Scattering structure of Vlasov equations around inhomogeneous Boltzmannian states

B. Després (LJLL/UPMC and IUF)

• Model problem : Vlasov-Poisson-Ampère 1d-1v $(x, v) \in I \times \mathbb{R}$

$$\begin{cases} \partial_t f + v\partial_x f - \varepsilon^2 E \partial_v f = 0, & & t > 0, & (x,v) \in I \times \mathbb{R}, \\ \partial_x E = \rho_{\mathrm{ref}}(x) - \int_{\mathbb{R}} f dv, & E = -\partial_x \varphi, & t > 0, & x \in I, \\ \partial_t E = 1^* \int_{\mathbb{R}} v f dv, & t > 0, & x \in I := [0,1]. & \end{cases}$$

The a priori small **scaling parameter** ε is for mathematical convenience.

• Understand/prove linear Landau damping $E(t) \underset{t \to +\infty}{\longrightarrow} 0$ around inhomogeneous Boltzmannian states in the **linear regime**

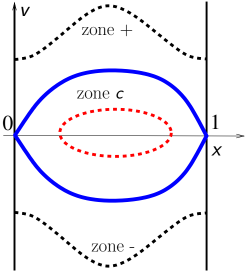
$$\begin{cases} f(x,v,t) = f_0(x,v) + g(x,v,t), \qquad f_0(x,v) = n_0(x) e^{-v^2/2}, \\ E(x,t) = E_0(x) + F(x,t), \\ E_0(x) = -\varphi_0'(x), \\ -\varphi_0''(x) + \underbrace{\int_{\mathbb{R}} \exp(-v^2/2 + \varepsilon^2 \varphi_0(x)) dv}_{= \sqrt{2\pi} \exp(\varepsilon^2 \varphi_0(x))} = \rho_{\mathrm{ref}}(x). \end{cases}$$

- Morrison, Hamiltonian Description of Vlasov Dynamics : Action-Angle Variables . . ., 2000.
- Lemou, Mehats and Raphael, A new variational approach to the stability of gravitational systems, 2011.
- Barré, Olivetti and Yamaguchi, Landau damping and inhomogeneous reference states, 2015 (2010).
- Campa, Chavanis : A dynamical stability criterion for inhomogeneous quasi-stationary states, 2010.

Set-up : simple electric potential

Electric potential $\begin{cases} \varphi_0'(x) > 0 \text{ for } 0 < x < x_0, \\ \varphi_0'(x) < 0 \text{ for } x_0 < x < 1, \end{cases}$

Sep : $\frac{v^2}{2} - \varepsilon^2 \varphi_0(x) = -\varepsilon^2 \varphi_0^-$.



Trapped particles

$$\begin{cases} \partial_t g + v\partial_x g - \varepsilon^2 E_0 \partial_v g - \varepsilon^2 F \partial_v f_0 = 0, \\ \partial_t F = 1^* \int_{\mathbb{R}} vg dv, \\ \partial_x F = -\int_{\mathbb{R}} g dv, \end{cases}$$

Energy identity with $G(v) = \exp(-v^2/2)$

$$\frac{d}{dt}\left(\int_I \int_{\mathbb{R}} \frac{g^2}{\varepsilon^2 n_0 G} dv dx + \int_I F^2 dx \right) = 0.$$

- Kruzkal-Obermann, On the stability of plasma in static equilibrium,1958.
- Antonov, Remarks on the problem of stability in stellar dynamics,1961.
- Barré, Olivetti and Yamaguchi, Landau damping and inhomogeneous reference states, 2015.

