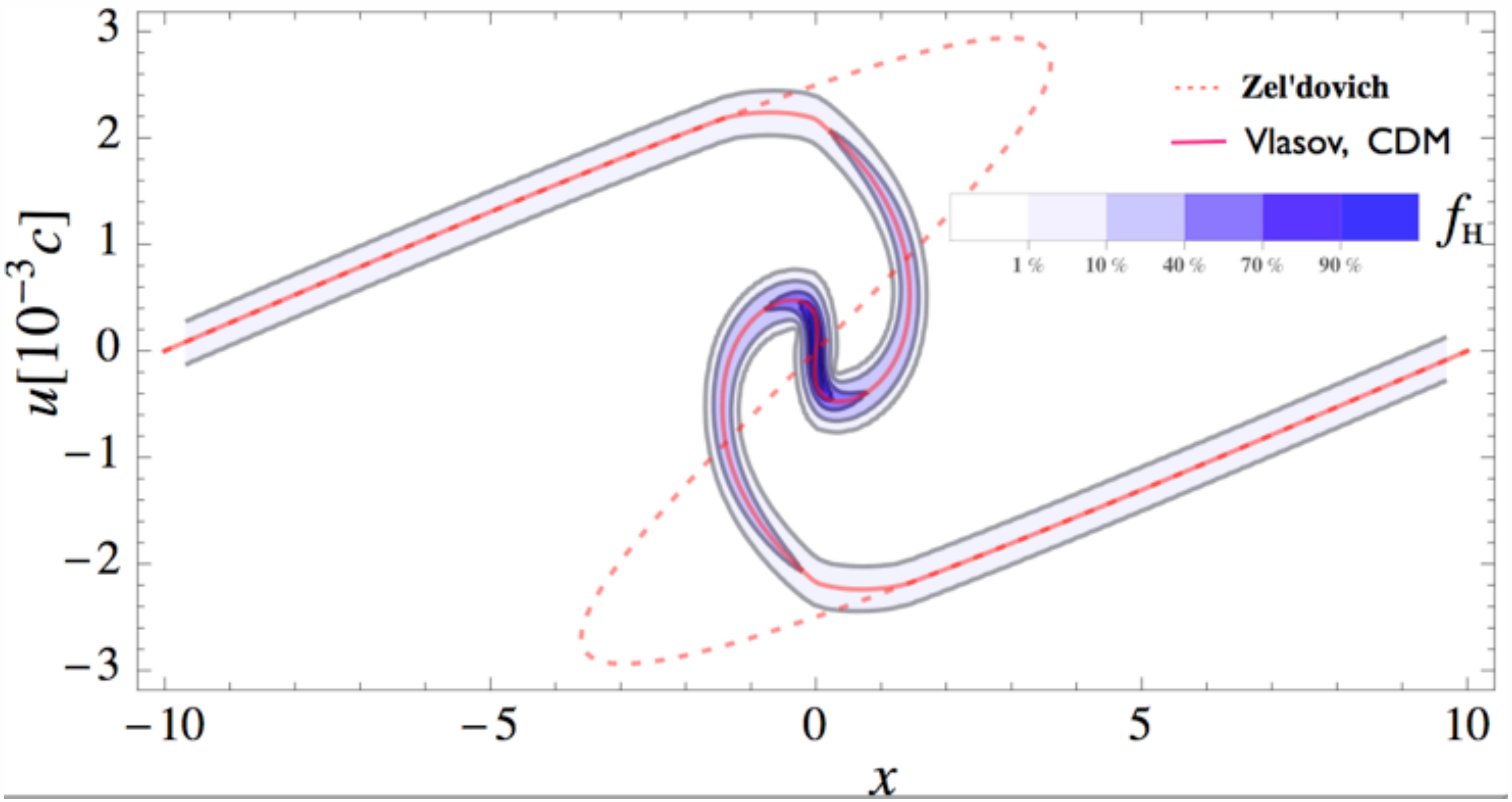
- unconditionally stable
- mass conserving
- error in time, space: $\mathcal{O}(\lambda^2\epsilon^2), \mathcal{O}(\epsilon^4)$

