Institute for Advanced Study

SPACE TELESCOPE SCIENCE INSTITUTE

Finite-N Effects and Secular Evolution of Self-Gravitating Systems

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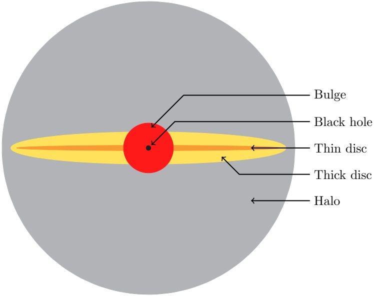
Secular dynamics

- What happens to **orbital structures** on **cosmic age**?

- Numerous systems of interest
  - Stellar discs
  - Galactic centers
  - Dark matter haloes

- How do they respond to:
  - to their stochastic environment?
    $\Longrightarrow$ **Dressed Fokker-Planck**
  - to their internal graininess?
    $\Longrightarrow$ **Balescu-Lenard**

- Properties
  - **Inhomogeneous** (= *complex traject.*)
  - **Relaxed** (= *short dynamical times*)
  - **Self-gravitating** (= *amplifying*)
  - **Perturbed** (= *envt.* + *discret.*)



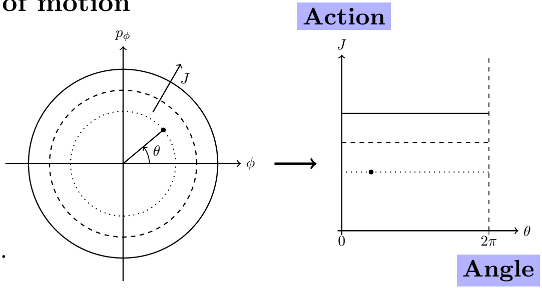
Galaxies are inhomogeneous

- Label orbits using **integrals of motion**

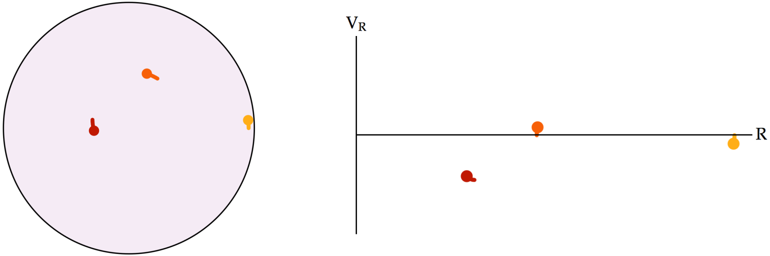
$$\begin{cases} \boldsymbol{\theta}(t) = \boldsymbol{\theta}_0 + t\,\boldsymbol{\Omega}\,, \\ \mathbf{J}(t) = \text{cst.} \end{cases}$$

$\Longrightarrow$ **Straight lines.**

$$\begin{cases} H(\mathbf{q},\mathbf{p}) = H(\mathbf{J})\,, \\ \textbf{Frequencies: } \boldsymbol{\Omega}(\mathbf{J}) = \dfrac{\partial H}{\partial \mathbf{J}}\,. \end{cases}$$



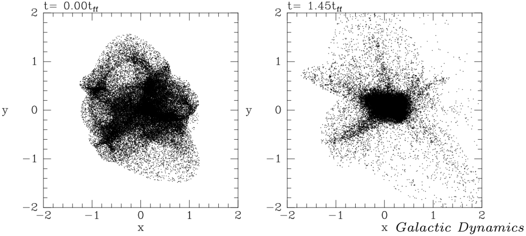
- **Razor-thin discs** (*epicyclic approximation*)



- $2D$ disc $\Longrightarrow$ Actions: $\boldsymbol{J} = (J_\phi, J_r)$.

