Studying the nonlinear suppression of Landau damping in the Vlasov–Poisson system using Hermite mode fluxes in a Fourier–Hermite spectral representation

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Overview

Fourier–Hermite spectral expansion for the 1+1D Vlasov–Poisson system

Free energy diagnostics and Landau damping

Propagation of free energy between Hermite modes

Suppression of Landau damping by nonlinearity

Further application to quasineutral drift (gyro) kinetics

The one-dimensional Vlasov–Poisson system

Start with the 1+1D Vlasov–Poisson system for $\tilde{f}$

$$\frac{\partial \tilde{f}}{\partial t} + v\frac{\partial \tilde{f}}{\partial z} - E\frac{\partial \tilde{f}}{\partial v} = C[\tilde{f}], \;\; E = -\frac{\partial \Phi}{\partial z}, \;\; -\frac{\partial^2 \Phi}{\partial z^2} = 1 - \int_{-\infty}^{\infty} dv \, \tilde{f}.$$

Decompose $\tilde{f} = f_0 + f$, where $f_0(v)$ is a stationary, spatially-uniform distribution function satisfying $\int_{-\infty}^{\infty} dv \, f_0 = 1$. Equivalent to writing $\tilde{f}$ as a particular integral f_0 plus a complementary function f (not necessarily small).

The Vlasov–Poisson system becomes

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial z} - E\frac{\partial}{\partial v}(f_0 + f) = C[f], \;\; E = -\frac{\partial \Phi}{\partial z}, \;\; \frac{\partial^2 \Phi}{\partial z^2} = \int_{-\infty}^{\infty} dv \, f.$$

The overall charge neutrality condition becomes

$$\int_0^L dz \int_{-\infty}^{\infty} dv \, f(z, v, t) = 0.$$

This makes E and Φ periodic functions of $z \in [0, L]$.

Relative entropy

The decomposition $\tilde{f} = f_0 + f$ motivates introducing the spatially-integrated relative entropy [*eg* Bardos *et al.* 1993]

$$\mathcal{R}[\tilde{f}|f_0] = \int_0^L dz \int_{-\infty}^{\infty} dv\ \tilde{f}\log(\tilde{f}/f_0) - \tilde{f} + f_0.$$

Expanding for small perturbations $f \ll f_0$ gives a positive-definite quadratic:

$$\mathcal{R}[\tilde{f}|f_0] = \int_0^L dz \int_{-\infty}^{\infty} dv\ f^2/(2f_0) + \mathcal{O}\left(f^3\right).$$

Taking $f_0 = \pi^{-1/2}\mathrm{e}^{-v^2}$ gives

$$\mathcal{R}[\tilde{f}|f_0] = \mathcal{H}[\tilde{f}] + \int_0^L dz \int_{-\infty}^{\infty} dv\ \overbrace{\left(\tfrac{1}{2}\log\pi - 1\right)\tilde{f} + v^2\tilde{f} + f_0}^{\text{collision invariants}},$$

a combination of $\langle n\rangle$, $\langle\mathcal{E}\rangle$, and the spatially-integrated Boltzmann entropy

$$\mathcal{H}[\tilde{f}] = \int_0^L dz \int_{-\infty}^{\infty} dv\ \tilde{f}\log\tilde{f}.$$

A quadratic approximate free energy

Relative entropy is not conserved in a collisionless plasma, since f_0 couples to the electric field through $-E\partial_v f_0$.

What is conserved (without collisions) is the free energy per unit length:

$$W_{\mathrm{exact}} = \frac{1}{L}\left[\frac{1}{2}\mathcal{R}[\tilde{f}|f_0] + \frac{1}{2}\int_0^L dz\,|E|^2\right],$$

with a $1/2$ because $f_0 = \pi^{-1/2}\mathrm{e}^{-v^2}$ has dimensionless temperature $1/2$.

Using the quadratic approximation for the relative entropy leads to

$$W = W_f + W_E.$$

Both pieces

$$W_f = \frac{1}{2L}\int_0^L dz\,\int_{-\infty}^{\infty} dv\,\frac{f^2}{2f_0}, \quad W_E = \frac{1}{2L}\int_0^L dz\,|E|^2,$$

will be expressed neatly later in terms of Fourier–Hermite expansion coefficients using Parseval's theorem (with the $1/L$).

Fourier–Hermite expansion

Following Armstrong (1967), Grant & Feix (1967) etc etc we expand

$$f(z,v,t) = \sum_{m=0}^{N_m-1} \sum_{j=-N_\vartheta}^{N_\vartheta} a_{jm}(t) \mathrm{e}^{\mathrm{i}k_j z} \phi_m(v),$$

where $k_j = 2\pi j/L$, and for $m = 0, 1, 2, \ldots$

$$H_m(v) = (-1)^m \mathrm{e}^{v^2} \frac{d^m}{dv^m} \mathrm{e}^{-v^2}, \; \phi^m(v) = \frac{H_m(v)}{\sqrt{2^m m!}}, \; \phi_m(v) = \frac{\mathrm{e}^{-v^2}}{\sqrt{\pi}} \phi^m(v).$$

Each $\phi_m(v) \to 0$ as $v \to \pm\infty$. The ϕ_m and ϕ_n satisfy the biorthogonality relations

$$\int_{-\infty}^{\infty} dv \, \phi_n(v) \phi^m(v) = \delta_{nm} \text{ for } m, n \geq 0.$$

The Poisson equation is $-k_j^2 \hat{\Phi}_j = a_{j0}$. The quadratic free energy terms are

$$W_f = \frac{1}{4} \sum_{j=-N_\vartheta}^{N_\vartheta} \sum_{m=0}^{N_m-1} |a_{jm}|^2, \quad W_E = \frac{1}{2} \sum_{j=-N_\vartheta}^{N_\vartheta} |\hat{E}_j|^2, \quad \hat{E}_j = -\mathrm{i}k_j \hat{\Phi}_j.$$

Evolution equations for the a_{jm}

Substituting this expansion into the Vlasov–Poisson system gives

$$\frac{da_{jm}}{dt} + \mathrm{i}k_j\left[\sqrt{\frac{m+1}{2}}a_{j,m+1} + \sqrt{\frac{m}{2}}a_{j,m-1}\right] + \sqrt{2}\hat{E}_j\delta_{m1} + \mathsf{N}_{jm} = \mathsf{C}_{jm},$$

for $|j| \leq N_\vartheta$ and $m < N_m$. It is closed by setting $a_{j,-1} = 0$ and $a_{j,N_m} = 0$. The nonlinear term is the discrete Fourier convolution

$$\mathsf{N}_{jm} = \sqrt{2m}\sum_{j'=-N_\vartheta}^{N_\vartheta}\hat{E}_{j'}a_{j-j',m-1}.$$

We consider model collision operators $\mathsf{C}_{jm} = -\nu\mathsf{c}_{jm}a_{jm}$ for constants c_{jm}, such as the Lenard–Bernstein (1958) with $A = v$, $D = 1/2$, and Kirkwood (1946) with $A = v - u$, $D = T$, Fokker–Planck collision operators

$$C[\tilde{f}] = \partial_v\big(A\tilde{f} + \partial_v(D\tilde{f})\big),$$

that yield

$$\mathsf{c}_{jm}^{\mathrm{LB}} = m, \quad \text{and } \mathsf{c}_{jm}^{\mathrm{Kirkwood}} = m\mathcal{I}_{\{m\geq 3\}}, \quad \text{where } \mathcal{I}_{\{m\geq 3\}} = \begin{cases} 1 & m \geq 3, \\ 0 & \text{otherwise.} \end{cases}$$

Energy diagnostics

The evolution equation for a_{j0} implies

$$\frac{d}{dt}W_E + \mathcal{F} = 0, \quad \text{where} \quad \mathcal{F} = \mathrm{Re}\left(\sum_{j=-N_\vartheta, j\neq 0}^{N_\vartheta} \frac{\mathrm{i}a_{j0}^* a_{j1}}{k_j\sqrt{2}} \right).$$

We find later that $\mathrm{i}a_{j0}^* a_{j1}$ is the free energy flux from the $m = 0$ to the $m = 1$ Hermite mode. The electric field can only decay via a net forward flux.

More generally, the equations for $d_t a_{jm}$ imply

$$\frac{d}{dt}(W_f + W_E + W_N) = \mathcal{C}, \quad \text{where} \quad \mathcal{C} = -\frac{\nu}{2} \sum_{j=-N_\vartheta}^{N_\vartheta} \sum_{m=0}^{N_m-1} \mathsf{c}_{jm}|a_{jm}|^2,$$

is the free energy dissipated by collisions (non-negative when the $\mathsf{c}_{jm} \geq 0$).

We account for the $O(f^3)$ difference between W_{exact} and $W_f + W_E$ using

$$W_N = \frac{1}{2}\mathrm{Re}\int_0^t dt' \sum_{j=-N_\vartheta}^{N_\vartheta} \sum_{m=0}^{N_m-1} a_{jm}^* \mathsf{N}_{jm}$$

Landau damping

Following previous work, we study Landau damping from the initial conditions

$$\tilde{f}(v) = (1 + A\cos kz)\mathrm{e}^{-v^2}/\sqrt{\pi}$$

with $A = 1/2$ and $k = 1/2$ in a box of length $L = 4\pi$.

We switch off the $E\partial_v f$ term to obtain linear Landau damping.