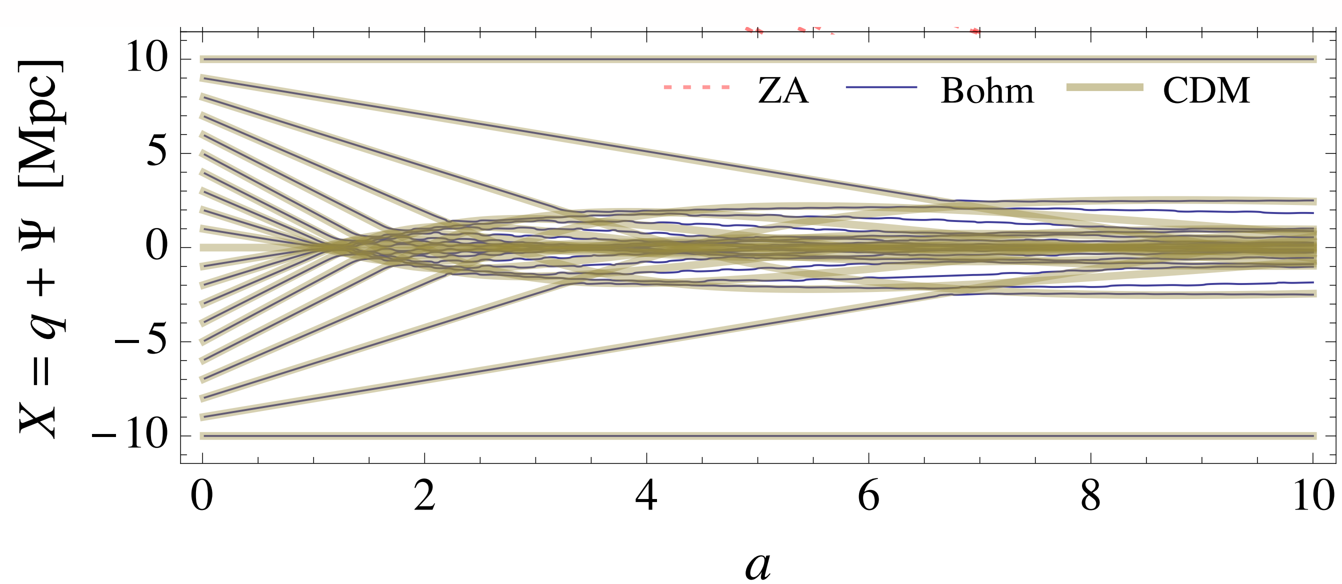
$$\left(1-\frac{i\lambda\tilde{\hbar}}{48\epsilon Ha^3}\hat{\delta}_i\right)S_{ij} = \left(1+\frac{i\lambda\tilde{\hbar}}{48\epsilon Ha^3}\hat{\delta}_i\right)\psi^n_{ij}$$

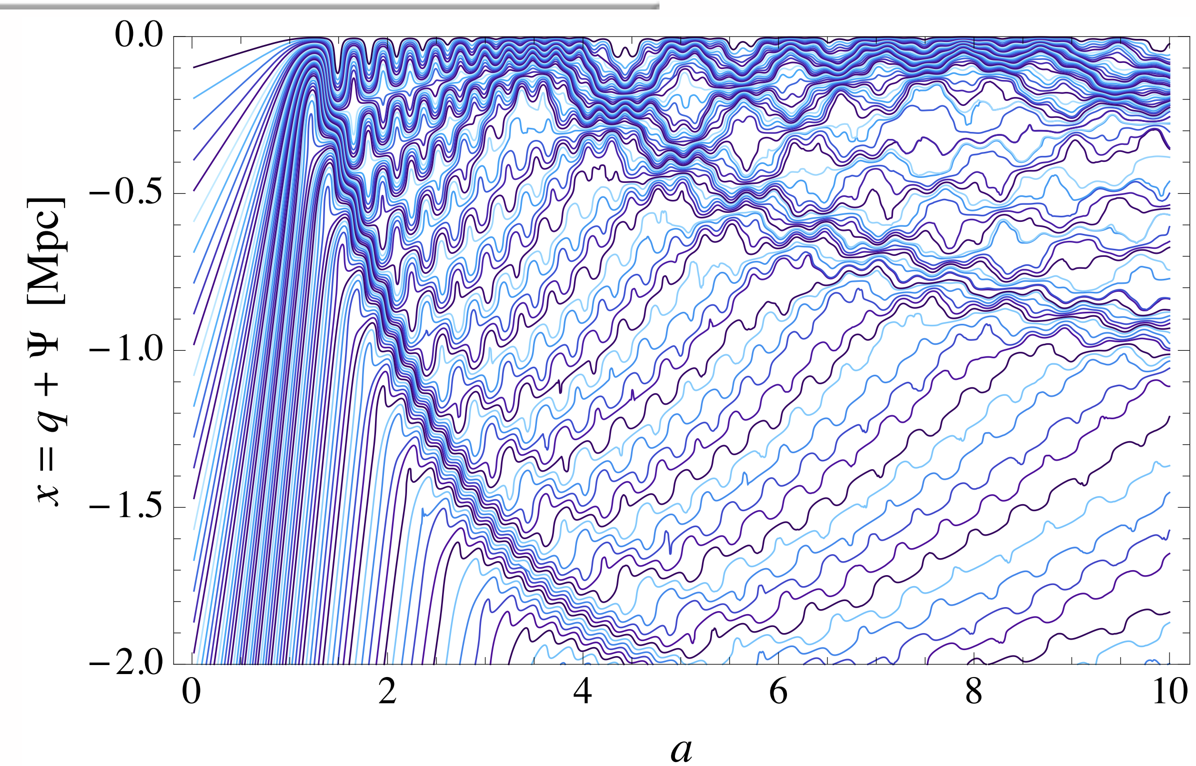
$$\left(1-\frac{i\lambda\tilde{\hbar}}{48\epsilon Ha^3}\hat{\delta}_j\right)T_{ij} = \left(1+\frac{i\lambda\tilde{\hbar}}{48\epsilon Ha^3}\hat{\delta}_j\right)S_{ij}$$

