Lagrangian theory of multi-transmission-line interacting with electron beam

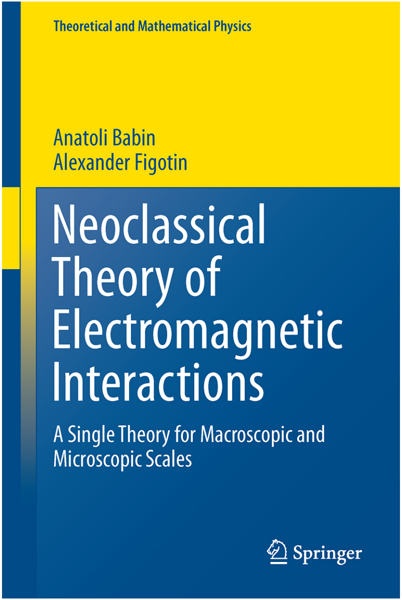
Alexander Figotin, joint work with Guillermo Reyes

University of California at Irvine

The work was supported by AFOSR

November, 2017

Outline


Theoretical and Mathematical Physics
Anatoli Babin
Alexander Figotin
Neoclassical Theory of Electromagnetic Interactions
A Single Theory for Macroscopic and Microscopic Scales
Springer

Neoclassical theory features

- The theory is a relativistic Lagrangian theory. It is a single theory for all spatial scales: macroscopic and atomic.
- An elementary charge is a material wave acquiring particle properties when its energy is localized.
- Every elementary charge has an individual wave function over four-dimensional space-time continuum, and there is no configuration space as in QM.
- The field equations for the elementary EM fields are exactly the Maxwell equations and the Lorenz expression for the forces is also exact.
- There is a new fundamental spatial scale - the size of a free electron. Its currently assessed value is 100 Bohr radii - 5 nm.

Neoclassical electron as a plasma "cloud"/"super-particle"

- The neoclassical electron can be thought of as a "jelly-like" continuously distributed entity. It has similarities with a conventional plasmonic excitation as well as the concept of "cloud" or "super-particle" in the plasma physics.
- The new fundamental scale in our theory - the size a of a free electron - is significant for collisions. Just as in the case of the "cloud" in plasma, neoclassical electrons have very different collision signature compare to that of the point charge. Consequently, the free electron size a (estimated currently as $5\mathrm{nm}$) may play an important role in plasma physics.

Some experimental evidence of electron size

- Electron emission theory. Experimental evidence of extremely large currents for nanometer-scale emitting areas: Fursey (pp. X-XI). "Current densities up to $5 \times 10^9\ \mathrm{A/cm^2}$ were demonstrated for field emission localized to nanometer-scale emitting areas. In experiments by V. N. Shrednik et al., current densities up to $\left(10^9 - 10^{10}\right)\ \mathrm{A/cm^2}$ are recorded from nanometer-sized tips under steady-state conditions. Recently, G. N. Fursey and D. V. Galazanov, using tips with an apex radius of $\sim 10\ \dot{A}$, were able to reach current densities of $\left(10^{10} - 10^{11}\right)\ \mathrm{A/cm^2}$. These current densities are close to the theoretical supply limit of a metal's conduction band when the electron tunneling probability is unity."
- It is conceivable that micro-machining at the scale comparable with the size of neoclassical free electron ($5\mathrm{nm}$) may significantly reduce the work function of the material surface with consequent extremely large currents.

Complexity of plasma physics phenomena

- Elskens Y. and Elscande D. wrote in their book "Microscopic Dynamics of Plasmas and Chaos":

  "Plasmas are highly complex media, especially when they are magnetized, and they exhibit both granular and collective aspects. A model which captures all this complexity is amenable neither to analytic treatment nor to physical intuition."

- How one can deal with plasma physics enormous complexity?
- We try to face the complexity of plasma physics phenomena by staying as close as possible to the solid ground of the Lagrangian field theory.