Scattering framework

Set $M(x,v) = \sqrt{n_0(x)G(v)}$, $u(x,v,t) = \frac{g(x,v,t)}{\varepsilon M(x,v)}$ and

$$U = \begin{pmatrix} u \\ F \end{pmatrix} \in X := L^2(I \times \mathbb{R}) \times L_0^2(I) \qquad \text{(Hilbert space).}$$

One has $\partial_t U(t) = \mathbf{i}H^\varepsilon U(t)$

$$\mathbf{i}H^\varepsilon = \left(\begin{array}{c|c} -v\partial_x + \varepsilon^2 E_0 \partial_v & -\varepsilon v M \\ \hline \varepsilon 1^* \int_v vM & 0 \end{array} \right) \qquad \text{(unbounded operator).}$$

Decomposition of operators $\mathbf{i}H^\varepsilon = \mathbf{i}H_0^\varepsilon + \varepsilon \mathbf{i}K$

$$\mathbf{i}H_0^\varepsilon = \left(\begin{array}{c|c} -v\partial_x + \varepsilon^2 E_0 \partial_v & 0 \\ \hline 0 & 0 \end{array} \right) \text{ and } \varepsilon \mathbf{i}K = \left(\begin{array}{c|c} 0 & -\varepsilon v M \\ \hline \varepsilon 1^* \int_v vM & 0 \end{array} \right)$$

Gauss law and averaging lemmas (Golse), $\mathbf{i}K$ is a compact operator.

- Diperna-Lions, Global weak solutions of Vlasov-Maxwell systems, 1989
- Golse-Lions-Perthame-Sentis, Regularity of the moments of the solution of a transport equation, 1988

- Reed-Simon, Methods of modern mathematical physics : **Scattering theory**, 1979.
- Yafaev, **Scattering Theory : Some Old and New Problems**, 2000

Moments

- To simplify, use Hermite functions $\psi_n(v) = (2\pi)^{-\frac{1}{4}} n!^{-\frac{1}{2}} H_n(v)\, G^{\frac{1}{2}}(v)$

$$u(t,x,v) = \sum_n u_n(t,x)\psi_n(v), \quad u_n = \int_{\mathbb{R}} u\psi_n dv.$$

The unknown is now $U(t,\cdot) = (F(t,\cdot), u_0(t,\cdot), u_1(t,\cdot), u_2(t,\cdot), \dots)^t$.

- The unperturbed operator becomes $\mathbf{i}H_0^\varepsilon = -A\partial_x U + \varepsilon^2 E_0(x) BU$ with

$$A = A^t = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & 2^{\frac{1}{2}} & 0 & \dots \\ 0 & 0 & 2^{\frac{1}{2}} & 0 & 3^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 3^{\frac{1}{2}} & 0 & 4^{\frac{1}{2}} \\ \dots & \dots & \dots & \dots & 4^{\frac{1}{2}} & \dots \end{pmatrix},$$

The perturbation is $\mathbf{i}KU = DU$ with

$$D = \begin{pmatrix} 0 & 0 & \alpha 1^* \sqrt{n_0(x)} & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ -\alpha\sqrt{n_0(x)} & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} = -D^t.$$

- Després, Symmetrization of Vlasov-Poisson equations, 2014.

Eigenstructure of $A \approx v$ and $B \approx \partial_v$

- Set

$$U_\mu = \begin{pmatrix} 0 \\ \psi_0(\mu) \\ \psi_1(\mu) \\ \psi_2(\mu) \\ \dots \end{pmatrix} \notin X.$$

With $\sum_{n=0}^{\infty} \psi_n(\lambda)\psi_n(\mu) = \delta(\lambda - \mu)$, one understands U_μ is a Dirac mass written with moments.

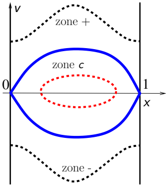
- Since $v\psi_n(v) = \sqrt{n+1}\psi_{n+1}(v) + \sqrt{n}\psi_{n-1}(v)$, one has

$$AU_\mu = \mu U_\mu$$

- Similarly $BU_\mu = -\partial_\mu U_\mu$.

Eigenstructure of $\mathbf{i}H_0^\varepsilon$

Generalized eigenvectors $\notin X$.