$$\left(1+\frac{i\lambda\epsilon}{2\tilde{\hbar}Ha}\Phi^n_{\psi\,ij}\right)\psi^{n+1}_{ij} = \left(1-\frac{i\lambda\epsilon}{2\tilde{\hbar}Ha}\Phi^n_{\psi\,ij}\right)T_{ij}$$

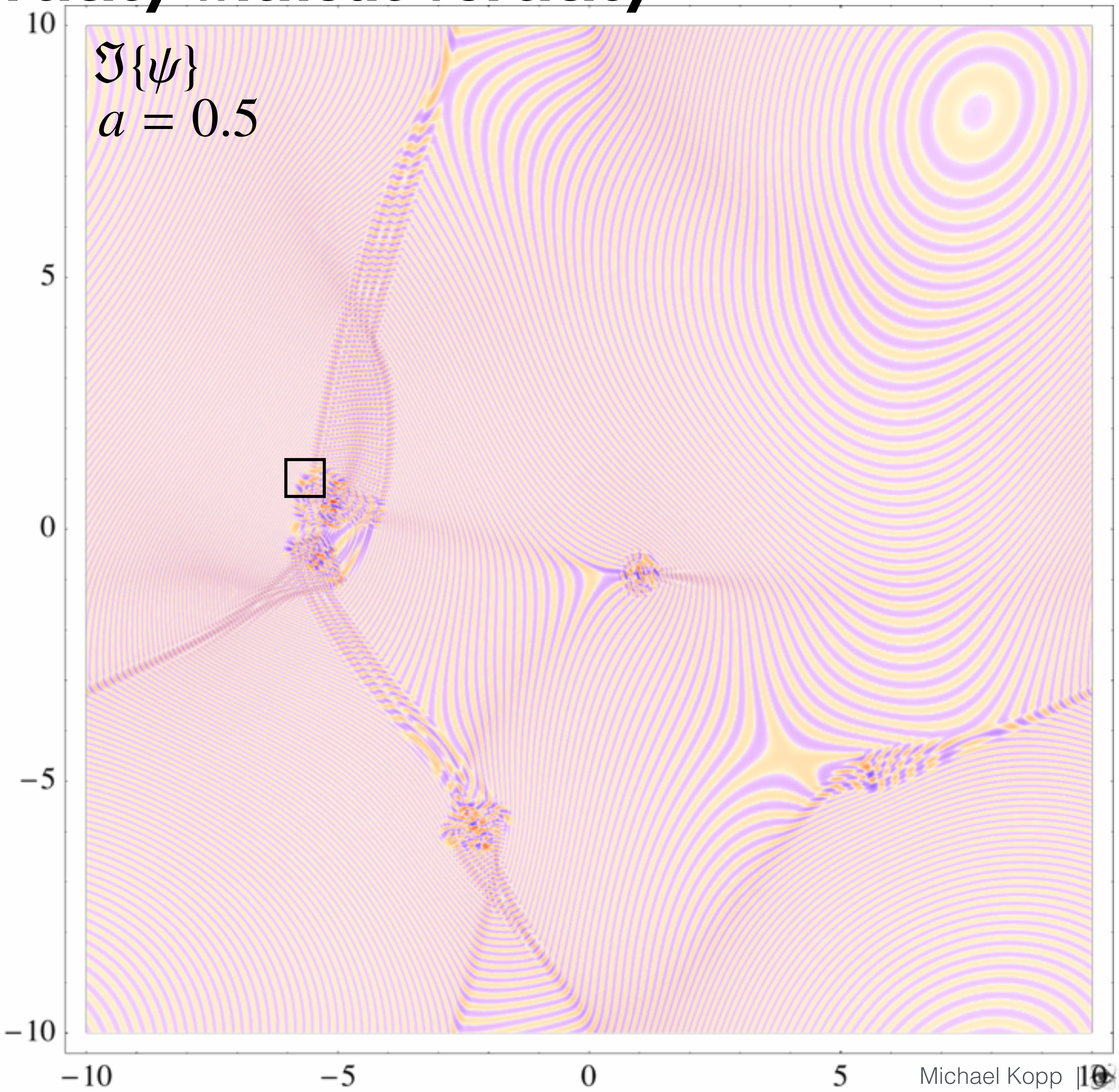
shell-crossing without shell-crossing: 1D pancake