Landau's Fourier–Laplace transform solution is

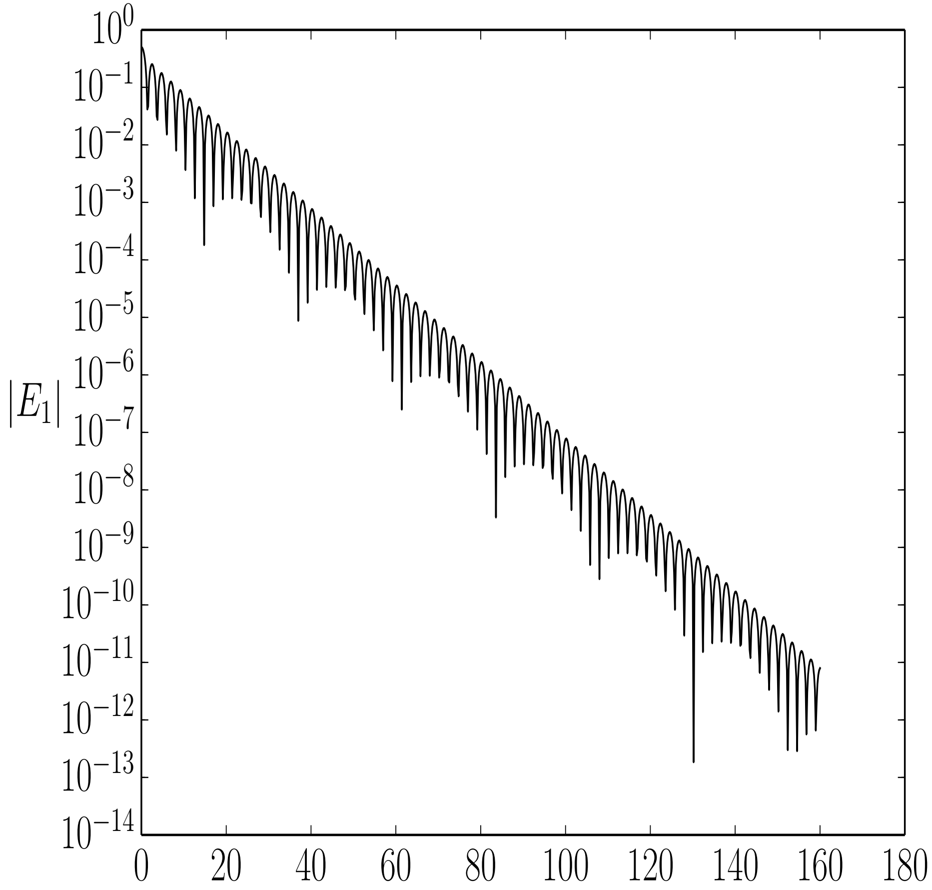
$$\hat{f}(k, v, t) = \frac{1}{2\pi\mathrm{i}}\int_\Gamma \Bigg(\underbrace{\frac{\hat{f}(k, v, t = 0)}{p + \mathrm{i}kv}}_{\text{phase mixes}} - \underbrace{\frac{\mathrm{i}k\overline{\Phi}\partial_v f_0(v)}{p + \mathrm{i}kv}}_{\text{decays}} \Bigg) \mathrm{e}^{pt} dp,$$

where $\hat{f}(k, v, t = 0)$ is the Fourier transform of the initial perturbation, and $\overline{\Phi}(k, p)$ is the Fourier–Laplace transform of the electrostatic potential.

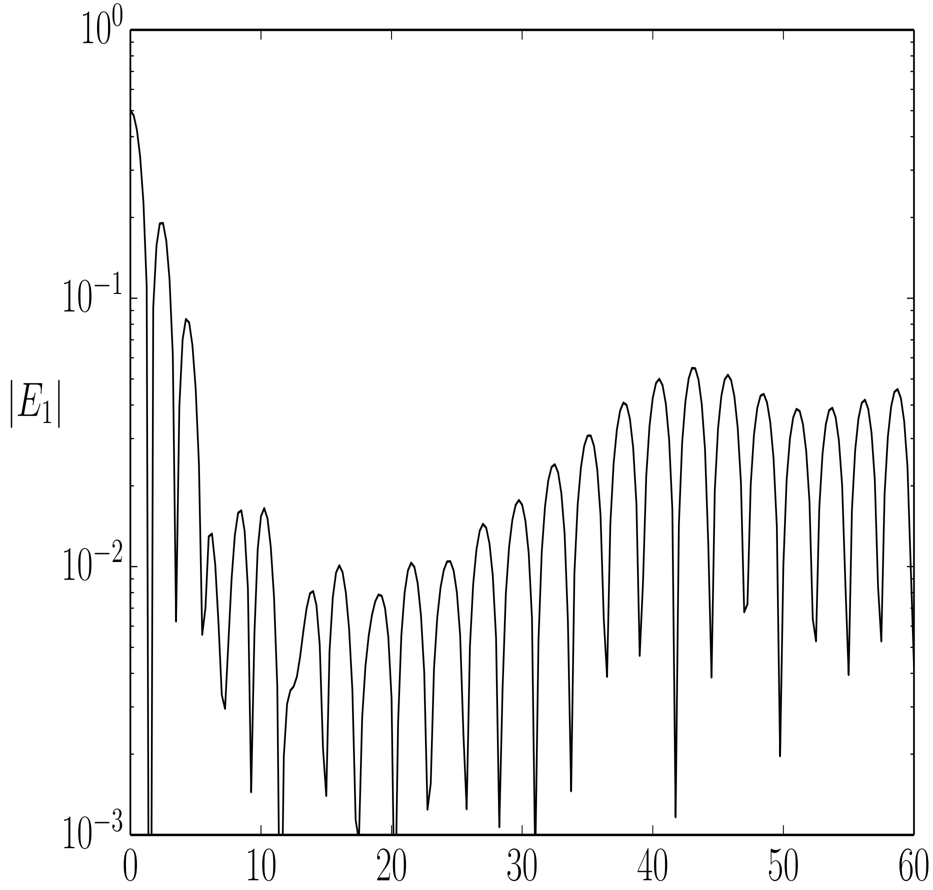
The first term phase-mixes but does not decay. The second term decays like $\mathrm{e}^{-\gamma t}$ at the Landau damping rate γ.