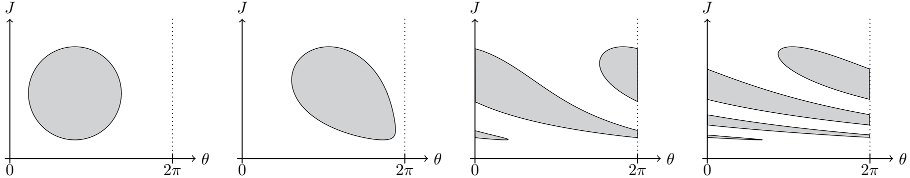
Galaxies are relaxed

- Initial DF $\xrightarrow{\text{(few) } t_{\text{cross}}}$ $F = F(\mathbf{J}, t)$.
  - **Violent relaxation**: $\mathrm{d}E/\mathrm{d}t = \partial\psi/\partial t|_{\mathbf{x}(t)}$



Strong **potential fluctuations** $\Longrightarrow$ Frozen **mean field**.

  - **Phase Mixing**

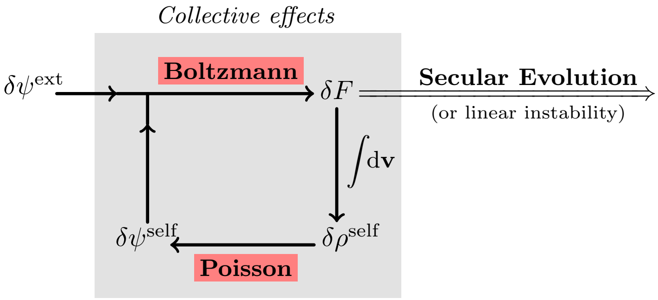


  - Evolution only possible through **fluctuations** $\Longrightarrow$ **Secular evolution**

Orbital distortion driven by resonances

Galaxies are self-gravitating

- **Self-gravitating amplification**



- **Matrix method** (Kalnajs 1976)
  $\Longrightarrow$ **Representative basis**

$$\Delta\psi^{(p)} = 4\pi G \rho^{(p)} \,.$$

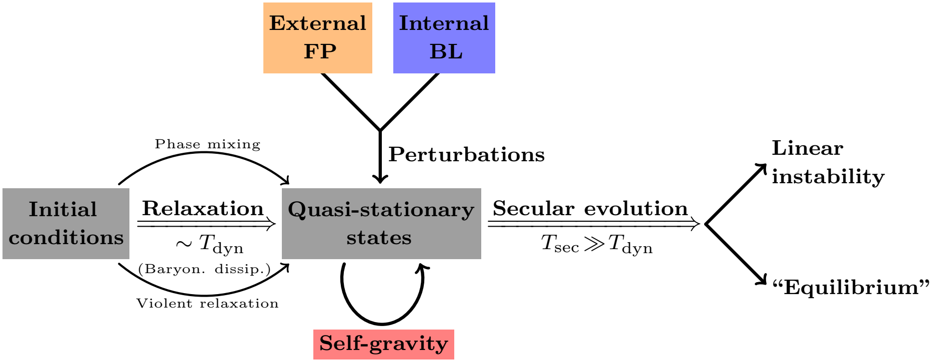
$$\begin{cases} \delta\psi^{\mathrm{ext}}(\mathbf{x},t) = \sum_p b_p(t)\, \psi^{(p)}(\mathbf{x}) \,, \\ \delta\psi^{\mathrm{self}}(\mathbf{x},t) = \sum_p a_p(t)\, \psi^{(p)}(\mathbf{x}) \,. \end{cases}$$

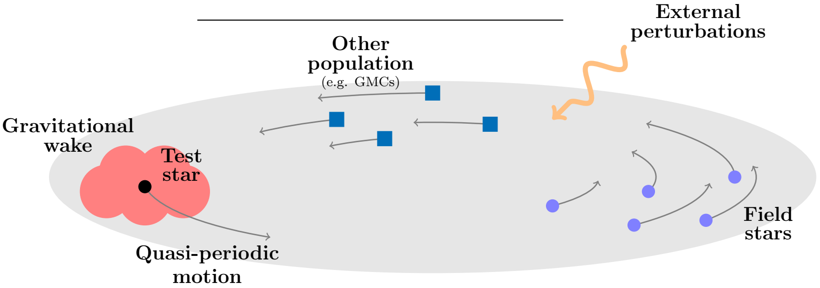
$$\underbrace{\big[\widehat{\boldsymbol{a}} + \widehat{\boldsymbol{b}}\big](\omega)}_{\substack{\textbf{Total} \\ \text{perturbations}}} = \underbrace{\big[\mathbf{I} - \widehat{\mathbf{M}}(\omega)\big]^{-1}}_{\textbf{Dressing}} \cdot \underbrace{\widehat{\boldsymbol{b}}(\omega)}_{\substack{\textbf{External} \\ \text{perturbations}}}$$

