Non-perturbative guiding center transformation: the gyro-phase is the Kaluza-Klein $\mathbf{5}^{th}$ dimension

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Outline

- Electromagnetism vs General Relativity
- Kaluza-Klein mechanism
- non perturbative guiding center transformation
- Electromagnetism within General Relativity

Electromagnetism vs General Relativity

Differences within a variational approach

General Relativity:

- The Einstein's equation is obtained by varying the metric tensor.
- The trajectories are geodesics in a curved (by masses) space-time manifold.
- Hilbert-Einstein action

$$S_{\mathrm{HE}} = -\int \frac{R}{16\pi G}\sqrt{-g}\,dt d^3x$$

Electromagnetism:

- The Maxwell's equations are not obtained by varying a metric tensor.
- The trajectories are solutions of the Lorentz's force law which are not geodesics:
  - ▷ Einstein's Teleparallelism Theory (metric with torsion, Eddington's affine geometry)
  - ▷ Kaluza-Klein Mechanism (extra dimension)
  - ▷ Weyl attempt (non Riemannian metric with scale invariance)
- The action for Maxwell's equations is different from the Hilbert-Einstein action:
  $S_f = -\int (F^2/4)\sqrt{-g}\,dt d^3x$
  - ▷ For taking into account both the interactions, the lagrangian density is the sum
    $\mathcal{L} = -\rho_m - A_\alpha J^\alpha - F^2/4 - R/(16\pi G)$

Kaluza-Klein mechanism

- A Hilbert-Einstein action in a 5D space-time is

$$S_{\rm HE} = -\int \frac{\tilde{R}}{16\pi\tilde{G}}\sqrt{-\tilde{g}}\,dtd^3x\,dy$$

- A 5D Kaluza-Klein metric tensor

$$\tilde{g}_{ab} = \left| \begin{array}{cc} g_{\alpha\beta} + \kappa^2\varphi^2 A_\alpha A_\beta & \kappa\varphi^2 A_\alpha \\ \kappa\varphi^2 A_\beta & \varphi^2 \end{array} \right| . \qquad (1)$$

- Considering a Ricci tensor in 5D and Christoffel symbols in 5D
- The cylinder condition: dropping all derivatives with respect to the fifth coordinate
- Then

$$S_{\rm HE} = -\int d^4x\sqrt{-g}\varphi\left[\varphi^2\frac{F_{\alpha\beta}F^{\alpha\beta}}{4} + \frac{R}{16\pi G} + \frac{2}{3\kappa}\frac{\partial_\alpha\varphi\partial^\alpha\varphi}{\varphi^2}\right]$$

  ▷ Kaluza-Klein set $\varphi = 1$.

- REJECTED: no observation of the 5^{th} dimension and inconsistency of mass scale in compactification schemes.

Plasma modeling

- The lagrangian for describing electrodynamics in plasmas is the sum of the single particle lagrangian, $\ell(t, x, v)$, times the distribution function of particles, $f(t, x, v)$, with the e.m. lagrangian. The action is often expressed like:

$$S_{\text{plasma}} = \int d\,t d^3\,x d^3\,v f(t, x, v)\ell(t, x, v) - \int \frac{F_{\alpha\beta}F^{\alpha\beta}}{4} d\,t d^3\,x, \tag{2}$$

- There is an asymmetry in the action (2) between the matter action and the field one.

- In principle, for restoring the symmetry between the two lagrangians, it should be simple to think at an action written as

$$S_{\text{plasma}} = \int d\,t d^3\,x d^3\,v \mathcal{L}_{\text{plasma}}, \tag{3}$$

where $\mathcal{L}_{\text{plasma}} = f(t, x, v)\ell(t, x, v) + \text{"somethingnew"}$ and the property that

$$\int \text{"somethingnew"}\, d^3\,v = -\frac{F_{\alpha\beta}F^{\alpha\beta}}{4}. \tag{4}$$

Non-perturbative guiding center transformation

For the general plasma modeling:

- ▷ general e.m. fields (tokamak magnetic fields)
- ▷ extended to relativistic energies (runaway electrons)
- ▷ Independent from ordering
  - $\rho_L \approx a$ (edge phenomena)
  - ω_c is finite (high frequency phenomena)
  - time dependency (non static equilibrium and transport time scale)
  - $E \neq 0$ (MHD ordering and plasma rotation)
- ▷ The ordering and the perturbative approach become relevant when fluctuations are considered (from guiding center to gyrocenter).
- ▷ The non perturbative guiding center transformation is based on:
  - Lorentz's force law solutions,
  - guiding center description,
  - scalar extended phase-space Lagrangian.