Amplitude of the first electric field Fourier mode $|\mathbf{E}_1|$

linear

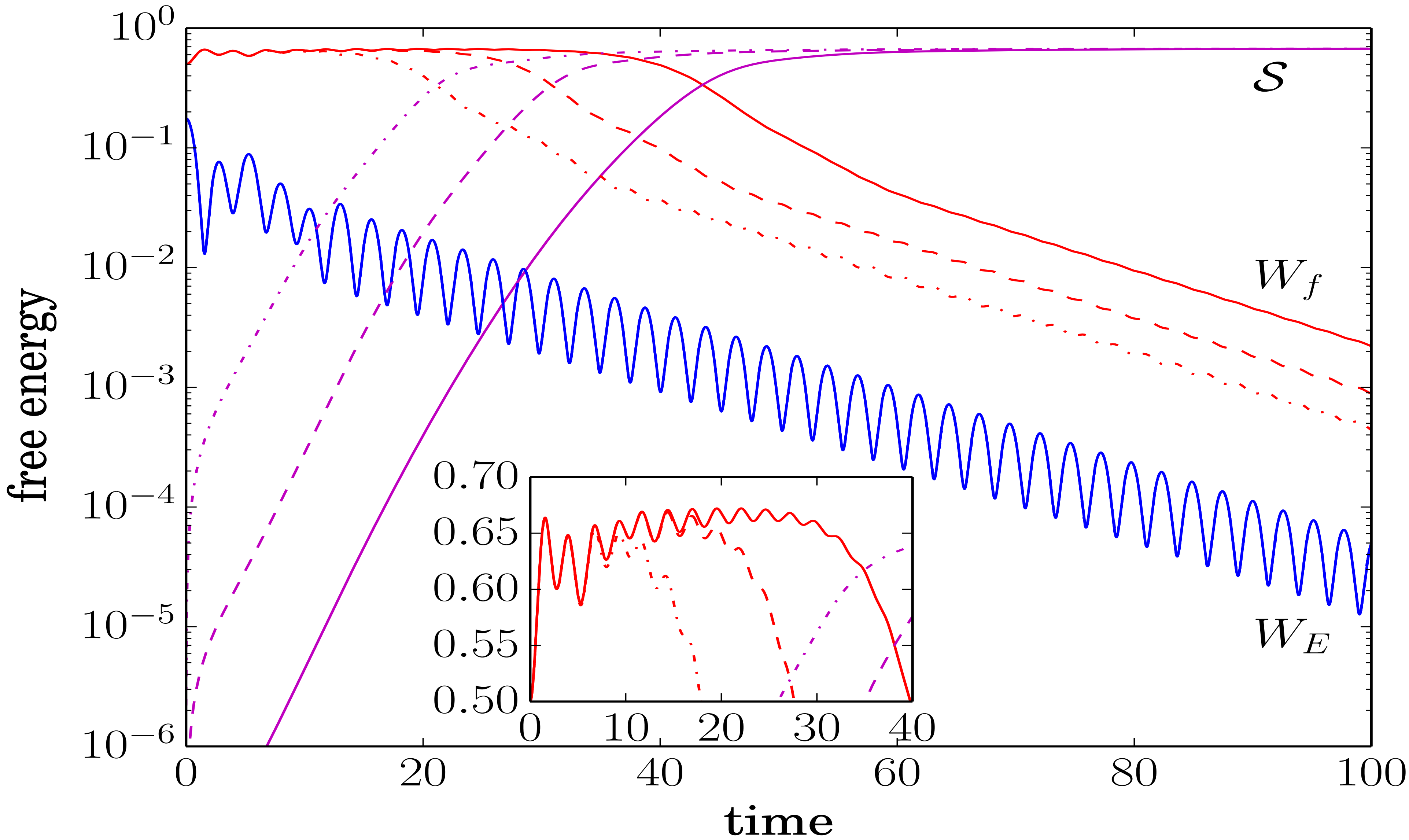


nonlinear



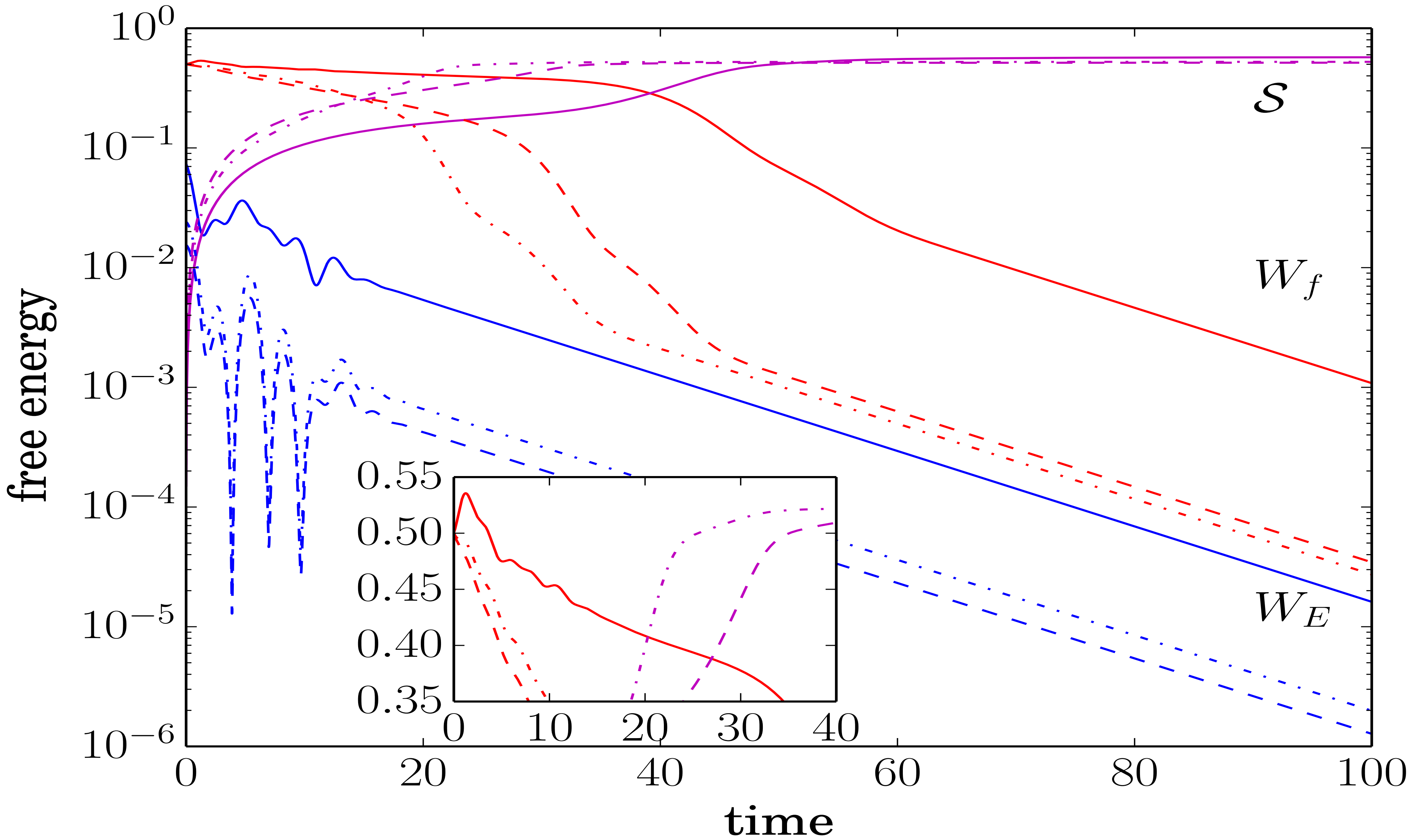
Oscillations due to beats between two counter-propagating modes $\omega = \pm\omega_R + i\gamma$

Evolution of energy diagnostics



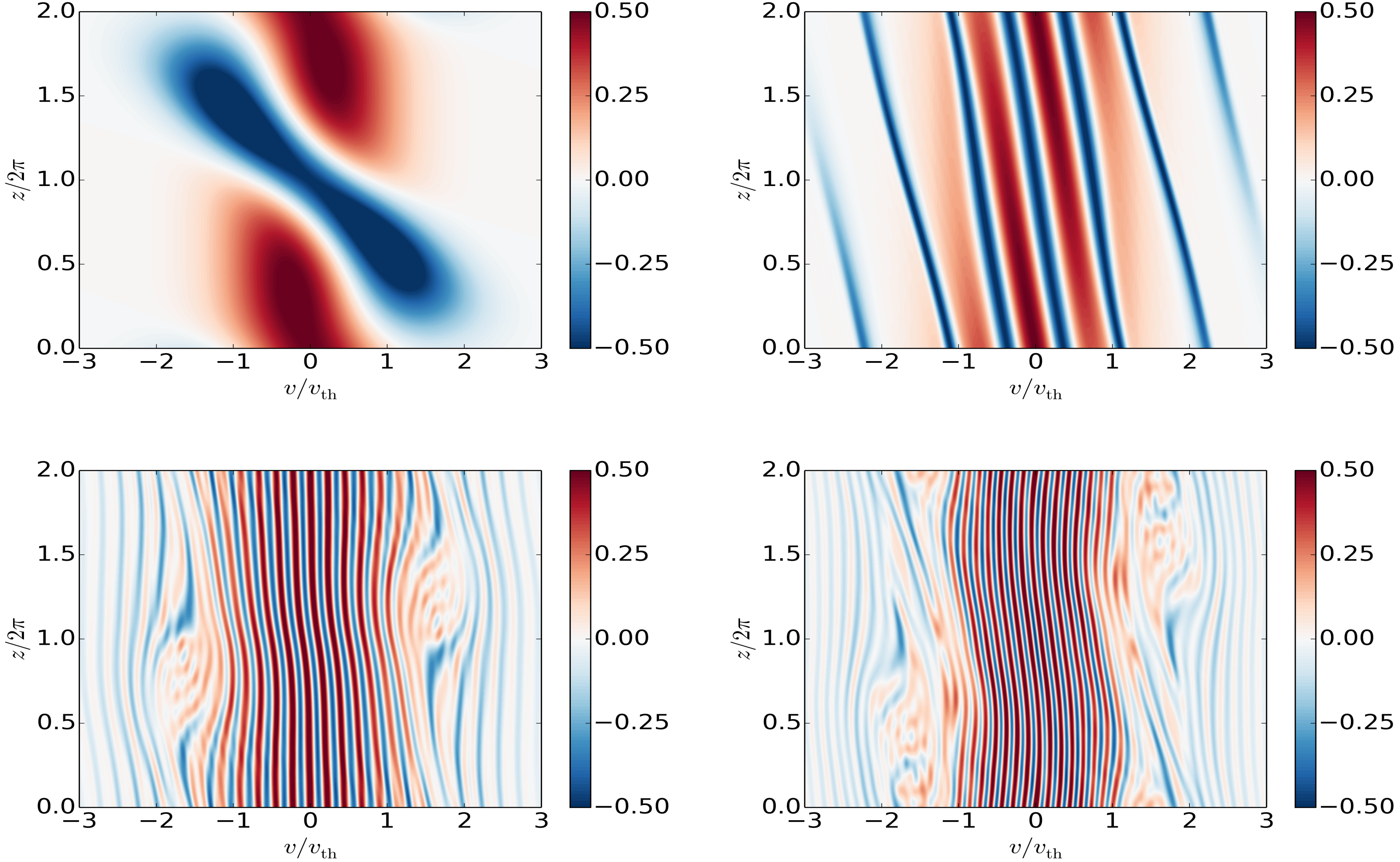
$N_m = 128$ (dot-dashed), $N_m = 256$ (dashed), $N_m = 512$ (solid), sink $\mathcal{S} = \int_0^t dt'\, \mathcal{C}$

Evolution of energy diagnostics (single mode)



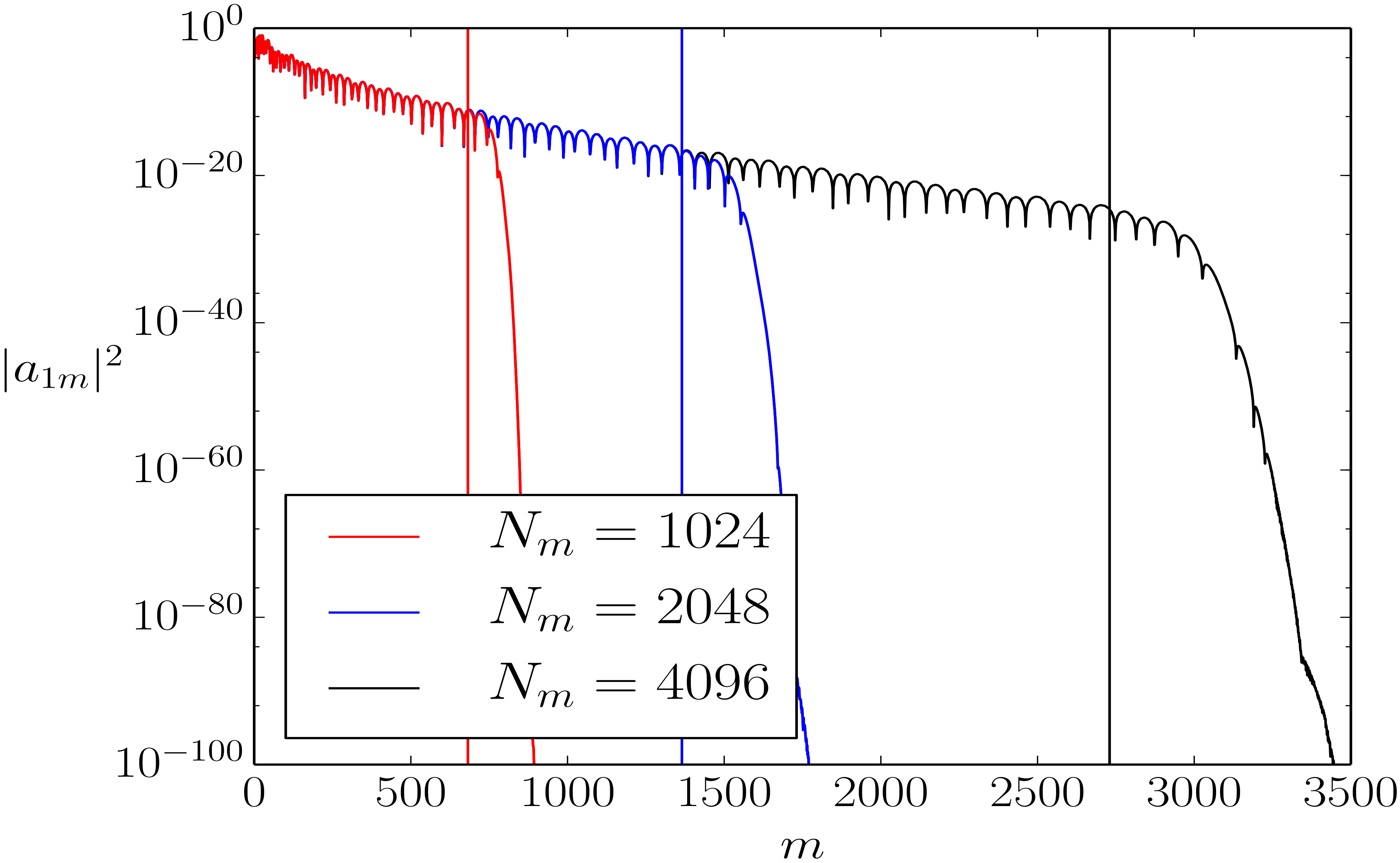
Initial conditions that project onto only one Landau damped mode give monotonic decay.