- **Response matrix**

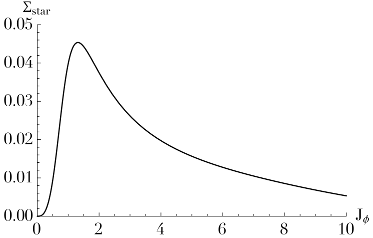
$$\widehat{\mathbf{M}}(\omega) \propto \sum_{\mathbf{m}} \int \mathrm{d}\mathbf{J}\, \frac{\mathbf{m}\cdot\partial F/\partial \mathbf{J}}{\omega - \mathbf{m}\cdot\boldsymbol{\Omega}(\mathbf{J})} \,.$$

The fate of self-gravitating systems

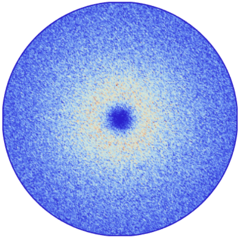




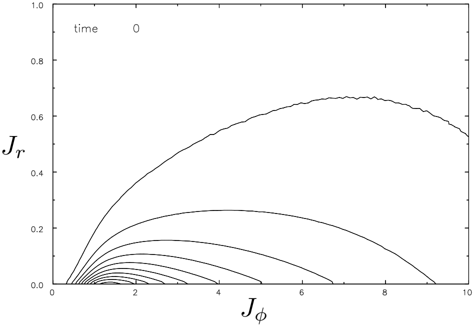
An example of secular evolution

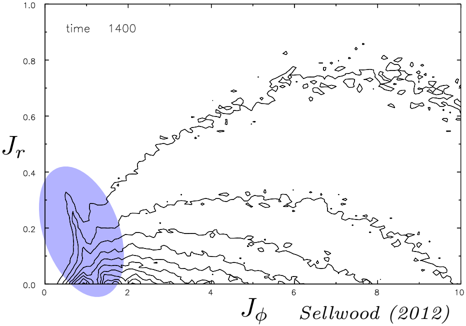


Initial **stable/stationary** DF



Evolved DF

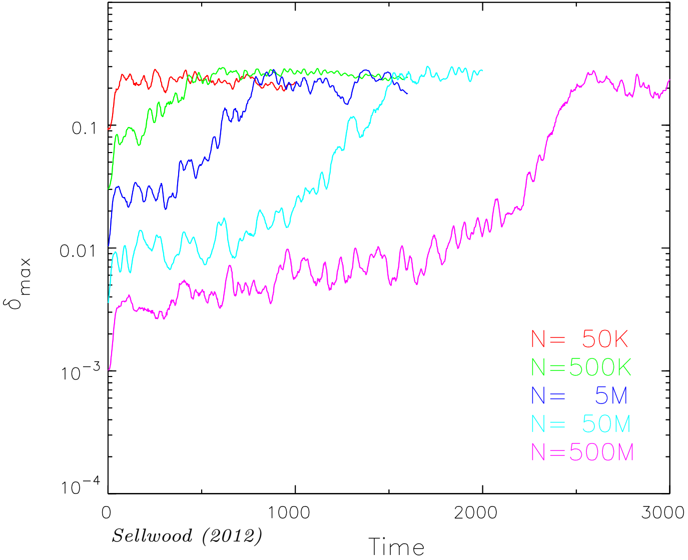




Sellwood (2012)

Secular diffusion in action-space

The role of finite−N effects



Sellwood (2012)

$\Longrightarrow$ The larger the number of particles, the slower the evolution.