Outline of presentation

- Published papers
- Electron beam interacting with a transmission line - the Pierce model.
- Multi-transmission-line interacting with the electron beam (MTLB) - a Lagrangian theory
- "Negative energy" as the physical origin and source of gain and amplification.
- Mathematical analysis of equations of the MTLB system.

List of published papers

- A. Figotin and G. Reyes, *Multi-transmission-line-beam interactive system*, Journal of Mathematical Physics **54**, 111901 (2013).
- V. Tamma, A. Figotin, and F. Capolino, *Concept for Pulse Compression Device Using Structured Spatial Energy Distribution*, IEEE Trans. Microw. Theory Tech., vol. 64, no. 3, pp. 742–755, Mar. 2016.
- M. Othman, M. Veysi, A. Figotin, and F. Capolino, *Giant Amplification in Degenerate Band Edge Slow-Wave Structures Interacting with an Electron Beam*, Phys. Plasmas 1994-Present, vol. 23, no. 3, p. 033112, Mar. 2016.
- M. Othman, V. A. Tamma, and F. Capolino, *Theory and New Amplification Regime in Periodic Multi Modal Slow Wave Structures with Degeneracy Interacting with an Electron Beam*, IEEE Trans Plasma Sci, vol. 44, no. 4, pp. 594 –611, April 2016.
- M. Othman, M. Veysi, A. Figotin, and F. Capolino, *Low Starting Electron Beam Current in Degenerate Band Edge Oscillators*, IEEE Trans Plasma Sci, vol. 44, no. 6, pp. 918-929 June 2016.

Acronims

- SWS - Slow wave Structure
- TWT - Traveling Wave Tube
- TL - Transmission Line
- MTL - Multy-Transmission-Line
- MTLB - Multy-Transmission-Line-Beam

The Traveling Wave Tube main components

- **Electron beam**, from which the energy is extracted through in-phase interaction with the structure.
- **Slow wave structure**. An RF signal is amplified as energy is extracted from the electron beam.

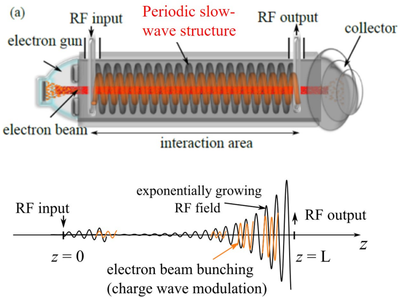


Figure: Traveling tube and its basic components

Pierce's model

- One-dimensional model: all quantities depend on (z, t)
- Charge wave ("jelly of charge"), characterized by
- Volumetric charge density $\rho_{tot}(z, t)$;
    - Electron velocity $u(z, t)$;
    - Cross section σ.
- Electron velocity $u(z, t)$;
- Cross section σ.
- Linearized theory: It is assumed that AC perturbations of the DC regime are small:

$$u(z, t) = u_0 + v(z, t); \quad \rho_{tot}(z, t) = \rho_0 + \rho(z, t);$$
$$I_{tot}(z, t) = I_0 + I_b \approx \rho_0 u_0 + \rho(z, t) u_0 + \rho_0 v(z, t).$$

- Space charge effects (electron-electron repulsion) can be included.

Pierce's model

Slow wave structure model and assumptions on interaction:

- The SWS is mimicked via a single TL with distributed parameters L, C constant along the line. The dynamical variables are the voltage and the current $V(z,t)$ and $I(z,t)$.
- The axial component of the electric field $E_z(z,t)$ acts upon the beam.
$$E_z = -\frac{\partial V}{\partial z}$$
- Magnetic effects and radiation are neglected.
- The beam induces a current upon the SWS. It is a mirror image of the AC current on the beam (Ramo's Theorem, quasistatic regime).