Phase space plots



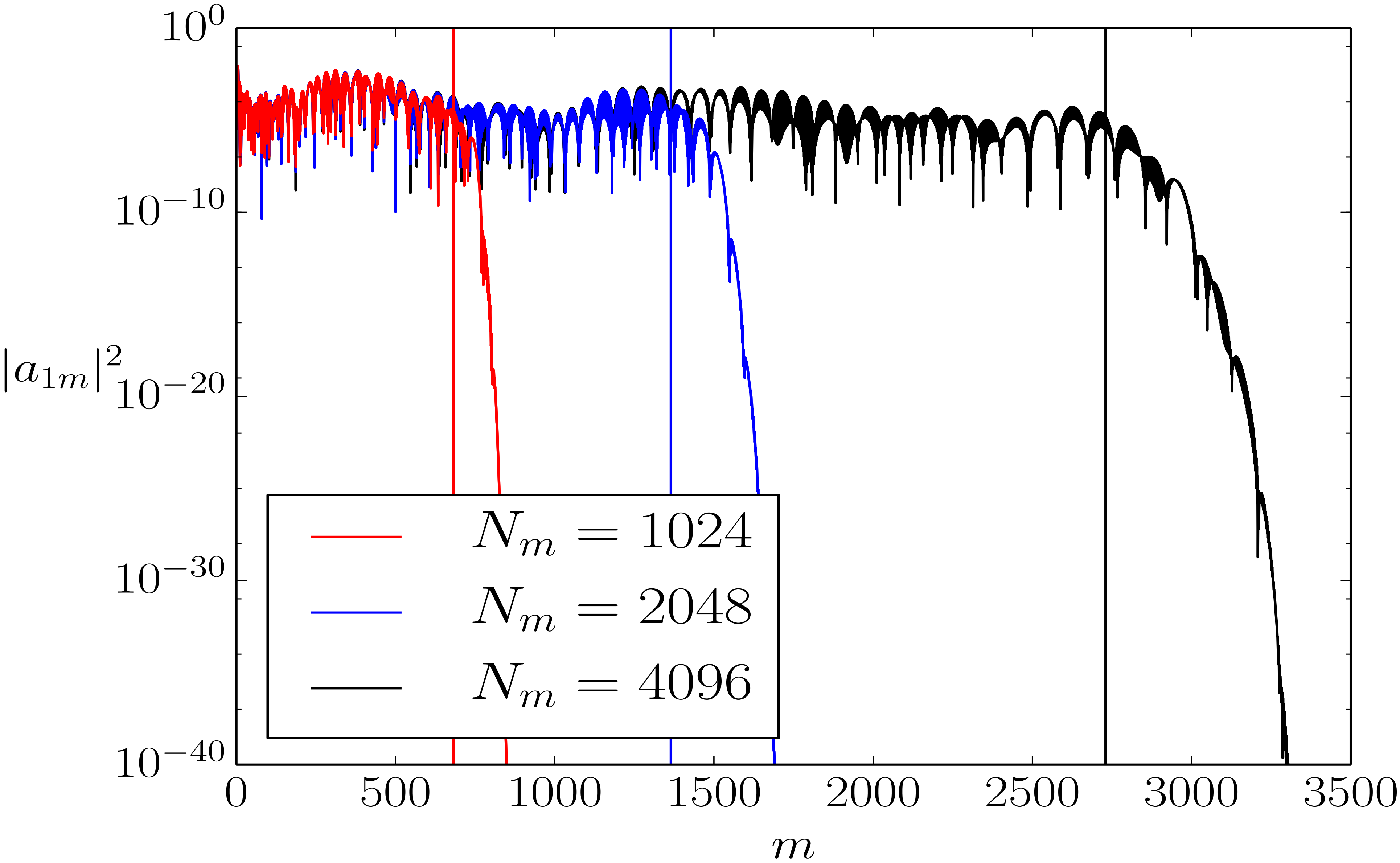
$f(x,v,t)$ at $t \in \{1, 10, 40, 80\}$ computed with $N_k = 256$ and $N_m = 2048$

Hermite spectra at $t = 10$



The vertical lines are at $N = (2/3)N_m$ for each N_m

Hermite spectra at $t = 40$



The vertical lines are at $N = (2/3)N_m$ for each N_m

Fluxes in Hermite space

The quadratic approximate free energy W_f is a sum of $|a_{jm}|^2$.

Writing $\tilde{a}_{jm} = (i\,\mathrm{sgn} k_j)^m a_{jm}$ gives an equation with real coefficients

$$\frac{d\tilde{a}_{jm}}{dt} + |k_j| \left(\sqrt{(m+1)/2}\; \tilde{a}_{j,m+1} - \sqrt{m/2}\; \tilde{a}_{j,m-1} \right) = 0, \qquad (\dagger)$$

for $m \geq 2$ in the linearised collisionless system [Zocco & Schekochihin 2011].

If $f(z, v, 0)$ is even in z and v, the $\tilde{a}_{jm}$ are initially real, and remain real.

The spectral free energy density thus obeys the discrete conservation law

$$\frac{1}{2}\frac{d\tilde{a}_{jm}^2}{dt} + \left(\Gamma_{j,m+1/2} - \Gamma_{j,m-1/2}\right) = 0,$$

with flux

$$\Gamma_{j,m-1/2} = |k_j| \sqrt{m/2}\, \tilde{a}_{jm} \tilde{a}_{j,m-1} = k_j \sqrt{m/2}\, \mathrm{Im} \left(a_{jm}^* a_{j,m-1} \right).$$

We set $\Gamma_{j,-1/2} = 0$ for consistency with $a_{j,-1} = 0$.

Characteristics in Hermite space

If $\tilde{a}_{jm}$ varies slowly in m, in the sense that $\tilde{a}_{jm} \approx \tilde{a}_{j,m+1}$, we can approximate the discrete flux

$$\Gamma_{j,m-1/2} = |k_j| \sqrt{m/2}\, \tilde{a}_{jm} \tilde{a}_{j,m-1}$$

by the slowly varying flux

$$\Gamma_{jm}^{\mathrm{SV}} = |k_j| \sqrt{m/2}\, \tilde{a}_{jm}^2$$

Treating m as a continuous variable leads to the conservation law

$$\frac{1}{2} \frac{\partial \tilde{a}_{jm}^2}{\partial t} + \frac{\partial \Gamma_{jm}^{\mathrm{SV}}}{\partial m} = 0,$$

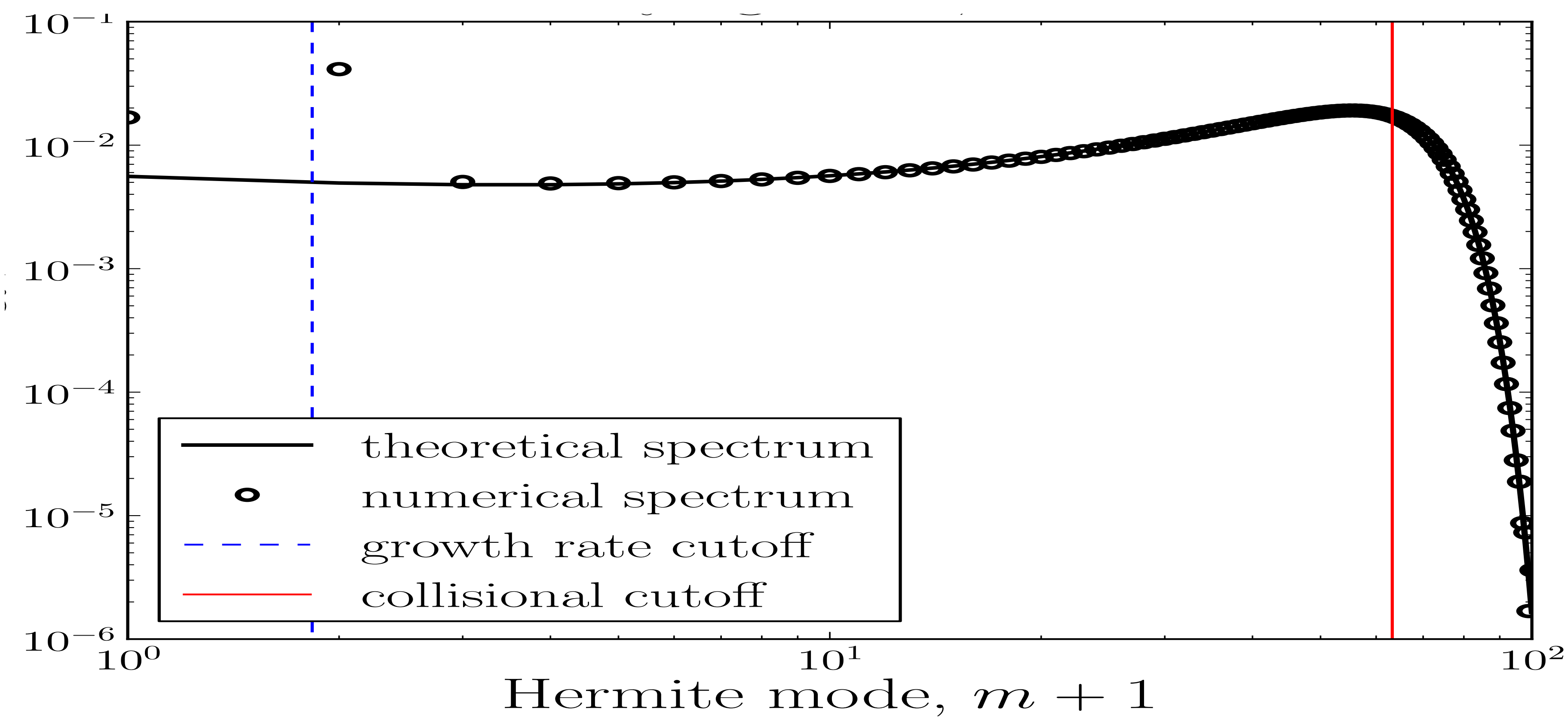
which may be rewritten as

$$\left(\frac{1}{2} \frac{\partial}{\partial t} + |k_j| \frac{\partial}{\partial \sqrt{2m}} \right) \left(\sqrt{2m} \tilde{a}_{jm}^2 \right) = 0.$$