The inhomogeneous Balescu-Lenard equation

- Describe the secular evolution driven by **finite−N effects** for a system
  - **inhomogeneous**
  - **stable**
  - **self-gravitating**
  - **isolated**
  - **discrete**

- Some references:
  - *Balescu (1960)*, *Lenard (1960)*: Plasma case
  - *Weinberg (1993)*: Homogeneous approximation
  - *Heyvaerts (2010)*: Angle-Action - BBGKY
  - *Chavanis (2012)*: Angle-Action - Klimontovitch
  - *Fouvry et al. (2015)*: $2D$ with full amplification
  - *Heyvaerts et al. (2017)*: Fokker-Planck approach
  - *Sridhar&Touma (2017)*, *Fouvry et al. (2017)*: Quasi-Keplerian case
  - *Benetti&Marcos (2017)*: HMF model

Inhomogeneous Balescu-Lenard equation

- **Inhomogeneous Balescu-Lenard equation**
  Heyvaerts (2010), Chavanis (2012)

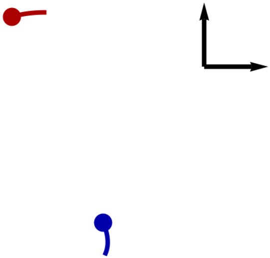
$$\frac{\partial F(\mathbf{J}_1, t)}{\partial t} = \pi (2\pi)^d \frac{M_{\mathrm{tot}}}{N} \frac{\partial}{\partial \mathbf{J}_1} \cdot \bigg[\sum_{\mathbf{m}_1, \mathbf{m}_2} \mathbf{m}_1 \int \mathrm{d}\mathbf{J}_2 \, \frac{\delta_{\mathrm{D}}(\mathbf{m}_1 \cdot \boldsymbol{\Omega}_1 - \mathbf{m}_2 \cdot \boldsymbol{\Omega}_2)}{|\mathcal{D}_{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{J}_1, \mathbf{J}_2, \mathbf{m}_1 \cdot \boldsymbol{\Omega}_1)|^2} \bigg[\mathbf{m}_1 \cdot \frac{\partial}{\partial \mathbf{J}_1} - \mathbf{m}_2 \cdot \frac{\partial}{\partial \mathbf{J}_2} \bigg] F(\mathbf{J}_1, t) \, F(\mathbf{J}_2, t) \bigg] .$$

- Some properties:
  - $F(\mathbf{J}, t)$: **Orbital distorsion** in action space.
  - $1/N$: Driven by **finite$-N$ effects**.
  - $\partial / \partial \mathbf{J}_1 \cdot$: **Divergence of a flux**, i.e. conservation.
  - $\mathbf{m}_1$: **Discrete** Fourier vectors - **Anistropic** diffusion.
  - δ_{D}: **Resonance condition** for distant encounters.
  - $1/\mathcal{D}_{\mathbf{m}_1, \mathbf{m}_2}$: **Self-gravitating dressing** (squared).
  - $\mathbf{m}_1 \cdot \boldsymbol{\Omega}_1$: Secular diffusion **at resonance**.

$\Longrightarrow$ **Master equation for self-induced orbital distortion.**

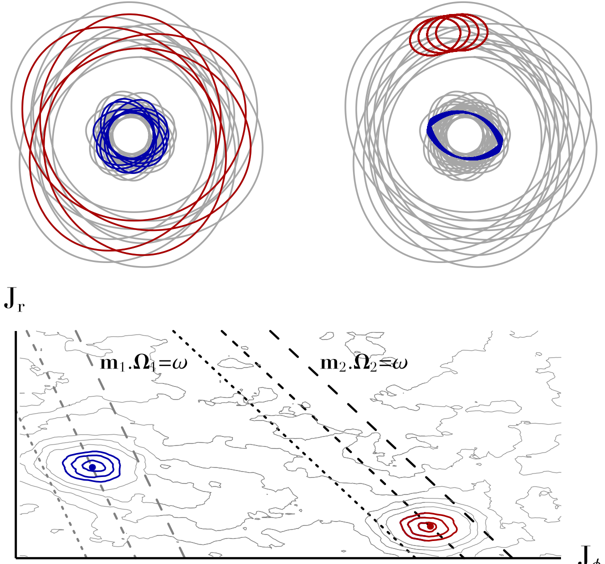
Resonant encounters

- **Resonance condition $\delta_{\mathrm{D}}(\mathbf{m}_1 \cdot \mathbf{\Omega}_1 - \mathbf{m}_2 \cdot \mathbf{\Omega}_2) \Longrightarrow$ Distant encounters.**