Pierce's model

System of equations for V, I, I_b :

$$\partial_z I = -C\partial_t V - \partial_z I_\mathrm{b} \qquad \partial_z V = -L\partial_t I$$
$$\left(\partial_t + u_0\partial_z\right)^2 I_\mathrm{b} = -\sigma\frac{e}{m}\rho_0\partial_t\partial_z V$$

- u_0, ρ_0 - electron DC velocity and density,
- e, m - electron charge and mass
- I, V - current and voltage along the TL
- e, m - electron charge and mass,
- I, V - current and voltage along the TL
- I_b - beam AC current.

Multi-Transmission-Line (MTL) model

- Assumptions on the beam are as in the Pierce model.
- The SWS is treated as a MTL.
  - The lines mutually interact.
  - Each line interacts independently with the beam.
  - The beam AC current is induced on each line.
- No uniformity is assumed on the MTL's parameters.

$$L = L(z), \qquad C = C(z),$$

  where $L(z)$ and $C(z)$ are $n \times n$ matrices of self- and mutual inductance and capacity.
- Explicit Lagrangian structure: the system is defined via a unique scalar function $\mathcal{L}$ of suitable generalized coordinates and their derivatives.

Advantages of the MTLB model

- It is an analytic model with an explicit representation of the dispersion relations and explicit amplification conditions.
- More flexibility:
  - General SWSs can be approximated with desired accuracy by means of a MTL
  - MTL can be non-uniform SWSs, in particular periodic structures made up of stacked layers of materials.
- Bunching as a primary amplification mechanism is incorporated phenomenologically.
- Simplicity and physical transparency due to the Lagrangian formalism.

Lagrangian formalism, a consistent energetic treatment

- The first critical step is an assignment "kinetic" T and "potential" V energies to any virtual configuration of involved fields. The kinetic energy T represents motion/dynamics, the potential energy V represents acting forces. The system Lagrangian $\mathcal{L}$ then is defined by $\mathcal{L} = T - V$.
- Dynamically admissible field configurations are found via the variational principle demanding the action $S = \int \mathcal{L} \, \mathrm{d}t\mathrm{d}x$ to be stationary. That yields the Euler-Lagrange field equations describing the system evolution in space time.
- Conserved quantities including the energy are found based on the Noether theorem that relates system symmetries and conserved quantities.
- As a practical matter expressions for the kinetic and the potential energies are obtained based on prior system knowledge including existing phenomenological theory.

Lagrangian for the MTLB system

MTLB Lagrangian for is

$$\mathcal{L} = \frac{1}{2}\left\{(\partial_t \boldsymbol{Q}, L\partial_t \boldsymbol{Q}) - \left(\partial_z \boldsymbol{Q} + \partial_z q\boldsymbol{B}, C^{-1}\left[\partial_z \boldsymbol{Q} + \partial_z q\boldsymbol{B}\right]\right)\right\} + \\ + \frac{\xi}{2}\left(\partial_t q + u_0 \partial_z q\right)^2$$

- $(\cdot)$: scalar product
- $\boldsymbol{Q}(z,t)$: charge at the point z of the MTL in the time interval $(0,t)$ ($n-$ dimensional vector)
- $q(z,t)$: charge at the point z on the beam in the time interval $(0,t)$.
- $\xi = \frac{m}{\sigma e \rho_0}$: beam parameter.
- $\boldsymbol{B} = (1, 1, \ldots 1)$: interaction constants (assuming perfect "mirroring").