• Zone $+$: e label of characteristics

$$\mu_e^\varepsilon(x) = \sqrt{2(e + \varepsilon^2 \varphi_0(x))} > 0, \quad t_e^\varepsilon = \int_0^1 \mu_e^\varepsilon(s)^{-1} ds,$$

$$\lambda_{e,k}^\varepsilon = -\frac{2\mathbf{i}\pi k}{t_e^\varepsilon} \in \mathbf{i}\mathbb{R}$$

$$U_{e,k,+}^\varepsilon(x) = \frac{1}{t_e^\varepsilon \mu_e^\varepsilon(x)} \exp\left(2\mathbf{i}\pi k \frac{\int_0^x \mu_e^\varepsilon(s)^{-1} ds}{t_e^\varepsilon}\right) U_{\mu_e^\varepsilon(x)} \in H^1(I)^{\mathbb{N}},$$

$$U_{e,k,+}^\varepsilon \cdot e_2 = \frac{1}{\alpha t_e} \exp\left(2\mathbf{i}\pi k \frac{\int_0^x \mu_e^\varepsilon(s)^{-1} ds}{t_e^\varepsilon}\right) \exp\left(-\frac{e+\varepsilon^2\varphi_0(x)}{2}\right).$$

Then $\mathbf{i}H_0^\varepsilon U_{e,k,+}^\varepsilon = \lambda_{e,k}^\varepsilon U_{e,k,+}^\varepsilon$.

• Zone c. Naturally $U_{e,k,c}^\varepsilon(x) = 0$ for $x \notin (a_e^\varepsilon, b_e^\varepsilon)$. Mutatis mutandi

$$\begin{aligned} U_{e,k,c}^\varepsilon(x) &= \frac{1}{t_e^\varepsilon \mu_e^\varepsilon(x)} \exp\left(2\mathbf{i}\pi k \frac{\int_{a_e^\varepsilon}^x \mu_e^\varepsilon(s)^{-1} ds}{t_e^\varepsilon}\right) U_{\mu_e^\varepsilon(x)} \\ &+ \frac{1}{t_e^\varepsilon \mu_e^\varepsilon(x)} \exp\left(-2\mathbf{i}\pi k \frac{\int_{a_e^\varepsilon}^x \mu_e^\varepsilon(s)^{-1} ds}{t_e^\varepsilon}\right) U_{-\mu_e^\varepsilon(x)} \in L^p(I)^{\mathbb{N}}, \quad 1 \le p < 2. \end{aligned}$$

Representation of the identity operator I_{d}

- Let $U \in X$.

The **spectral decomposition** holds

$$U = I_{\mathrm{d}} U = U \cdot e_0 e_0 + \sum_z \sum_{k \in \mathbb{Z}} \int_{e \in I_z^\varepsilon} \left(U, U_{e,k,z}^\varepsilon\right) U_{e,k,z}^\varepsilon t_e^\varepsilon de$$

with the Plancherel identity

$$\|U\|^2 = (U, e_0)^2 + \sum_z \sum_{k \in \mathbb{Z}} \int_{e \in I_z^\varepsilon} \left|\left(U, U_{e,k,z}^\varepsilon\right)\right|^2 t_e^\varepsilon de.$$

- The spectral measure t_e^ε is explicitly provided.

Eigenstructure of $\mathbf{i}H^{\varepsilon}$

Start from
$V^{\varepsilon}_{e,k,z}(x) = U^{\varepsilon}_{e,k,z}(x) + a^{\varepsilon}_{e,k,z}(x)e_0 + \sum_{z'}\sum_{p\in\mathbb{Z}}\int_{s\in I^{\varepsilon}_{z'}} b^{\varepsilon}_{\substack{s,p,z'\\e,k,z}} U^{\varepsilon}_{s,p,z'}(x) t^{\varepsilon}_s\, ds.$