Resonant stellar encounters

- **Resonance condition**: $\delta_{\mathrm{D}}(\mathbf{m}_1\cdot\mathbf{\Omega}_1-\mathbf{m}_2\cdot\mathbf{\Omega}_2)$

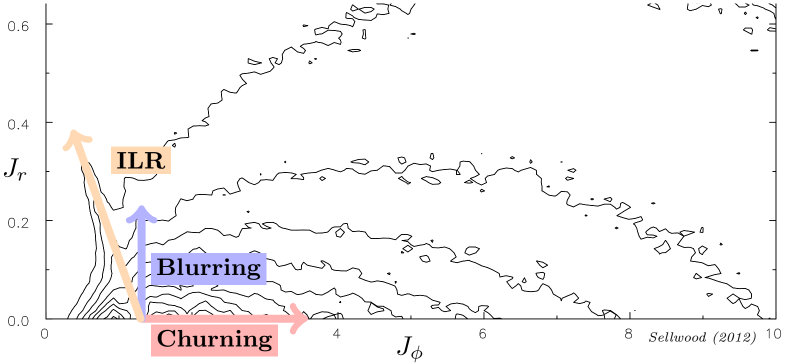


Fouvry et al. (2015)

Diffusion is anisotropic

$$\boxed{\frac{\partial F(\mathbf{J},t)}{\partial t} = \frac{\partial}{\partial \mathbf{J}} \cdot \left[\sum_{\mathbf{m}} \mathbf{m} \; \mathcal{F}_{\mathbf{m}}(\mathbf{J},t) \right].}$$

m: Resonance vector



Fokker-Planck equation

- **Test particle** of mass μ_{t} - $P(\mathbf{J}, t)$.
- **Bath particles** of mass $\mu_{\mathrm{b}}(\sim 1/N_{\mathrm{b}})$ - $F_{\mathrm{b}}(\mathbf{J}, t)$.

$$\frac{\partial P(\mathbf{J}_1, t)}{\partial t} = \pi(2\pi)^d \frac{\partial}{\partial \mathbf{J}_1} \cdot \bigg[\sum_{\mathbf{m}_1, \mathbf{m}_2} \mathbf{m}_1 \int \mathrm{d}\mathbf{J}_2 \, \frac{\delta_{\mathrm{D}}(\mathbf{m}_1 \cdot \mathbf{\Omega}_1 - \mathbf{m}_2 \cdot \mathbf{\Omega}_2)}{|\mathcal{D}_{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{J}_1, \mathbf{J}_2, \mathbf{m}_1 \cdot \mathbf{\Omega}_1)|^2} \bigg[\mu_{\mathrm{b}} \, \mathbf{m}_1 \cdot \frac{\partial}{\partial \mathbf{J}_1} - \mu_{\mathrm{t}} \, \mathbf{m}_2 \cdot \frac{\partial}{\partial \mathbf{J}_2} \bigg] \, P(\mathbf{J}_1, t) \, F_{\mathrm{b}}(\mathbf{J}_2, t) \bigg] .$$

- $\mu_{\mathrm{b}} \, \mathbf{m}_1 \cdot \dfrac{\partial}{\partial \mathbf{J}_1}$: **Diffusion**
  - Vanishes in the **collisionless limit** $N_{\mathrm{b}} \to \infty$.
  - Sourced by the **correlation of the bath fluctuations**

$$D_{\mathbf{m}}(\mathbf{J}) \propto \int_{-\infty}^{+\infty} \mathrm{d}t \, \mathrm{e}^{-\mathrm{i}\mathbf{m} \cdot \mathbf{\Omega}(\mathbf{J}) t} \, \langle \delta\psi_{\mathrm{b}}(0) \, \delta\psi_{\mathrm{b}}(t) \rangle \; .$$

- $\mu_{\mathrm{t}} \, \mathbf{m}_2 \cdot \dfrac{\partial}{\partial \mathbf{J}_2}$: **Dynamical Friction**
  - Induces **mass segregation**
  - Sourced by the **retroaction of the test particle** on the bath.