MTLB Lagrangian components

MTLB Lagrangian

$$L = \frac{1}{2}\left\{(\partial_t \boldsymbol{Q}, L\partial_t \boldsymbol{Q}) - \left(\partial_z \boldsymbol{Q} + \partial_z q \boldsymbol{B}, C^{-1}\left[\partial_z \boldsymbol{Q} + \partial_z q \boldsymbol{B}\right]\right)\right\} + \\ + \frac{\xi}{2}\left(\partial_t q + u_0 \partial_z q\right)^2$$

- $\frac{1}{2}(\partial_t \boldsymbol{Q}, L\partial_t \boldsymbol{Q})$: "kinetic energy" term represents the magnetic energy stored in inductive elements of MTL.
- $\frac{1}{2}\left(\partial_z \boldsymbol{Q} + \partial_z q \boldsymbol{B}, C^{-1}\left[\partial_z \boldsymbol{Q} + \partial_z q \boldsymbol{B}\right]\right)$: "potential energy" represents the electric field energy stored in capacitance of the MTL with the integrated in interaction term containing the electron beam contribution.
- $\frac{\xi}{2}\left(\partial_t q + u_0 \partial_z q\right)^2$: "kinetic energy" associated the electron beam electrons.

MTLB Lagrange-Euler field equations

MTLB Euler-Lagrange field equations:

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{Q}_t}\right)+\frac{\partial}{\partial z}\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{Q}_z}\right)-\frac{\partial \mathcal{L}}{\partial \boldsymbol{Q}}=0; \quad \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial q_t}\right)+\frac{\partial}{\partial z}\left(\frac{\partial \mathcal{L}}{\partial q_z}\right)-\frac{\partial \mathcal{L}}{\partial q}=0.$$

For our Lagrangian field equations turns into

$$L\partial_t^2\boldsymbol{Q}-\partial_z\left[C^{-1}(\partial_z\boldsymbol{Q}+\partial_z q\boldsymbol{B})\right]=0;$$
$$\xi\left(\partial_t+u_0\partial_z\right)^2 q-\left(\boldsymbol{B},\partial_z\left[C^{-1}(\partial_z\boldsymbol{Q}+\partial_z q\boldsymbol{B})\right]\right)=0,$$

which is a system of $n+1$ ODE equations of the second order for $\boldsymbol{Q}=(Q_1,Q_2,...Q_n)$ and q.

Advantages of Lagrangian formalism

- Simplicity and physical transparency: a single scalar function $\mathcal{L}$ assigns energies to all possible system's configurations. The field equations and the conservation laws are obtained by canonical procedures in a systematic way.
- Perfect mathematical consistency assured by the variational principle.
- Identification of the source of amplification as negative, unbounded potential energy associated with the beam.
- Exact explicit formulas for energies and their fluxes.
- Mathematically sound ways for incorporating additional features such as space charge (debunching), losses, saturation, etc.

Negative unbounded potential energy.

Source of amplification: The beam Lagrangian

$$\mathcal{L}_b = \frac{\xi}{2} \left(\partial_t q + u_0 \partial_z q \right)^2$$

has a term $\frac{\xi}{2}(u_0 \partial_z q)^2$ which can be interpreted as negative unbounded from below potential energy. This negative unbounded from below potential energy is an ideal energy supply that accounts for instability, gain and amplification.

The Euler-Lagrange equation for Lagrangian $\mathcal{L}_b$ is

$$\left(\partial_t + u_0 \partial_z \right)^2 q = 0.$$

To see clearly the role played by the unbounded negative potential energy let us compare side by side normal oscillatory system with positive potential energy and unstable one with negative potential energy.

Negative unbounded potential energy.

Normal oscillatory system: an example of a simple oscillator with positive potential energy and oscillatory modes

Potential Energy	Lagrangian	Modes
$V(x) = \frac{kx^2}{2} \geq 0, \;\; k > 0$	$\mathcal{L} = \frac{m\dot{x}}{2} - \frac{kx^2}{2}$	$x = e^{\pm \mathrm{i}\omega t}$, $\omega = \sqrt{\frac{k}{m}}$

Normal oscillatory system: an example of simple LC-circuit with positive potential energy and oscillatory modes

Potential Energy	Lagrangian	Modes
$V(q) = \frac{q^2}{2C} \geq 0, \;\; C > 0$	$\mathcal{L} = \frac{L\dot{q}^2}{2} - \frac{q^2}{2C}$	$q = e^{\pm \mathrm{i}\omega t}, \;\; \omega = 1/\sqrt{LC}$

