Mean field limit for stochastic systems with $W^{-1,\infty}$ forces

P.-E. Jabin, Z. Wang

Interacting particles

Consider N particles, identical and interacting two by two through the kernel K. Denote $X_i(t) \in \Pi^d$ the position of the i-th particle. Then

$$d\,X_i = \frac{1}{N} \sum_{j \neq i} K(X_i - X_j)\,dt + \varepsilon_N\,dW_i,$$

for N independent Brownian motions W_i^t.
The $\frac{1}{N}$ is a renormalization to get the correct time scale.
One of the most important case is the Biot-Savart law in dimension 2

$$K(x) = -\nabla^\perp \Phi, \quad \Phi(x) = C\,\log|x| + \textit{regular}.$$

Many other kernels of interest exist however...

Some of the questions to answer

- Well posedness for a finite N. Difficulty: Singularity of the force kernel, see for example Flandoli, Gubinelli, Priola.
- The Mean Field limit: Trying to find and justify that a continuum equation provides a good approximation to the system. Difficulty: The large number N of particles and the singularity of the kernel. See Osada and Fournier, Hauray, Mischler.
- Many other important problems: Quantum models, more complex interactions (mean field games, "self-rescaling" models), large time behaviors, corrections in large times to the MF limits...

How large is N?

- In physics, N ranges from 10^{10} to $10^{20} - 10^{25}$; some models of dark matter even predict up to 10^{60} particles.
- When used for numerical purposes (particles' methods...), the number is of order $10^9 - 10^{12}$.
- In biology or Life Sciences, typical population of micro-organisms include between 10^6 and 10^{12}.
- In other applications such as Social Sciences or Economics, numbers can be much lower of order 10^3.

Whenever possible, it is critical to quantify how fast the convergence to the continuous limit holds in terms of N, i.e. to quantify the mean field limit.

Brief mention of the existing Literature

The mean field limit and propagation of chaos has been proved in some stochastic cases

- For the Lipschitz case, $K \in W^{1,\infty}$, see McKean or even Itô.
- For the Biot-Savart law, see Osada, and Fournier, Hauray, Mischler. But no quantitative estimates.
- For Coulomb potential and the Keller-Segel model, see Fournier, Jourdain or Haskovec, Schmeiser...

But we have more results for deterministic systems

- If $K \in W^{1,\infty}$, now very famous results by Neunzert and Wick, Braun and Hepp, Dobrushin, and later Spohn. Still important case to further understand the framework (see for example Golse, Hauray, Mischler, Mouhot, Ricci...).
- For the Biot-Savart law (2d Euler): Goodman, Hou and Lowengrub, Schochet...
- $K \sim |x|^{-\alpha}$, $|\nabla K| \sim |x|^{-\alpha-1}$ with $\alpha < d-1$, in Hauray.

The classical idea for quantitative estimates

When they provide quantitative estimates, most of the previous results rely on a trajectorial approach. Defining the empirical measure

$$\mu_N(t,x,v) = \frac{1}{N}\sum_{i=1}^{N}\delta(x - X_i^N(t)),$$

it solves the approximate equation

$$\partial_t \mu_N + v\cdot\nabla_x \mu_N + (K \star_x \mu_N)\cdot\nabla_v \mu_N = \text{vanishing martingales.}$$

Then one tries to bound some MKW distance, $W_p(\mu_N, f)$ with f the smooth solution to the Vlasov Eq. at the limit. This consists mostly in comparing the two characteristic flows.

Some comments

- The argument always boils down to a Gronwall estimate, with the classical

$$W_1(\mu_N, \rho) \leq e^{\|\nabla K\|_{L^\infty}\, t} \, (W_1(\mu_N^0, \rho^0) + \text{corrections}),$$

  requiring to justify some kind of Lipschitz regularity (directly, through dispersion, weak-strong...).
- Quantitative but slow convergence: $W_1(\mu_N^0, \rho^0) \geq C\, N^{-1/d}$.
- Provides a precise control on the trajectories (good and bad).
- Very unlikely to work for more singular kernels.