Difficulties of the Balescu-Lenard equation

- Balescu-Lenard equation

$$\frac{\partial F(\mathbf{J}_1,t)}{\partial t} = \pi(2\pi)^d \frac{M_{\mathrm{tot}}}{N} \frac{\partial}{\partial \mathbf{J}_1} \cdot \bigg[\sum_{\mathbf{m}_1,\mathbf{m}_2} \mathbf{m}_1 \int \mathrm{d}\mathbf{J}_2 \, \frac{\delta_{\mathrm{D}}(\mathbf{m}_1 \cdot \mathbf{\Omega}_1 - \mathbf{m}_2 \cdot \mathbf{\Omega}_2)}{|\mathcal{D}_{\mathbf{m}_1,\mathbf{m}_2}(\mathbf{J}_1,\mathbf{J}_2,\mathbf{m}_1 \cdot \mathbf{\Omega}_1)|^2} \bigg[\mathbf{m}_1 \cdot \frac{\partial}{\partial \mathbf{J}_1} - \mathbf{m}_2 \cdot \frac{\partial}{\partial \mathbf{J}_2} \bigg] F(\mathbf{J}_1,t) \, F(\mathbf{J}_2,t) \bigg] .$$

- **Dressed susceptibility coefficients**

$$\frac{1}{\mathcal{D}_{\mathbf{m}_1,\mathbf{m}_2}(\mathbf{J}_1,\mathbf{J}_2,\omega)} = \sum_{p,q} \psi^{(p)}_{\mathbf{m}_1}(\mathbf{J}_1) \, \big[\mathbf{I} - \widehat{\mathbf{M}}(\omega)\big]^{-1}_{pq} \, \psi^{(q)\,*}_{\mathbf{m}_2}(\mathbf{J}_2) .$$

Difficulties:

- **Inhomogeneous system**
  - **Angle-action** $(\mathbf{x},\mathbf{v}) \mapsto (\boldsymbol{\theta},\mathbf{J})$.
- **Long-range system**
  - **Basis elements** $\psi^{(p)}$.
- **Self-gravitating system**
  - **Response matrix** $\widehat{\mathbf{M}}(\omega)$.
- **Resonant encounters**
  - **Resonance** $\delta_{\mathrm{D}}(\mathbf{m}_1 \cdot \mathbf{\Omega}_1 - \mathbf{m}_2 \cdot \mathbf{\Omega}_2)$.

Balescu-Lenard - Global approach

- Balescu-Lenard equation

$$\frac{\partial F(\mathbf{J}_1,t)}{\partial t} = \pi(2\pi)^d \frac{M_{\rm tot}}{N} \frac{\partial}{\partial \mathbf{J}_1} \cdot \left[\sum_{\mathbf{m}_1,\mathbf{m}_2} \mathbf{m}_1 \int \mathrm{d}\mathbf{J}_2 \, \frac{\delta_{\rm D}(\mathbf{m}_1\cdot\mathbf{\Omega}_1 - \mathbf{m}_2\cdot\mathbf{\Omega}_2)}{|\mathcal{D}_{\mathbf{m}_1,\mathbf{m}_2}(\mathbf{J}_1,\mathbf{J}_2,\mathbf{m}_1\cdot\mathbf{\Omega}_1)|^2} \left[\mathbf{m}_1 \cdot \frac{\partial}{\partial \mathbf{J}_1} - \mathbf{m}_2 \cdot \frac{\partial}{\partial \mathbf{J}_2} \right] F(\mathbf{J}_1,t) \, F(\mathbf{J}_2,t) \right] .$$

- **Dressed susceptibility coefficients**

$$\frac{1}{\mathcal{D}_{\mathbf{m}_1,\mathbf{m}_2}(\mathbf{J}_1,\mathbf{J}_2,\omega)} = \sum_{p,q} \psi^{(p)}_{\mathbf{m}_1}(\mathbf{J}_1) \left[\mathbf{I} - \widehat{\mathbf{M}}(\omega) \right]^{-1}_{pq} \psi^{(q)\,*}_{\mathbf{m}_2}(\mathbf{J}_2) .$$

Difficulties:

- **Inhomogeneous system**
  - **Angle-action** $(\mathbf{x},\mathbf{v}) \mapsto (\boldsymbol{\theta},\mathbf{J}) \Longrightarrow$ $2D$ discs are **explicitly integrable**.
- **Long-range system**
  - **Basis elements** $\psi^{(p)} \Longrightarrow$ **Global** basis elements.
- **Self-gravitating system**
  - **Response matrix** $\widehat{\mathbf{M}}(\omega) \Longrightarrow$ Numerical **linear theory**.
- **Resonant encounters**
  - **Resonance** $\delta_{\rm D}(\mathbf{m}_1\cdot\mathbf{\Omega}_1 - \mathbf{m}_2\cdot\mathbf{\Omega}_2) \Longrightarrow$ Integrate along **resonant lines**.