The free energy density propagates along characteristics labelled by m_0:

$$\sqrt{m} = \sqrt{m_0} + \sqrt{2} |k_j| t$$

Eigenmodes based on "continuous m" approximation



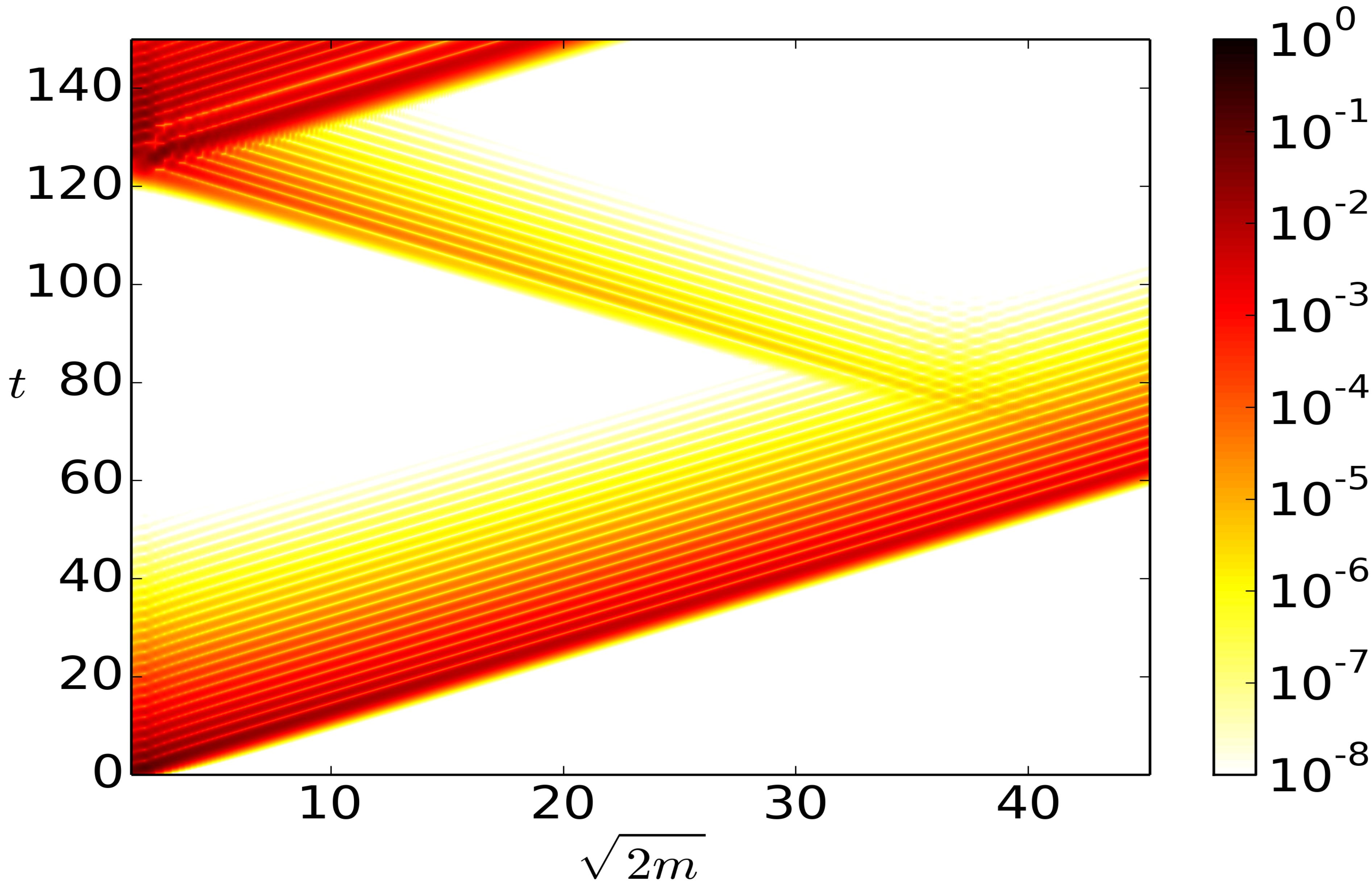
Matrix eigenvector versus

$$|a_{jm}|^2 = (c/\sqrt{2m}) \exp\left(-\mathrm{sgn}(\gamma)\,(m/m_\gamma)^{1/2} - (m/m_c)^{n+1/2}\right)$$

where the growth rate cutoff m_γ and collisional cutoff m_c are

$$m_\gamma = 1/(8\gamma^2), \quad m_c = N^n(n+1/2)^{1/(n+1/2)}/(\sqrt{2}\,\nu)$$

A forward propagating “phase-mixing” mode



Backwards propagating “anti-phase-mixing” modes

The previous result approximates the semi-discrete system

$$\tfrac{1}{2}d_t\,\tilde{a}_{jm}^2 + \left(\Gamma_{j,m+1/2} - \Gamma_{j,m-1/2}\right) = 0,$$

under the assumption that $\tilde{a}_{jm} \approx \tilde{a}_{j,m+1}$, by the PDE

$$\tfrac{1}{2}\partial_t \tilde{a}_{jm}^2 + \partial_m \Gamma_{jm}^{\mathrm{SV}} = 0.$$

Numerical analysis of the semi-discrete “finite difference scheme” shows that it has a second set of “parasitic” solutions with $\tilde{a}_{jm} \approx -\tilde{a}_{j,m+1}$.
Writing $\tilde{a}_{jm} = (-1)^m \hat{a}_{jm}$ transforms (†) into

$$\frac{d\hat{a}_{jm}}{dt} - |k_j|\left(\sqrt{(m+1)/2}\;\hat{a}_{j,m+1} - \sqrt{m/2}\;\hat{a}_{j,m-1}\right) = 0.$$

Assuming $\hat{a}_{jm}$ is slowly varying $\big($while $\tilde{a}_{jm} \sim (-1)^m\big)$ leads to

$$\tfrac{1}{2}\partial_t \hat{a}_{jm}^2 - \partial_m \Gamma_{jm}^{\mathrm{SV}} = 0.$$

Free energy propagates backwards (towards low m) along characteristics

$$\sqrt{m} = \sqrt{m_0} - \sqrt{2}|k_j|t.$$

Hermite flux diagnostics

We introduce the decomposition [Schekochihin *et al.* '14, Kanekar *et al.* '15]

$$\tilde{a}^{+}_{jm} = \tfrac{1}{2}\left(\tilde{a}_{jm} + \tilde{a}_{j,m+1}\right), \quad \tilde{a}^{-}_{jm} = (-1)^m \tfrac{1}{2}\left(\tilde{a}_{jm} - \tilde{a}_{j,m+1}\right).$$

These modes evolve according to

$$d_t \tilde{a}^{\pm}_{jm} + S^{\pm}_{jm} + \mathsf{E}^{\pm}_{jm} + N^{\pm}_{jm} = 0.$$

The streaming term is

$$\begin{aligned} S^{\pm}_{jm} = &\pm \tfrac{1}{2}|k_j|\Big((s_{m+2} + s_{m+1})\tilde{a}^{\pm}_{j,m+1} - (s_{m+1} + s_m)\tilde{a}^{\pm}_{j,m-1}\Big) \\ &\pm \tfrac{1}{2}|k_j|(-1)^m\Big((s_{m+2} - s_{m+1})\tilde{a}^{\mp}_{j,m+1} - (s_{m+1} - s_m)\tilde{a}^{\mp}_{j,m-1}\Big), \end{aligned}$$

where $s_m = \sqrt{m/2}$. The second line is $O(1/m)$ smaller than the first line.

We compare the Hermite flux Γ_{jm} to $\Gamma^{\mathrm{SV}}_{jm}$ using the normalized flux

$$\Gamma^{\mathrm{N}}_{jm} = \frac{\Gamma_{jm}}{\Gamma^{\mathrm{SV}}_{jm}} = \frac{(\tilde{a}^{+}_{jm})^2 - (\tilde{a}^{-}_{jm})^2}{(\tilde{a}^{+}_{jm} + (-1)^m \tilde{a}^{-}_{jm})^2}.$$

The approximation $\Gamma_{jm} \approx \Gamma^{\mathrm{SV}}_{jm}$ is thus valid when $|\tilde{a}^{+}_{jm}| \gg |\tilde{a}^{-}_{jm}|$.

Dissipation and collision operators

We absorb any cascade of free energy to the highest resolved Fourier modes using the Hou–Li (2007) spectral filter

$$\exp\left(-36(|k_j|/k_{\max})^{36}\right)$$

applied to the a_{jm} coefficients every timestep. This filter is highly selective in k_j but produces no noticable reflections and has no tunable parameters.