Plug in $\mathbf{i}H^{\varepsilon}V = \lambda^{\varepsilon}_{e,k,z}V$ and rearrange

$$\sum_{z'}\sum_{p\in\mathbb{Z}}\int_{s\in I^{\varepsilon}_{z'}} b^{\varepsilon}_{\substack{s,p,z'\\e,k,z}} \left(\lambda^{\varepsilon}_{s,p}-\lambda^{\varepsilon}_{e,k}\right) U^{\varepsilon}_{s,p,z'} t^{\varepsilon}_s\, ds = \alpha\varepsilon \exp\left(\varepsilon^2\varphi_0/2\right) a^{\varepsilon}_{e,k,z} e_2$$

$$+\alpha\left(\frac{1}{\alpha}\lambda^{\varepsilon}_{e,k} a^{\varepsilon}_{e,k,z}(x) - \varepsilon 1^*\left(e^{\varepsilon^2\varphi_0/2}U^{\varepsilon}_{e,k,z}\cdot e_2\right)\right.$$

$$\left.-\varepsilon\sum_{z'}\sum_{p\in\mathbb{Z}}\int_{s\in I^{\varepsilon}_{z'}} b^{\varepsilon}_{\substack{s,p,z'\\e,k,z}} 1^*\left(e^{\varepsilon^2\varphi_0/2}U^{\varepsilon}_{s,p,z'}\cdot e_2\right) t^{\varepsilon}_s\, ds\right) e_0.$$

Crux : use the representation of the identity

$$\begin{cases} b^{\varepsilon}_{\substack{s,p,z'\\e,k,z}}\left(\lambda^{\varepsilon}_{s,p}-\lambda^{\varepsilon}_{e,k}\right) = \varepsilon\left(\alpha\exp\left(\varepsilon^2\varphi_0/2\right) a^{\varepsilon}_{e,k,z}e_2, U^{\varepsilon}_{s,p,z'}\right), \\ \frac{1}{\alpha}\lambda^{\varepsilon}_{e,k}a^{\varepsilon}_{e,k,z} - \varepsilon\sum_{z'}\sum_{p\neq 0}\int_{s\in I^{\varepsilon}_{z'}} b^{\varepsilon}_{\substack{s,p,z'\\e,k,z}} 1^*\left(\exp\left(\varepsilon^2\varphi_0/2\right)U^{\varepsilon}_{s,p,z'}\cdot e_2\right) t^{\varepsilon}_s\, ds = \varepsilon 1^*\left(\exp\left(\varepsilon^2\varphi_0/2\right) U^{\varepsilon}_{e,\right. \end{cases}$$

Keystone : solve the Lipmann-Schwinger equation

- **Strong form :** for $k \neq 0$, $z \in \{+,-,c\}$ and $e \in I_z^\varepsilon$,

solve the integral equation for $a = a_{e,k,z}^\varepsilon \in L_0^2(I)$

$$\frac{1}{\alpha}\lambda_{e,k}^\varepsilon a$$
$$-\varepsilon^2 \sum_{z'} \sum_{p\neq 0} P.V. \int \frac{\left(\alpha a e_2, \exp\left(\varepsilon^2\varphi_0/2\right) U_{s,p,z'}^\varepsilon\right)}{\lambda_{s,p}^\varepsilon - \lambda_{e,k}^\varepsilon} 1^* \left(\exp\left(\varepsilon^2\varphi_0/2\right) U_{s,p,z'}^\varepsilon \cdot e_2\right) t_s^\varepsilon ds$$
$$= \varepsilon 1^* \left(\exp\left(\varepsilon^2\varphi_0/2\right) U_{e,k,z}^\varepsilon \cdot e_2\right).$$