Our new result

Theorem

Assume $K = K_1 + K_2$ *with* $K_1 \in L^\infty$, $K_2 = \mathrm{div}\, V$ *with* $\mathrm{div}\, K_2$, $V \in L^\infty$, ρ *a smooth solution to the limiting equation. Consider the law* ρ_N *on* Π^d *of any solution to the SDE system. Then for* $\bar\rho_N = \Pi_{i=1}^N \rho(t, x_i)$, *and for some constant* C *depending on* $\bar\rho$

$$H_N(\rho_N|\bar\rho_N)(t) = \frac{1}{N}\int_{\Pi^{d\,N}} \rho_N \log\left(\frac{\rho_N}{\bar\rho_N^0}\right)$$

$$\leq e^{C\,(\|K_1\|_{L^\infty}+\|V\|_{L^\infty}+\|K_2\|_{L^\infty})\,t}\left(H_N(\rho_N|\bar\rho_N)(t=0) + \frac{C}{N}\right)$$

Consequently for any fixed k, *the marginals* $\rho_{N,k}$ *satisfy*

$$\|\rho_{N,k} - \Pi_{i=1}^k \bar\rho(t, x_i, v_i)\|_{L^1(\Pi^{k\,d})} \leq C_{T,\bar\rho,K}\, N^{-1/2}.$$

Application to 2d Navier-Stokes

At first, the previous results does not apply since

$$\frac{x^\perp}{|x|^2} = C\,\nabla^\perp \log|x| + \text{smooth}, \quad \log|x| \notin L^\infty \ldots$$

But in fact

$$\frac{x^\perp}{|x|^2} = C \begin{bmatrix} \arctan\frac{x_1}{x_2} & 0 \\ 0 & \arctan\frac{x_2}{x_1} \end{bmatrix}.$$

This is connected to difficult mathematical question about which kernels K can be represented this way. See Bourgain, Brézis and Phuc, Torres proving that any $K \in L^{2,\infty}$ is admissible.
In particular, any $|K| \le C/|x|$ works.

The Liouville equation

The Liouville eq. describes the evolution of the law $rho_N(t, x_1, \ldots, x_N)$ of the distribution of the particles

$$\partial_t \rho_N + L_N\, \rho_N = 0,$$

where

$$L_N\, \rho_N = \sum_{i=1}^{N} \frac{1}{N} \sum_{j \neq i} K(x_i - x_j) \cdot \nabla_{v_i} \rho_N - \frac{\epsilon_N^2}{2} \sum_i \Delta_{x_i} \rho_N.$$

The Liouville equation encompasses all the relevant statistical information about the dynamics.

The marginals

The marginals are defined from ρ_N

$$\rho_{N,k}(t,x_1,\ldots,x_k)=\int_{(\Pi^d\times\mathbb{R}^d)^{N-k}}\rho_N(t,\ldots)\,dx_{k+1}\ldots dx_N.$$

Big advantage: The $\rho_{N,k}$ live in a fixed space $\Pi^{d\,k}$. They encompass the relevant, observable physical quantities.

Propagation of chaos holds if for any fixed k

$$\rho_{N,k}\rightharpoonup \Pi_{i=1}^k\bar\rho(t,x_i).$$

It is actually enough to prove it for any $k\geq 2$.
Here we actually prove a strong form of propagation of chaos (as per Hauray-Mischler-Mouhot), controlling a strong norm of

$$\rho_N-\Pi_{i=1}^N\bar\rho(t,x_i).$$

Why is the entropy critical

There are many conserved quantities in the Liouville Eq., for example

$$\|\rho_N(t,\ldots)\|_{L^\infty} \leq \|\rho_N(t=0,\ldots)\|_{L^\infty} \sim C^N.$$

But this does not say anything about the marginals and in fact there is no reason why $\rho_{N,k} \in L^\infty$ uniformly in N. But due to the additive nature of the entropy

$$\frac{1}{k}\int_{\Pi^{k\,d}} \rho_{N,k}\,\log \rho_{N,k} \leq \frac{1}{N}\int_{\Pi^{N\,d}} \rho_N\,\log \rho_N.$$

But ρ_N can be of bounded entropy, while propagation of chaos does not hold ($\rho_N = Z_N\,\mathbb{I}_{\Omega_N}$, $\Omega_N = \bigcup_{x\in K} \Pi_{i=1}^N B(x,1/2)$).