Unstable gain system: an example of LC-circuit with negative capacitance. It has negative potential energy - an ideal unlimited energy supply - and exponentially growing/decaying modes

Potential Energy	Lagrangian	Modes
$V(q) = \frac{q^2}{2C} \leq 0, \; C < 0$	$\mathcal{L} = \frac{L\dot{q}^2}{2} - \frac{q^2}{2C}$	$q = e^{\pm \lvert\omega\rvert t}, \omega = \mathrm{i}/\sqrt{L\lvert C\rvert}$

Other aspects of negative potential energy.

- In linear oscillatory systems, the virial is bounded and the Virial Theorem implies

$$\langle T \rangle = \langle V \rangle \Rightarrow \langle \mathcal{L} \rangle = 0. \qquad \langle \cdot \rangle \text{ - time average.}$$

  In our case, $\mathcal{L}$ is positive along any solution different from a traveling wave.

- Consequently there is a manifest departure from the Virial Theorem.
- The potential energy term $-\frac{\xi}{2}(u_0 \partial_z q)^2$ is related to non-homogeneous distribution of space charge describing bunching.
- An interesting aspect of the MTLB system is that properties of a gain medium are achieved through negative potential energy. This approach to gain differs markedly from the conventional one. In the later case the system gain modeled effectively as "negative dissipation" in terms of negative Rayleigh dissipation function *complementing* the Lagrangian formalism.

Negative potential energy in Plasma Physics

Waves with negative potential energy are well known in Plasma Physics.

"the wave energy density is actually the change in the total system energy density in going from a situation where there is no wave to a situation where there is a wave. Typically, negative energy waves occur when the equilibrium has a steady-state flow velocity and there exists a mode which causes the particles to develop a slower mean velocity than in steady state. Wave growth taps free energy from the flow"

Paul M. Bellan, Fundamentals of Plasma Physics

SUMMARY

- Analytic model: explicit dispersion relation, expressions for energies and amplification conditions.
- More flexibility: description of general SWS with desired accuracy.
- Allows to easily incorporate new features: debunching, saturation, nonlinear effects.
- Insight into the amplification mechanism.

Noether's theorem symmetries - conservation laws

Conserved quantities $\Leftrightarrow$ Continuous symmetry groups of $\mathcal{L}$:

Symmetry	Conservation Law
Time invariance	Energy
Space translation invariance	Momentum
Space rotation invariance	Angular momentum

The energy conservation

$$\mathcal{L}(t,\ldots) = \mathcal{L}(t+\tau,\ldots) \Leftrightarrow \frac{\partial \mathcal{L}}{\partial t} = 0 \Rightarrow \textbf{Energy conservation}$$

It has the form

$$\partial_t H + \partial_z S = 0, \qquad H \text{ - energy;} \qquad S \text{ - energy flux}$$

The change of energy inside a given volume is due only to energy flux through its boundary.

Energy conservation for MLTB system.

Emmy Noether Theorem provides for explicit expressions for the total energy H and its flux S:

$$H = \frac{\partial \mathcal{L}}{\partial(\partial_t \boldsymbol{Q})}\partial_t \boldsymbol{Q} + \frac{\partial \mathcal{L}}{\partial(\partial_t q)}\partial_t q - \mathcal{L}; \quad S = \frac{\partial \mathcal{L}}{\partial(\partial_z \boldsymbol{Q})}\partial_t \boldsymbol{Q} + \frac{\partial \mathcal{L}}{\partial(\partial_z q)}\partial_t q.$$

- The system can be split into subsystems with explicit formulas for the energy exchange between them.
- We are not aware of any systematic way to find conserved quantities based on the system of field equations alone.
- Even if such a conservative quantity is found, its nature is not clear. That is one can't determine if it is energy, momentum, etc.