Balescu-Lenard - Global approach

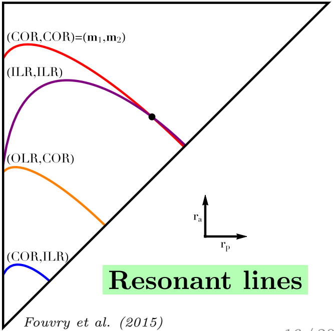
- Balescu-Lenard equation

$$\frac{\partial F(\mathbf{J}_1,t)}{\partial t} = \pi(2\pi)^d \frac{M_{\mathrm{tot}}}{N} \frac{\partial}{\partial \mathbf{J}_1}\cdot\bigg[\sum_{\mathbf{m}_1,\mathbf{m}_2} \mathbf{m}_1 \int \mathrm{d}\mathbf{J}_2 \, \frac{\delta_{\mathrm{D}}(\mathbf{m}_1\cdot\mathbf{\Omega}_1 - \mathbf{m}_2\cdot\mathbf{\Omega}_2)}{|\mathcal{D}_{\mathbf{m}_1,\mathbf{m}_2}(\mathbf{J}_1,\mathbf{J}_2,\mathbf{m}_1\cdot\mathbf{\Omega}_1)|^2} \bigg[\mathbf{m}_1\cdot\frac{\partial}{\partial \mathbf{J}_1} - \mathbf{m}_2\cdot\frac{\partial}{\partial \mathbf{J}_2}\bigg] F(\mathbf{J}_1,t)\, F(\mathbf{J}_2,t)\bigg] .$$

- **Dressed susceptibility coefficients**

$$\frac{1}{\mathcal{D}_{\mathbf{m}_1,\mathbf{m}_2}(\mathbf{J}_1,\mathbf{J}_2,\omega)} = \sum_{p,q} \psi^{(p)}_{\mathbf{m}_1}(\mathbf{J}_1) \, \Big[\mathbf{I} - \widehat{\mathbf{M}}(\omega)\Big]^{-1}_{pq} \, \psi^{(q)\,*}_{\mathbf{m}_2}(\mathbf{J}_2) .$$

$$\widehat{\mathbf{M}}_{pq}(\omega) \sim \sum_{\mathbf{m}} \int \mathrm{d}\mathbf{J} \, \frac{\mathbf{m}\cdot\partial F/\partial \mathbf{J}}{\omega - \mathbf{m}\cdot\mathbf{\Omega}} \, \psi^{(p)*}_{\mathbf{m}}(\mathbf{J}) \, \psi^{(q)}_{\mathbf{m}}(\mathbf{J}) .$$



Resonant lines

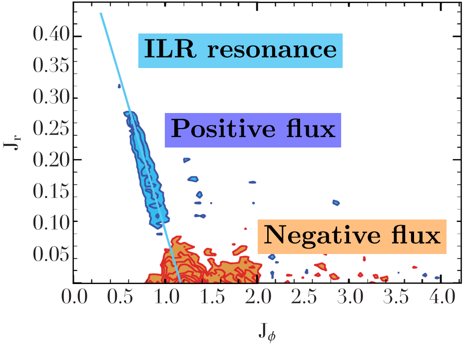
Fouvry et al. (2015)

Balescu-Lenard Global - Application

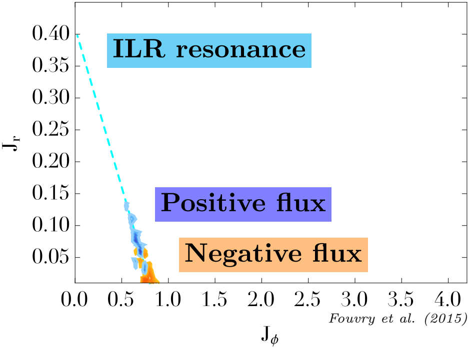
- **Diffusion flux** in action space

$$\frac{\partial F}{\partial t} = \text{div}\Big(\mathbf{F}_{\text{tot}}(\mathbf{J})\Big) \,.$$