A first attempt at a stability estimate

Define $\bar\rho_N = \Pi_{i=1}^N \bar\rho(t, x_i, v_i)$ and observe that

$$\partial_t \bar\rho_N + L_N\, \bar\rho_N = R_N\, \bar f_N,$$

with

$$R_N = \frac{1}{N} \sum_{i,j} (K(x_i - x_j) - K \star \bar\rho(t, x_i))\ \nabla_{x_i} \log \bar\rho(t, x_i).$$

Calculate the relative entropy thanks to the convexity of the norm and the dissipation due to diffusion. After integrations by parts

$$\frac{d}{dt} \int \rho_N \log \frac{\rho_N}{\bar\rho_N}\, dx \leq \int \rho_N\, \tilde R_N\, dx + \text{lower order terms},$$

with for $K = \operatorname{div} V$

$$\tilde R_N = \frac{1}{N} \sum_{i,j} (V(x_i - x_j) - V \star \rho(t, x_i))\ \frac{\nabla^2_{x_i} \bar\rho(t, x_i)}{\bar\rho(t, x_i)}.$$

Problem: A priori $R_N = O(N)$.

A modified law of large numbers

The main goal throughout all the proof is to justify that in some sense $\tilde{R}_N = O(1)$. Here

$$\int |\tilde{R}_N|^2 \, \bar{\rho}_N = \frac{1}{N^2} \sum_{i_1,i_2,j_1,j_2} \int (V(x_{i_1} - x_{j_1}) - V \star \bar{\rho}(t, x_{i_1}))$$

$$(V(x_{i_2} - x_{j_2}) - V \star \bar{\rho}(t, x_{i_2})) \, \frac{\nabla^2_{x_{i_1}} \bar{\rho}(t, x_{i_1})}{\bar{\rho}(t, x_{i_1})}$$

$$\frac{\nabla^2_{x_{i_1}} \bar{\rho}(t, x_{i_1})}{\bar{\rho}(t, x_{i_1})} \Pi_{k=i_1,i_2,j_1,j_2} \bar{\rho}(t, x_k, v_k)$$

$$\leq \|V\|^2_{L^2} \sup_x \int \frac{|\nabla^2 \bar{\rho}|^2}{\bar{\rho}} \, dv,$$

as the integral vanish if $i_1 \neq i_2$ or if $j_1 \neq j_2$, i.

The relative entropy estimate II

We recall that

$$\frac{d}{dt}H_N \leq \frac{1}{N}\int \tilde{R}_N\,\rho_N \leq \frac{1}{N}\int \tilde{R}_N\,\frac{\rho_N}{\bar{\rho}_N}\,\bar{\rho}_N$$
$$\leq \frac{L}{N}\int \rho_N\,\log\frac{\rho_N}{\bar{f}_N} + \frac{L}{N}\log\int e^{|\tilde{R}_N|/L}\,\bar{\rho}_N,$$

simply by using the positivity of relative entropy.
Therefore the whole question revolves around proving that

$$\int e^{|R_N|/L}\,\bar{\rho}_N = O(1).$$

This is in essence a large deviation estimate which we can prove in this case through a direct combinatorics approach.

A new combinatorics result

Theorem

Consider $\bar\rho \in L^1(\Pi^d)$ *with* $\bar\rho \geq 0$ *and* $\int_{\Pi^d} \bar\rho\,dx = 1$. *Consider further any* $\phi(x,z) \in L^\infty$ *s.t. for some given universal constant*

$$\gamma := C\,\left(\sup_{p\geq 1} \frac{\|\sup_z |\phi(.,z)|\|_{L^p(\bar\rho dx)}}{p}\right)^2 < 1.$$

Assume that ϕ *satisfies the following cancellations*

$$\int_{\Pi^d} \phi(x,z)\,\bar\rho(x)\,dx = 0 \quad \forall z, \qquad \int_{\Pi^d} \phi(x,z)\,\bar\rho(z)\,dz = 0 \quad \forall x.$$

Then

$$\int_{\Pi^{d\,N}} \bar\rho_N \exp\left(\frac{1}{N}\sum_{i,j=1}^N \phi(x_i,x_j)\right) dX^N \leq \frac{3}{1-\gamma} < \infty,$$

Conclusion

- As many recent results, based on a weak-strong principle.
- First step towards more statistical approaches?
- Is there a way to extend the large deviation argument to even more singular kernels providing the right structure is given?