To absorb free energy at the highest resolved Hermite modes we used either a Hermite version of the Hou–Li filter

$$\exp\left(-36(m/(N_m-1))^{36}\right)$$

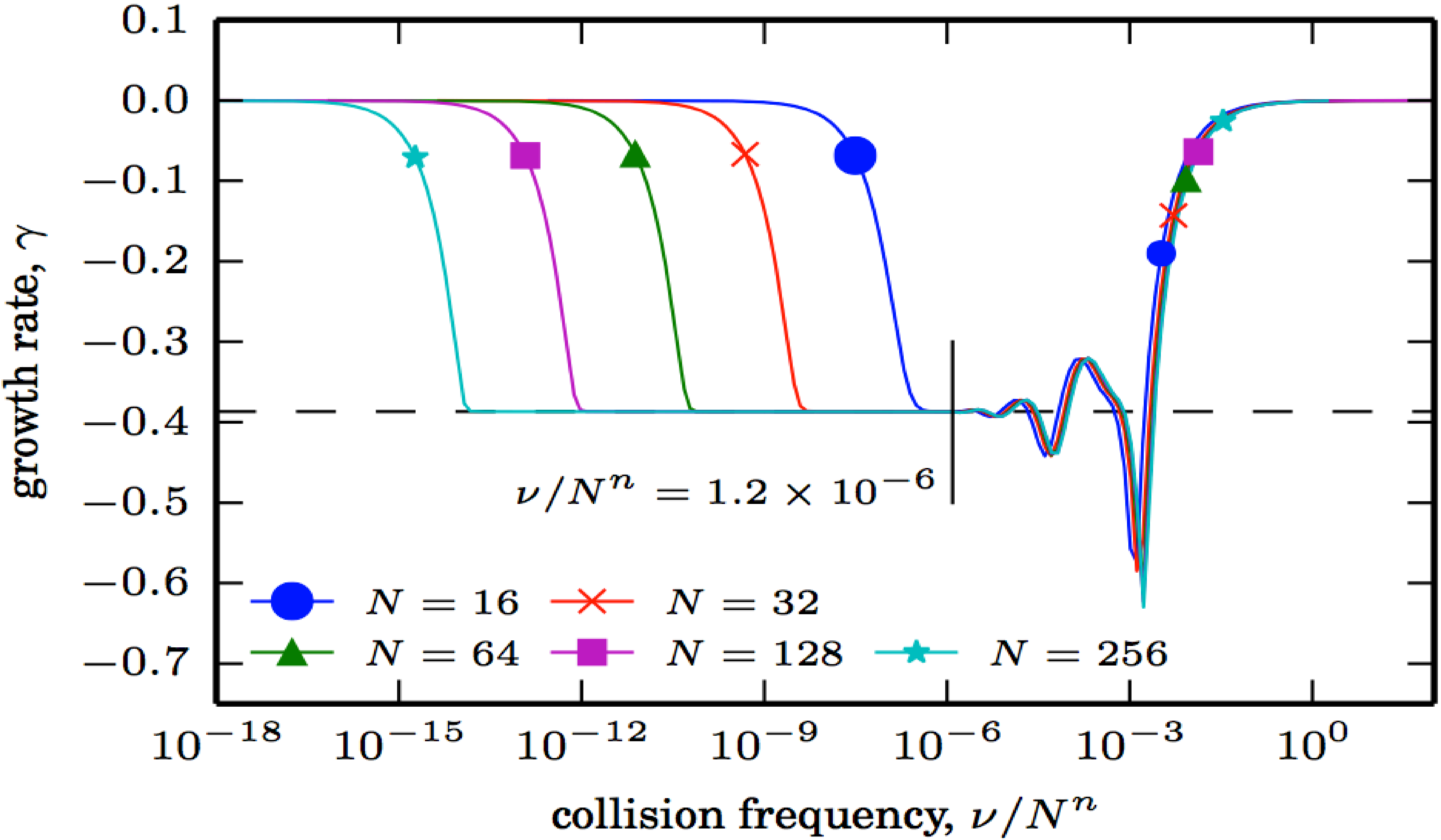
or an iterated Lenard–Bernstein collision operator

$$\mathsf{c}_{jm}^{\text{hyper}} = \nu(m/N_m)^{\alpha} a_{jm}.$$

The latter with $\alpha = 6$ was effective for computing the correct growth and decay rates in a linear gyrokinetic model, even with $N_m = 10$, over a large range of ν values.

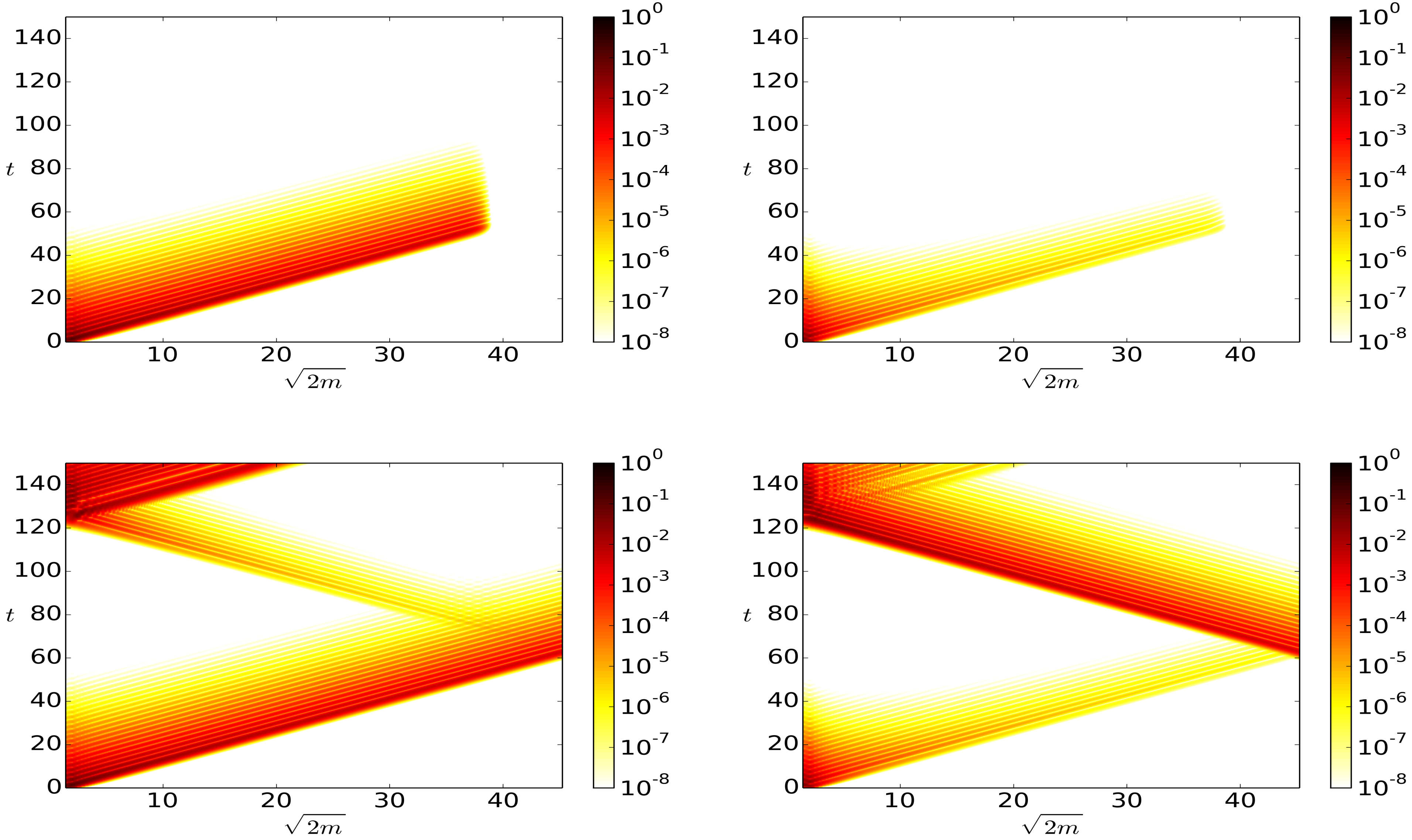
However, it is harder to establish convergence with increasing N_m.

Plateau in hypercollision operator parameters



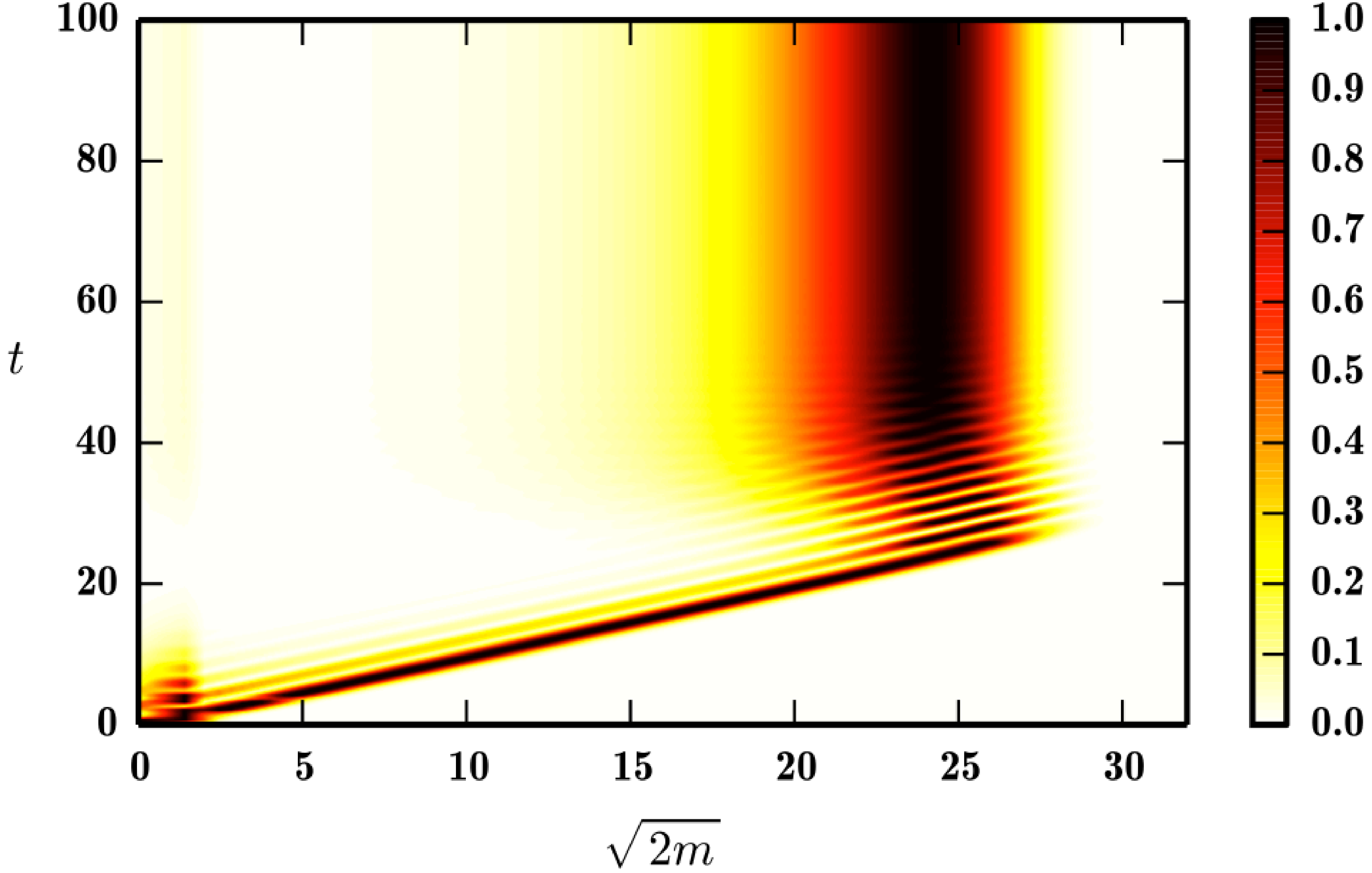
Landau damping rate in a 1D drift kinetics problem with $n = 6$ hypercollisions.