- **Variational form :** find $a \in L_0^2(I)$ such that

$$(a,b) + \varepsilon^2 \sum_{p\neq 0} \frac{1}{p^2\pi^2} P.V. \int_{\mathbb{R}} \frac{m_{a,b,p}^\varepsilon(\lambda)}{\lambda - \mathbf{i}\mu/2\pi p} d\lambda$$
$$+\varepsilon^2 \sum_{p>0} \frac{1}{p^2\pi^2} P.V. \int_{\mathbb{R}} \frac{n_{a,b,p}^\varepsilon(\lambda)}{\lambda - \mathbf{i}\mu/2\pi p} d\lambda = \frac{1}{\mu}\varepsilon l(b), \quad \forall b \in L_0^2(I)$$

where $\lambda = 1/t_e^\varepsilon$, the kernel is

$$m_{a,b,p}^\varepsilon(\lambda) = \left(a, v_{s^\varepsilon(|\lambda|),p,\mathrm{sign}(\lambda)}^\varepsilon\right)\left(v_{s^\varepsilon(|\lambda|),p,\mathrm{sign}(\lambda)}^\varepsilon, b\right) \exp(-s^\varepsilon(|\lambda|))(s^\varepsilon)'(|\lambda|)$$

with $v_{s,p,+}^\varepsilon(x) = \exp\left(2\mathbf{i}\pi p \frac{\int_0^x \mu_s^\varepsilon(t)^{-1} dt}{t_s^\varepsilon}\right)$.

Trick 1 : $P.V. \int_{\mathbb{R}} \frac{\lambda f(\lambda)}{\lambda - \mu} d\lambda = \mu P.V. \int_{\mathbb{R}} \frac{f(\lambda)}{\lambda-\mu} d\lambda + \int_{\mathbb{R}} f(\lambda) d\lambda$.

Trick 2 : $\|g\|_{L^\infty(\mathbb{R})} \leq \sqrt{\|g\|_{L^2(\mathbb{R})} \|g'\|_{L^2(\mathbb{R})}}$

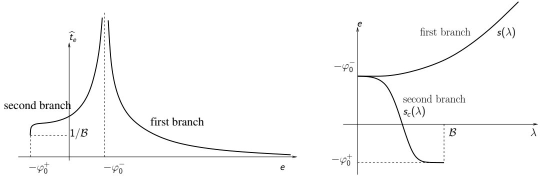
Well-posedness of the transformation $e \mapsto \widehat{t}_e := t_e^\varepsilon$

Introduction

Technical material

Lipmann-Schwinger equation

Issue is the region of trapped particles, i.e. the second branch



Elementary calculations show that

$$\frac{d}{de}\frac{\widehat{t}_e}{2} = \frac{d}{de}\int_{x_0}^{\widehat{b}_e} \frac{dx}{\sqrt{e-\psi(x)}} = -\int_0^1 \frac{4u^4 e}{\sqrt{1-u^2}} \frac{(\sqrt{\psi})''}{(\psi')^3}(\psi^{-1}(eu^2))du > 0$$

where $\psi(x) = -\varphi_0(x) + \max_y \varphi_0(y) \geq 0$.

- $\sqrt{\psi}'' > 0$ for HMF model.

Statement of the claim

Usual assumptions + specific assumptions
1) Boltzmanian hypothesis,
2) one region of trapped particles,
3) the time needed for particles to travel along characteristics is a monotone function of the characteristic label

$$\sup_{x\neq 0, x_0, 1} \left(\frac{d^2}{dx^2} \sqrt{\varphi_0^+ - \varphi_0(x)} \right) < 0$$

4) the initial data has zero mean value along the characteristics curves of the transport operator $v\partial_x - \varepsilon^2 E_0 \partial_v$.

There exists a constant ε_* such that for all $0 < \varepsilon < \varepsilon_* \leq 1$, then the electric field tends to zero in strong norm $\lim_{t\to\infty} \|F(t)\|_{L^\infty(I)} = 0$. Moreover if $\varphi_0'' < \sqrt{2\pi}$ then the ion density ρ_{ref} is positive (and so the solution is physically admissible).

- If $\varphi_0 \equiv 0$, one solves the Lipmann-Schwinger equation Fourier mode per Fourier mode and explicitly constructs a modified Möller wave operator. It is equal to the Morrisson integro-differential transform.
- The small parameter ε^* is only for mathematical convenience. My feeling is it should be removed.
- Preprint available.