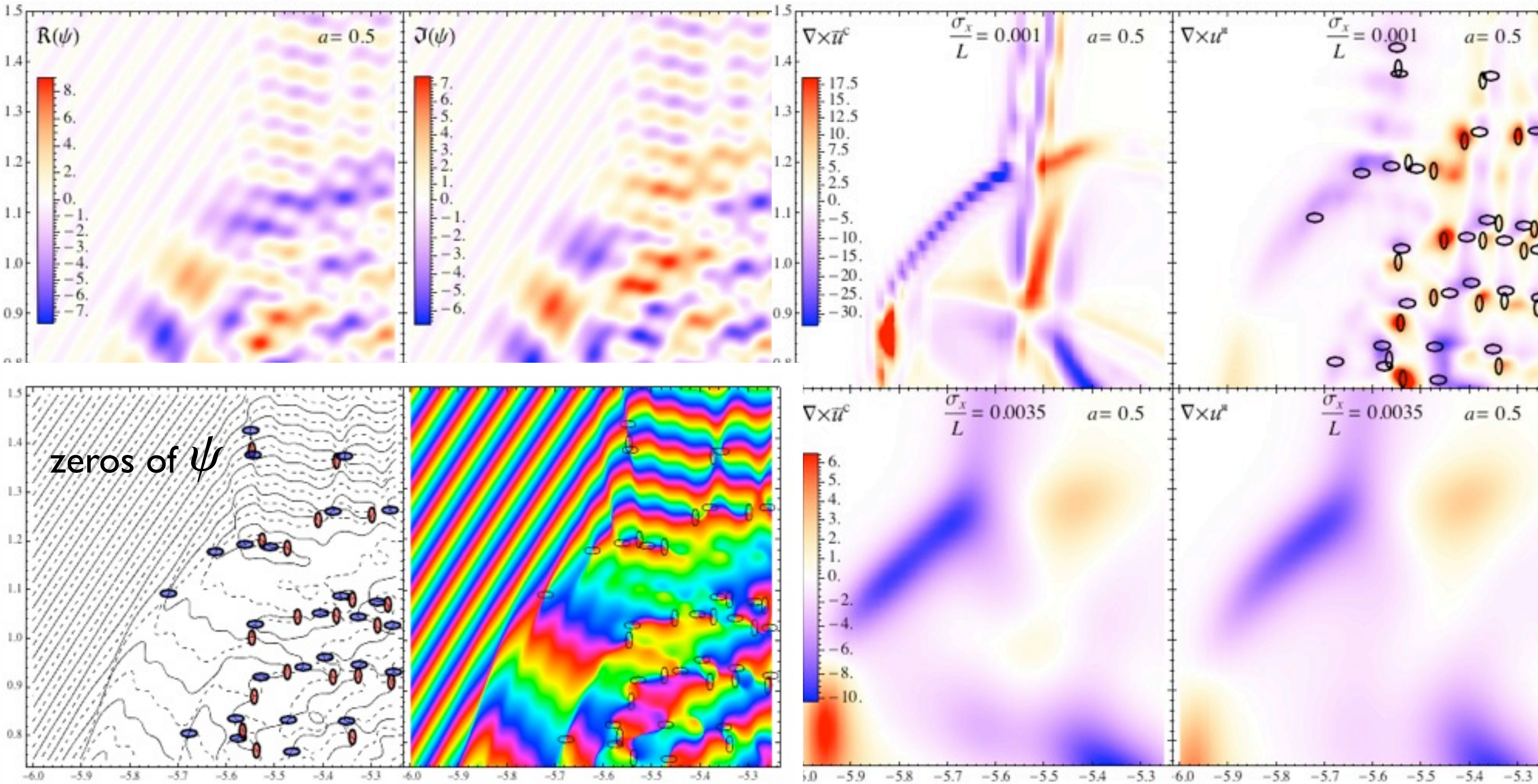




vorticity without vorticity



vorticity without vorticity



ψ single valued $\Rightarrow \frac{1}{2\pi\tilde{\hbar}} \oint_C \nabla\phi \cdot d\boldsymbol{l} = m \in \mathbb{Z} \overset{\text{Stokes}}{\Rightarrow} \boldsymbol{\nabla} \times (\boldsymbol{\nabla}\phi) = \hat{\boldsymbol{z}}\, 2\pi\, \tilde{\hbar}\, m\, \delta_{\mathrm{D}}(\boldsymbol{x}_{\mathrm{vort}})$

$$\boldsymbol{\nabla} \times \boldsymbol{u}_{\mathrm{H}} = \hat{\boldsymbol{z}}\, 2 \frac{\sigma_u}{\sigma_x} \sum_i^{N_{\mathrm{vort}}} m_i e^{\frac{(\boldsymbol{x}-\boldsymbol{x}_i)^2}{2\sigma_x{}^2}} + \sigma_x^2 \left(\boldsymbol{\nabla} \frac{n^{\mathrm{H}}_{,i}}{n^{\mathrm{H}}} \times \boldsymbol{\nabla}\bar{\phi}_{,i} \right) + \mathcal{O}(\sigma_x{}^4)$$

Initial conditions

$\phi_P(\boldsymbol{q})$ here: gaussian random field, or product of sines

$$\boldsymbol{P}(\boldsymbol{q}) = \nabla_q \, \phi_P(\boldsymbol{q})$$

$$\boldsymbol{\Psi} = D(a)\, \boldsymbol{P}(\boldsymbol{q})$$