Propagation of Hermite modes $|a^{\pm}_{jm}|$

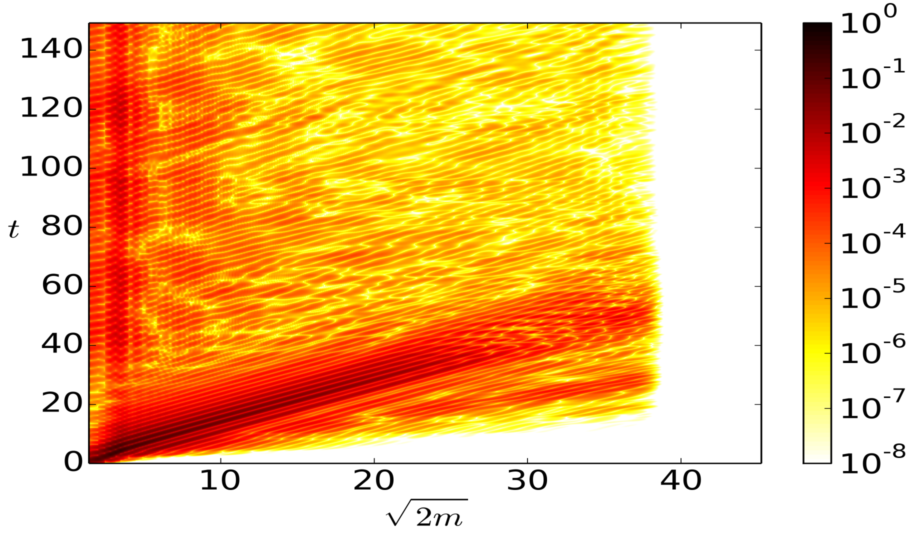


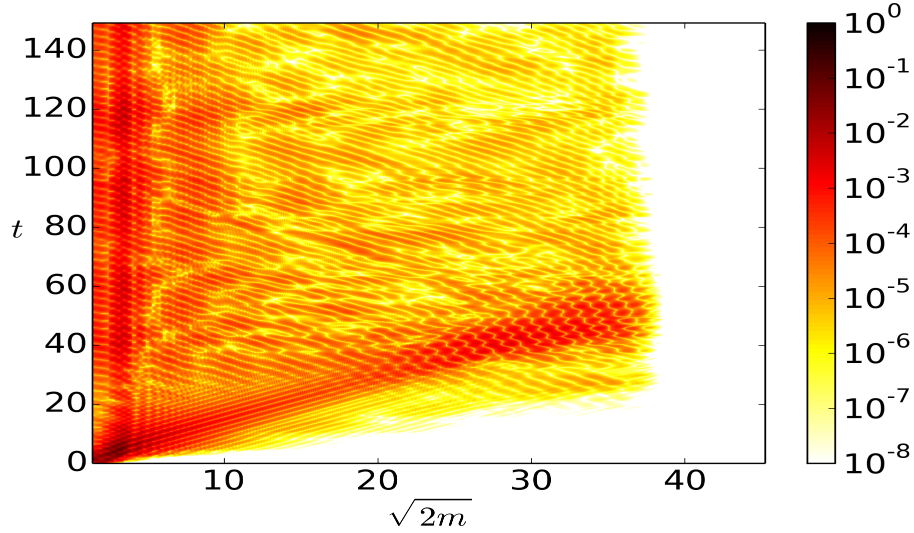
Left $|a^{+}_{jm}|$ and right $|a^{-}_{jm}|$. Top with Hou-Li filter in m, bottom without.