Lorentz's force law solutions of motion

- A charge e with mass m that (non-relativistically) moves in a externally given and static e.m. field is classically described by the system of equations:

$$\dot{x} = v; \tag{5}$$

$$\dot{v} = (e/m)(E + v \times B) \tag{6}$$

within eulerian description $\dot{x} = v(t, x, \alpha)$ with $\dot{\alpha} \cdot \nabla_\alpha v = 0$. Substituting $E = -\partial_t A - \nabla\Phi$ and $B = \nabla \times A$, eq. (6) becomes

$$\partial_t v + \dot{x} \cdot \nabla v + \dot{\alpha} \cdot \nabla_\alpha v = (e/m)(-\partial_t A - \nabla\Phi + v \times \nabla \times A). \tag{7}$$

or

$$\partial_t[v + (e/m)A] + \nabla[v^2/2 + (e/m)\Phi] = v \times \nabla \times [v + (e/m)A]. \tag{8}$$

Equivalent to the velocity law:

$$E_c + v \times B_c = 0, \tag{9}$$

if the canonical e.m. fields are defined as

$$(e/m)E_c = -\partial_t p - \nabla\varepsilon \qquad \text{and} \qquad (e/m)B_c = \nabla \times p \tag{10}$$

The guiding center description of motion

- The guiding center description of motion is not eulerian nor lagrangian.

$$x = X + \rho(\gamma), \tag{11}$$

$$\dot{x} = V + \sigma(\gamma). \tag{12}$$

- $\dot{X} = V(t, X, \mu, \varepsilon)$ with $E_c + V \times B_c = 0$.
- If $\mathcal{E} = V^2/2 + (e/m)\Phi(t, X)$, then the cyclotron frequency, $\dot{\gamma}$, is chosen to be

$$(m/e)\mu\dot{\gamma} = \varepsilon - \mathcal{E} \tag{13}$$

- the guiding center momentum is $P = V + (e/m)A(t, X)$.
- The lagrangian doesn't explicitly depend on the gyro-phase, γ is cyclic.
- The constant of motion for such symmetry is the magnetic moment, μ.
- The differences are that in the perturbative approach
  - ▷ the latter items are considered at each perturbative order,
  - ▷ $v = v_{\|} b_{(0)} + v_{\perp}$ instead of $v = V + \sigma$, (unfortunate choice if B departs too much from being straight).

The guiding center Lagrangian

- The single (charged) particle Lagrangian, L, must be gauge independent: if

$$L \to L + \dot{\mathcal{S}}, \tag{14}$$

the equations of motion are the same.

- L must be a scalar: if the index A runs over the extended phase-space dimensions then

$$L = \gamma_A \dot{z}^A \tag{15}$$

  - ▷ Canonical coordinates:

$$z^A = (t, x, v) \qquad \text{and} \qquad \gamma_A = (-\varepsilon, p, 0).$$

  - ▷ Non canonical, guiding center coordinates:

$$Z^A = (t, X, \gamma, \mu, \varepsilon) \qquad \text{and} \qquad \Gamma_A = (-\mathcal{E}, P, -(m/e)\mu, 0, 0).$$

- $L = p \cdot \dot{x} - \varepsilon = P \cdot \dot{X} - \mathcal{E} - (m/e)\mu\dot{\gamma} \Longleftrightarrow p \cdot \dot{x} = P \cdot \dot{X}$ if $\varepsilon = \mathcal{E} + (m/e)\mu\dot{\gamma}$.
- If $\mathcal{S} = (m/e)\mu\gamma$ then guiding center Lagrangian is

$$L_{gc} = L + \dot{\mathcal{S}} = P \cdot \dot{X} - \varepsilon + (m/e)\mu\dot{\gamma}.$$