Energy balance.

Energy exchange between subsystems.

$$H = H_1 + H_2, \qquad S = S_1 + S_2,$$

H_1, S_1 - energy stored in the SWS and flux into the SWS;
H_2, S_2 - energy stored in the beam and flux into the beam;

- Power flowing from the beam to the SWS:

$$P_{\mathrm{beam}\to\mathrm{SWS}} = \partial_t H_1 + \partial_z S_1 = \partial_t \left[\frac{1}{2}(CV, V) + \frac{1}{2}(LI, I) \right] + \partial_z (I, V)$$

- Positive and increasing power time average on the growing mode

$$\langle P_{\mathrm{beam}\to\mathrm{SWS}} \rangle (z) = -\left[\omega\xi \left|k_0\right|^2 \left|\widehat{q}\right|^2 (\mathrm{Re}\, v_0 - u_0) \mathrm{Im}\, v_0 \right] \mathrm{e}^{-2(\mathrm{Im}\, k_0) z},$$

where v_0 is the complex root with $\mathrm{Im}\, v_0 > 0$, $k_0 = \omega / v_0$.

Debunching (space charge) effect

Space charge (debunching) effects can be introduced as an additional term in the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left\{\left(\partial_t \boldsymbol{Q}, L\partial_t \boldsymbol{Q}\right) - \left(\partial_z \boldsymbol{Q} + \partial_z q\boldsymbol{B}, C^{-1}\left[\partial_z \boldsymbol{Q} + \partial_z q\boldsymbol{B}\right]\right)\right\} + \frac{\xi}{2}\left(\partial_t q + u_0\partial_z q\right)^2 - \frac{2\pi R_{\mathrm{sc}}^2}{\sigma} q^2.$$

R_{sc}- plasma reduction factor,

$$R_{\mathrm{sc}} \approx \left[1 + \left(\frac{\Lambda u_0}{\omega}\right)^2\right]^{-\frac{1}{2}}, \quad \Lambda\text{- related to the SWS geometry.}$$

- $\Lambda = 0 \Leftrightarrow R_{\mathrm{sc}} = 1$: Infinite beam.
- $\Lambda > 0 \Leftrightarrow R_{\mathrm{sc}} < 1$: Confined beam.

The sign of the new term is negative, corresponding to standard potential energy linked to repulsion.

Spatially homogeneous SWS. Amplification Analysis

If C and L are spatially uniform, we look for solutions of the form

$$Q = \widehat{Q}e^{-i(\omega t - kz)}; \qquad q = \widehat{q}e^{-i(\omega t - kz)}, \qquad \omega \text{ real, } k \text{ complex.}$$

Such solutions exist if and only if ω, k satisfy the dispersion relation:

$$-\xi(v - u_0)^2 = \sum_{i=1}^{n} \frac{\widetilde{D}_i^2}{v_i^2 - v^2} - d; \quad v = \frac{\omega}{k},$$

where $\widetilde{D}_i, d$ are functions of the SWS parameters L, C and

$\pm v_i$ - characteristic velocities of the MTL:

$$\left|C^{-1} - v_i^2 L\right| = 0; \;\; 0 < v_1 \leq v_2 \leq ... \leq v_n$$

Growing modes exist $\Leftrightarrow$ There are complex solutions for v (hence for k) $\Leftrightarrow$ number of real solutions $\leq 2n$.