Convergence to an eigenfunction after rescaling amplitude

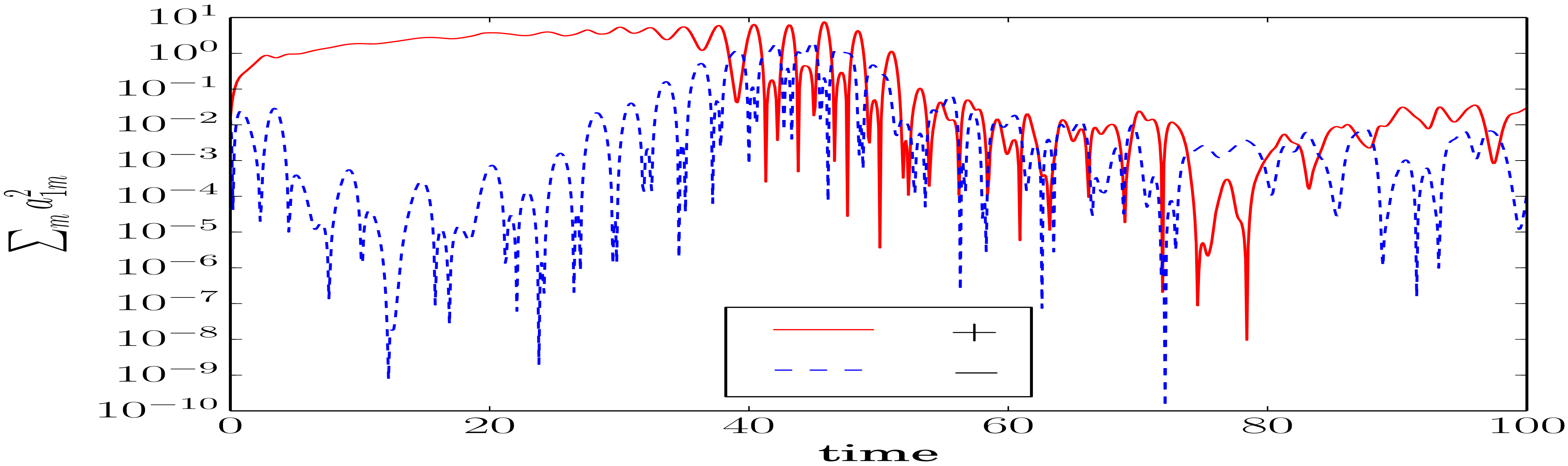


Quasi-steady amplitudes of Hermite modes $|a^{\pm}_{jm}|$

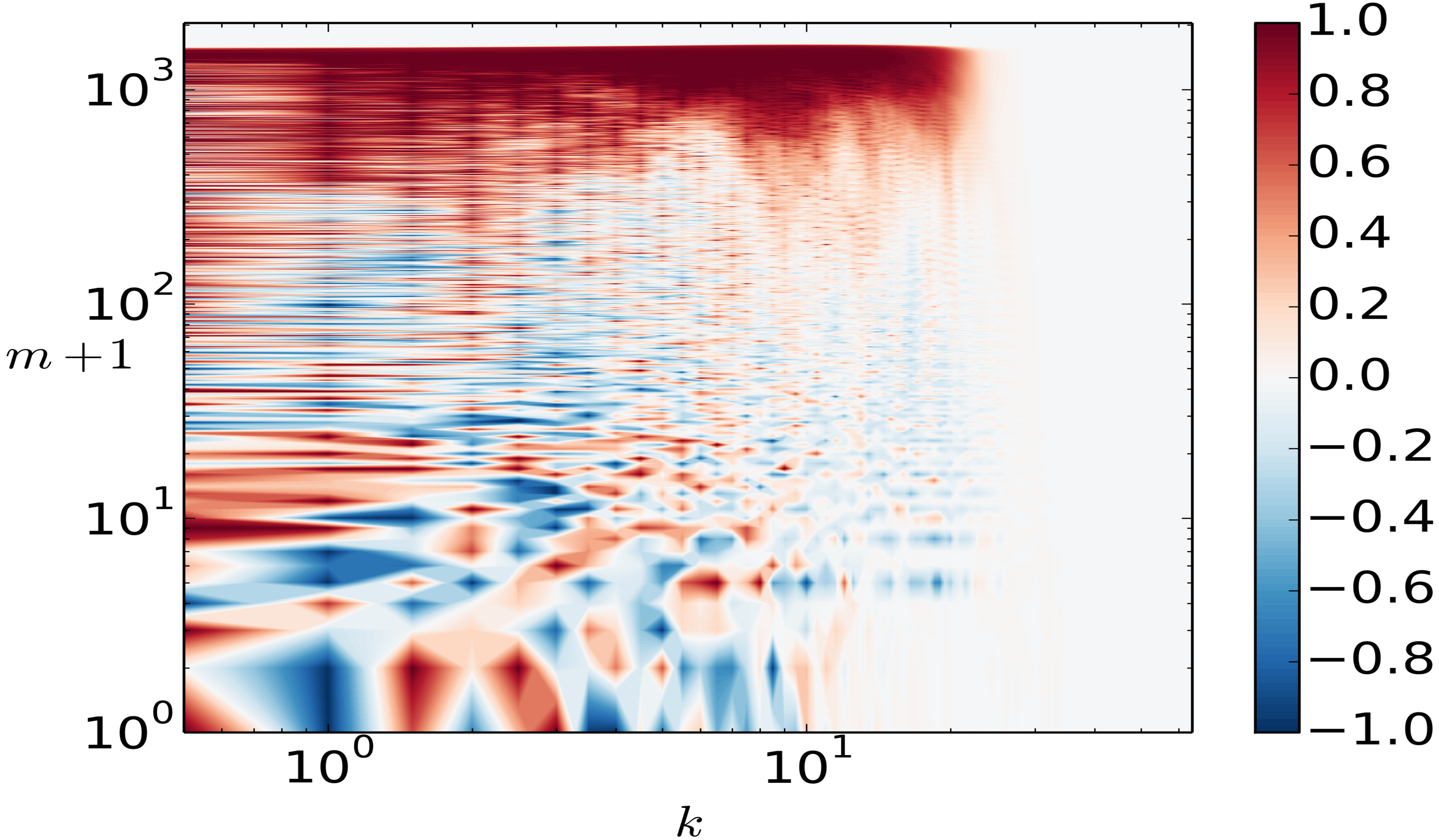




Free energy in $k = 1$ and forwards (+) and backwards (-) modes



Time-averaged normalised Hermite flux

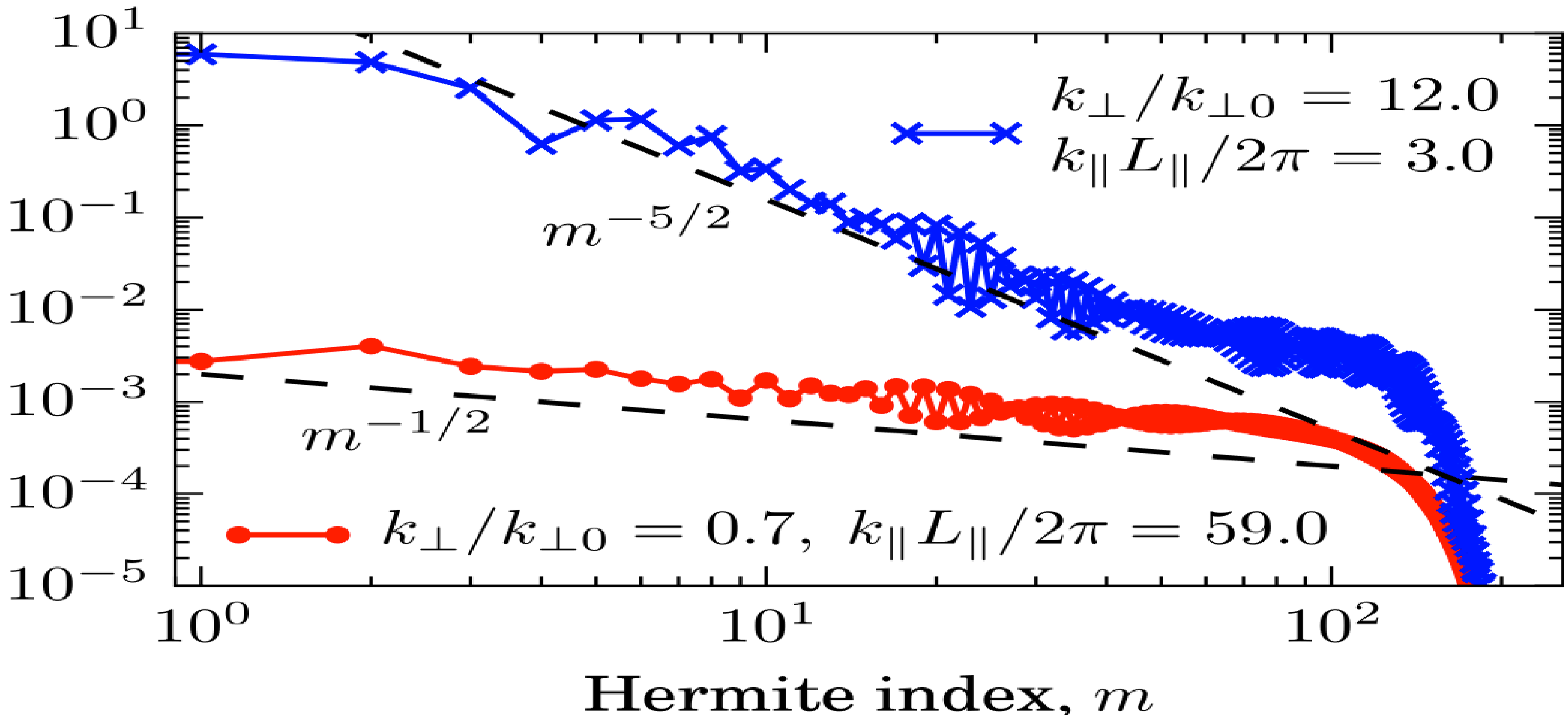


$\Gamma^{\mathrm{N}}_{jm} = \left((\tilde{a}^{+}_{jm})^2 - (\tilde{a}^{-}_{jm})^2\right) / \left(\tilde{a}^{+}_{jm} + (-1)^m \tilde{a}^{-}_{jm}\right)^2$ averaged over $t \in [40, 80]$

A similar story in drift (gyro)kinetics

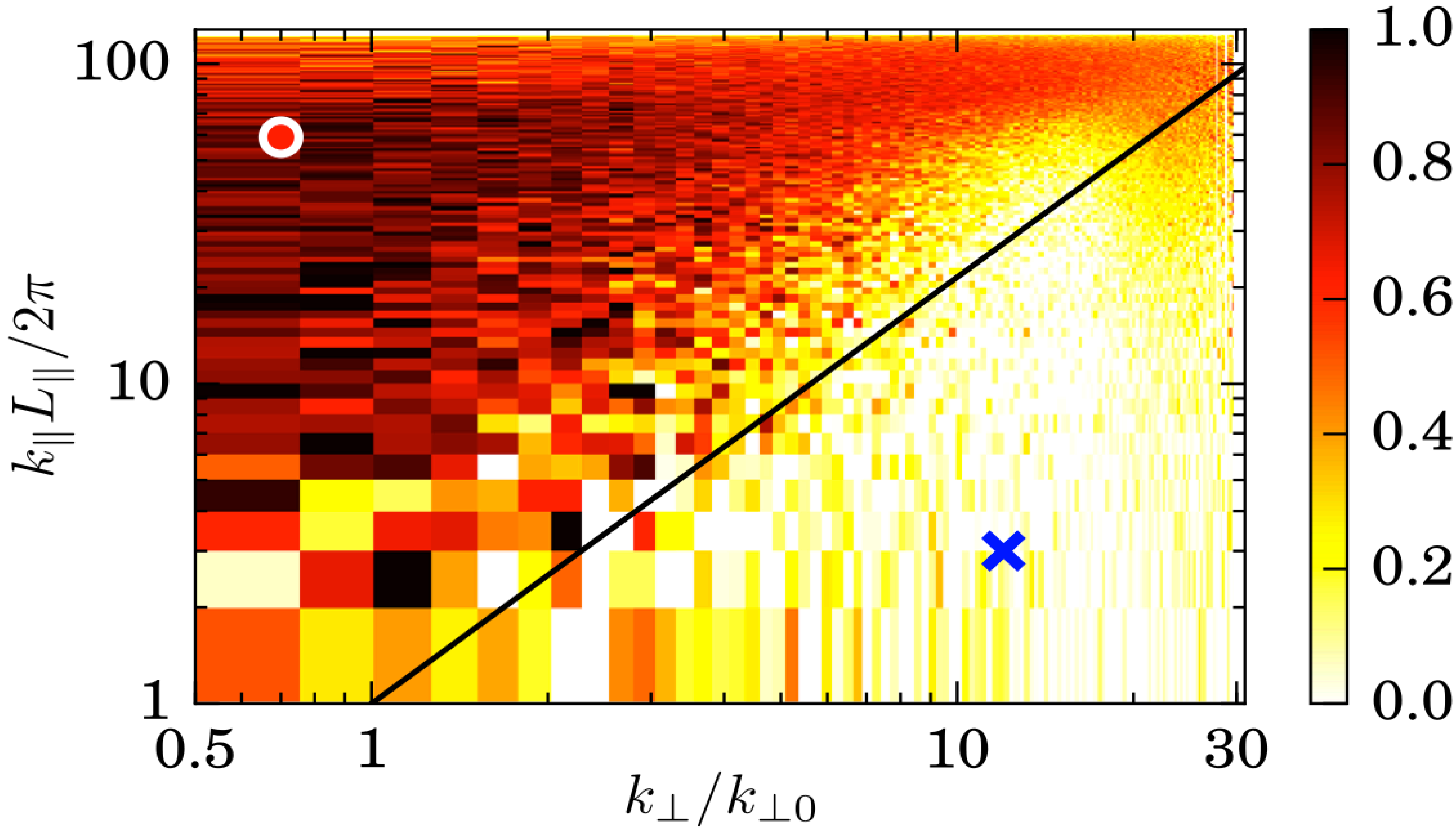
$$\partial_t f + v_\parallel \partial_\parallel (f + \Phi f_0) + \boldsymbol{u}_\perp \cdot \nabla_\perp f = C[f] + \text{ITG forcing}$$

$$\Phi = ZT_e/T_i \int_{-\infty}^{\infty} f, \quad \text{and} \quad \boldsymbol{u}_\perp = \tfrac{1}{2}\rho_i\, \hat{\boldsymbol{z}} \times \nabla_\perp \Phi$$



Steeper $m^{-5/2}$ spectrum due to diversion of free energy into $k_\perp$ instead of higher m.

Normalised flux Γ^N in drift (gyro)kinetics



Normalised flux Γ^N from $m = 30$ to $m = 31$ plotted against $k_{\parallel}$ & $k_{\perp}$

Line on which parallel streaming time equals perpendicular eddy turn-over time

Conclusions

Phase-space diagnostics based on a Fourier–Hermite spectral expansion of f.

Formation of fine scales in velocity space (Landau damping) appears as free energy propagating towards larger Hermite modes along straight characteristics.

The system also supports backwards propagating “anti-phase-mixing” modes.

Recurrence occurs when the fine-scale cut-off generates these modes by reflection. Can be avoided with a “hypercollision operator” or Hermite filter.

Nonlinearity produces almost perfect statistical reflection of phase-mixing modes into anti-phase-mixing modes, leaving no net flux to fine scales in velocity space, and no Landau damping.

Holds both for Vlasov–Poisson and drift gyrokinetics with different nonlinearities.

J. T. Parker & PJD (2015) *Fourier–Hermite spectral representation for the Vlasov–Poisson system in the weakly collisional limit*, J. Plasma Phys. **81** 305810203

J. T. Parker *et al.* (2016) *Suppression of phase mixing in drift-kinetic plasma turbulence* Phys. Plasmas **23** 070703

J. T. Parker (2016) DPhil thesis, University of Oxford, arXiv:1603.04727