General relativity on extended phase-space

- Let's indicate with $\hat{L}$ the phase-space relativistic Lagrangian: $\hat{L} = \gamma_A (dz^A/d\hat{s}) \equiv \gamma_A z'^A$.
- Thus, let's introduce a metric tensor: $\hat{g}_{AB}$ so that $\hat{L}d\hat{s} = \hat{g}_{AB}\gamma^A dz^B$. γ^A are the contravariant momenta in the extended phase space (6+1)D.
- An Hilbert-Einstein lagrangian is added to address the metric tensor. The total lagrangian (distribution) becomes:

$$\ell a = f_m \hat{L} - \frac{\hat{\mathcal{R}}}{16\pi\hat{G}}, \tag{16}$$

- f_m is the scalar distribution function of masses, $\hat{G}$ and $\hat{\mathcal{R}}$ are the gravitational constant and the scalar curvature for the extended phase space, respectively.
- A field theory on the extended phase-space is simply given by the action

$$S = \int \ell a \, d\mathcal{M}, \tag{17}$$

- If the guiding center coordinates are used then $d\mathcal{M} = \sqrt{-\hat{g}}\, dt\, d^3X\, d\gamma\, d\varepsilon\, d\mu$.

The misleading symmetry

- Relativistic guiding center transformation:

$$x^\alpha = X^\alpha + \rho^\alpha(\gamma) \tag{18}$$

$$x'^\alpha = U^\alpha + \nu^\alpha(\gamma). \tag{19}$$

  - ▷ If $\mu \neq 0$ the guiding center is not a particle: $U^\alpha U_\alpha \neq 1$.
  - ▷ The time is separated in slow and fast component: $t = t_{\text{slow}} + t_{\text{fast}}(\gamma)$, being $X^0 = t_{\text{slow}}$ and $\rho^0 = t_{\text{fast}}(\gamma)$.

- The misleading symmetry is the condition: $(m/e)\mu\gamma' = 1 - U^\alpha U_\alpha$.
  (it corresponds to $(m/e)\mu\dot{\gamma} = \varepsilon - \mathcal{E}$ in n.r.)

- The relativistic Lagrangian has the same form after the transformation:
  - ▷ In general: $\hat{L} = \gamma_\alpha u^\alpha = \Gamma_\alpha U^\alpha - (m/e)\mu\gamma'$.
  - ▷ In particular: $\hat{L} = -1 + (e/m)A_\alpha(x^\beta)u^\alpha = -1 + (e/m)A_\alpha(X^\beta)U^\alpha$.

- The form of the Lagrangian remains the same after a translation on phase-space $\Rightarrow$ The extended energy momentum tensor, T_{AB}, is conserved:

$$\hat{G}_{AB} = 8\pi\hat{G}\,\hat{T}_{AB}. \qquad \text{Einstein's equation on the extended phase space} \tag{20}$$

Kaluza-Klein solution

- In 5D: $\tilde{L} = \tilde{g}_{ab}\Gamma^a Z'^b$, $a, b = 0, 1, 2, 3, 4$ and $\ell a = f_m \tilde{L} - \tilde{R}/(16\pi\tilde{G})$.
- The misleading symmetry: $U^a U_a = 1$.
- The following KK metric tensor is adopted:

$$\tilde{g}_{ab} = \left| \begin{array}{cc} g_{\alpha\beta} - k_G^2 A_\alpha A_\beta & k_G^2 (m/e)^2 \mu A_\alpha \\ k_G^2 (m/e)^2 \mu A_\beta & -k_G^2 (m/e)^4 \mu^2 \end{array} \right| . \tag{21}$$

being $k_G = \sqrt{16\pi G}$.

- if $\sqrt{-\tilde{g}} = \sqrt{-g}(m/e)^2 k_G \mu$ and $\tilde{G} = G \int (m/e)^2 k_G \mu \tilde{J}_{\mathcal{P}} d\gamma d\varepsilon d\mu$.
- then

$$S_{\mathrm{f}} = -\int \sqrt{-g} dt d^3 X \, \frac{R}{16\pi G} - \int \sqrt{-g} dt d^3 X \, \frac{F_{\alpha\beta} F^{\alpha\beta}}{4}. \tag{22}$$

- Even if the terms in the lagrangian density, $\mathcal{L}$, are the desired ones, they are referring to fields on (t, X) where X is the guiding center position and it doesn't indicate the position of a particle $\Rightarrow$ the present theory is NON LOCAL
- In the pasma action, the term "somethingnew" is Hilbert-Einstein in extended phase-space.

Conclusion

- In general relativity theory, it is possible to think at a consistency between the gravitational field and the motion of masses. Indeed, what is said is that the space-time coincides with the gravitational field thanks to the Einstein's equation.
    - ▷ Finally, the gravitational field coincides with the extended phase-space not with the only space-time. The important difference with the standard approach is that from giving a geometry to the extended phase-space it is possible to obtain both gravitation and electromagnetism.