Spatially homogeneous SWS. Amplification Analysis

Geometric interpretation: $n = 2$

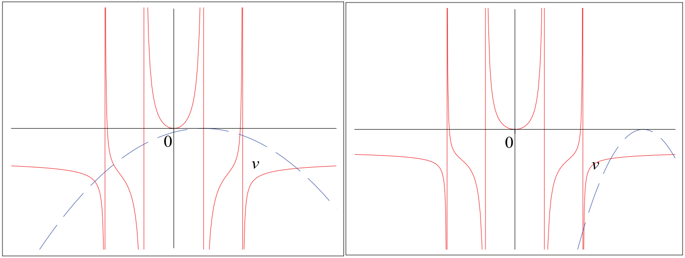


Figure: Blue graph: Parabola $y = -\xi(v - u_0)^2$ with vertex $(u_0, 0)$. Red graph: Function $y = \sum_{i=1}^{n} \frac{\widetilde{D}_i^2}{v_i^2 - v^2} - d$ with asymptotes $v = \pm v_i$.

- Left figure: Exactly four intersections if $u_0 \leq v_1$ (smallest MTL eigenvelocity) $\Rightarrow$ amplification.
- Right figure: Six intersections if u_0 is too large $\Rightarrow$ no amplification.
- We recover four solutions if ξ is small enough.

Spatially homogeneous SWS. Amplification Result

Theorem

If $u_0 \leq v_1$ there is a unique pair of complex conjugate roots $\Leftrightarrow$ a unique growing wave and a unique evanescent wave. If $u_0 > v_1$, same result for sufficiently small ξ. The needed smallness of ξ can be estimated.

Pierce's model, $n = 1$: $\xi < \xi_0 := \dfrac{L\gamma^2}{1-\gamma^{2/3}}$; $\quad \gamma = \dfrac{v_1}{u_0} = \dfrac{1}{u_0\sqrt{LC}}$ *(exact)*

$-n \geq 2 - \xi < \widetilde{\xi}_0 := \dfrac{\min_{v\in(v_1,u_0)} R'(v)}{2u_0}$, $R(v) = \displaystyle\sum_{i=1}^{n} \frac{\widetilde{D}_i^2}{v_i^2 - v^2} - d$ *(sufficeint)*

Spatially homogeneous SWS. Behavior of the amplification parameter

The number $\delta = \operatorname{Im} k > 0$ is a measure of amplification as z grows. The growing solution has the form

$$\mathcal{Q} = A(z,t)\, e^{\delta z}, \qquad A - \text{ bounded;} \qquad \mathcal{Q} = (\boldsymbol{Q}, q)$$

The behavior of δ as $\xi \to 0$ and $\xi \to \infty$ can be found analytically:

$$\frac{K'}{\sqrt[3]{\xi}} \text{ as } \xi \to \infty, \qquad K' > 0.$$

$$\frac{K'}{\sqrt[3]{\xi}} \text{ as } \xi \to \infty, \qquad K' > 0.$$

Pierce's model revisited

Dispersion relation: $n = 1$

$$\boxed{-\xi(v-u_0)^2 = \frac{\widetilde{D}^2}{v_1^2 - v^2} - d; \quad \widetilde{D}^2 = L^{-1}C^{-2}, \quad d = C^{-1}, \quad v_1 = 1/\sqrt{LC}}$$

Pierce's regime: $u_0 = v_1$, $\xi \gg 1$: three solutions have real part close to u_0. In terms of $k = \omega/v$,

$$k = k_{\mathrm{b}} + \mathrm{i}\delta, \qquad k_{\mathrm{b}} = \frac{\omega}{u_0}, \qquad \delta \text{ small compared to } k_{\mathrm{b}}.$$

Dispersion relation in δ :

$$(\mathrm{i}\delta)^3\left(2 + \mathrm{i}\delta k_{\mathrm{b}}^{-1}\right) = -L\xi^{-1}k_{\mathrm{b}}^2\left(1 + \mathrm{i}\delta k_{\mathrm{b}}^{-1}\right)^2.$$

Pierce's third degree equation for δ :

$$\delta/k_{\mathrm{b}} \ll 1 \Rightarrow \delta^3 \approx -\frac{Lk_{\mathrm{b}}^2\xi^{-1}}{2}\mathrm{i} : \text{oscill.} + \text{evanescent} + \text{growing (forward)}$$