displacement field in the Zel'dovich appr.

initial velocities and positions

$$\boldsymbol{u}_{\mathrm{d}}(\boldsymbol{X}) = \partial_\eta\big|_q \boldsymbol{\Psi}(\boldsymbol{q}),$$

$$\boldsymbol{x} = \boldsymbol{X}(t,\boldsymbol{q}) = \boldsymbol{q} + \boldsymbol{\Psi}(t,\boldsymbol{q}),$$

Eulerian density and velocity potential of a dust fluid

$$n_{\mathrm{d}}(t,\boldsymbol{x}) = \left\{\det\left[\delta_{ij} + D(a)\partial_{q^i}\partial_{q^j}\phi_P(\boldsymbol{q})\right]\right\}^{-1}\Big|_{\boldsymbol{q}=\boldsymbol{Q}(t,\boldsymbol{x})}$$

$$\phi_{\mathrm{d}}(t,\boldsymbol{x}) = a^2 H f D \left(\phi_P(\boldsymbol{q}) + \frac{1}{2} D(a)\,|\boldsymbol{P}(\boldsymbol{q})|^2\right)\Big|_{\boldsymbol{q}=\boldsymbol{Q}(t,\boldsymbol{x})},$$

Initial wave function

$$\psi_{\mathrm{ini}}(\boldsymbol{x}) = \sqrt{n_{\mathrm{d}}^{\mathrm{ini}}}\exp(i\phi_{\mathrm{d}}^{\mathrm{ini}}/\tilde{\hbar})$$

Computation with a single GPU

- Nvidia K20X GPU (Kepler architecture)
- CUDA C
- CuFFT
- 6GB memory
- ⇒8192^2 maximum resolution