- Predicted contours for $\text{div}\big(\mathbf{F}_{\text{tot}}\big)(t\!=\!0^{+})$



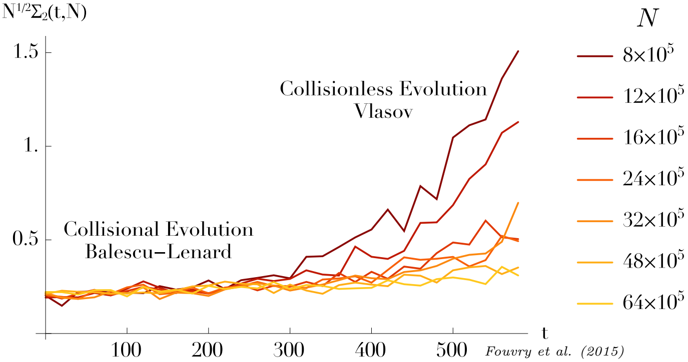
Sellwood (2012)



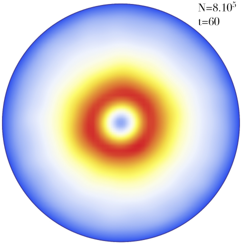
Balescu-Lenard

Late time evolution

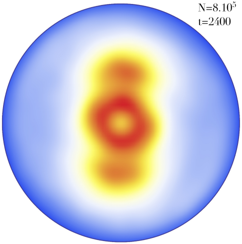
- **Phase transition: BL $\Longrightarrow$ Vlasov.**



Fouvry et al. (2015)



Initial times

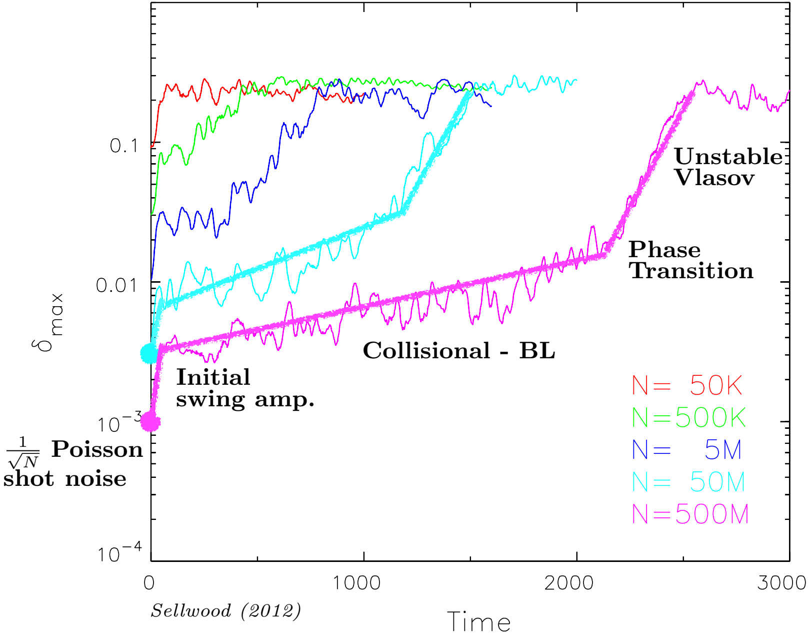


Late times

- 2-body (resonant) relaxation $\Longrightarrow$ small-scale structures in the DF

Destabilisation at the collisionless level

The fate of secular evolution



Sellwood (2012)

CONCLUSIONS

- From **linear response** to **secular evolution**.

Galactic dynamics enters the cosmic framework.

- Balescu-Lenard can be implemented!
  - (Thin/Thick) Stellar discs
  - (Inhomogeneous) HMF
  - Quasi-Keplerian Systems
  - Globular clusters
- Approach complementary to $N-$body and Monte Carlo methods