Kinetic theory of stellar systems and plasmas

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The Hamiltonian system

We consider an isolated system of N stars with identical mass m in Newtonian gravitational interaction. Their dynamics is fully described by the Hamilton equations

$$\boxed{\begin{gathered} m\frac{d\mathbf{r}_i}{dt} = \frac{\partial H}{\partial \mathbf{v}_i}, \qquad m\frac{d\mathbf{v}_i}{dt} = -\frac{\partial H}{\partial \mathbf{r}_i}, \\ H = \frac{1}{2}\sum_{i=1}^{N} m v_i^2 - Gm^2 \sum_{i<j} \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|}. \end{gathered}} \tag{1}$$

The Liouville equation

The evolution of the N-body distribution function $P_N(\mathbf{r}_1, \mathbf{v}_1, ..., \mathbf{r}_N, \mathbf{v}_N, t)$ is governed by the Liouville equation

$$\boxed{\frac{\partial P_N}{\partial t} + \sum_{i=1}^{N} \left(\mathbf{v}_i \cdot \frac{\partial P_N}{\partial \mathbf{r}_i} + \mathbf{F}_i \cdot \frac{\partial P_N}{\partial \mathbf{v}_i} \right) = 0,} \tag{2}$$

where

$$\mathbf{F}_i = -\frac{\partial \Phi_d}{\partial \mathbf{r}_i} = -Gm \sum_{j \neq i} \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} = \sum_{j \neq i} \mathbf{F}(j \to i) \tag{3}$$

is the gravitational force by unit of mass experienced by the i-th star due to its interaction with the other stars.

Remark : another possible starting point is the Klimontovich equation.

The BBGKY hierarchy

From the Liouville equation (2) we can construct the complete BBGKY hierarchy for the reduced distribution functions

$$P_j(\mathbf{x}_1,...,\mathbf{x}_j,t)=\int P_N(\mathbf{x}_1,...,\mathbf{x}_N,t)\,d\mathbf{x}_{j+1}...d\mathbf{x}_N, \tag{4}$$

where the notation $\mathbf{x}$ stands for $(\mathbf{r},\mathbf{v})$. The generic term of this hierarchy reads

$$\frac{\partial P_j}{\partial t}+\sum_{i=1}^{j}\mathbf{v}_i\cdot\frac{\partial P_j}{\partial \mathbf{r}_i}+\sum_{i=1}^{j}\sum_{k=1,k\neq i}^{j}\mathbf{F}(k\to i)\cdot\frac{\partial P_j}{\partial \mathbf{v}_i}$$
$$+(N-j)\sum_{i=1}^{j}\int \mathbf{F}(j+1\to i)\cdot\frac{\partial P_{j+1}}{\partial \mathbf{v}_i}\,d\mathbf{x}_{j+1}=0. \tag{5}$$

Remark : the BBGKY hierarchy was applied to self-gravitating systems by Gilbert (1968). Hierarchy not closed. Expansion in powers of $\epsilon=1/N\ll 1$.

The truncation of the BBGKY hierarchy at the order $1/N$

If we introduce the notations $f = NmP_1$ (distribution function) and $g = N^2 m^2 P_2'$ (two-body correlation function), we get at the order $1/N$:

$$\frac{\partial f}{\partial t}(1) + \mathbf{v}_1 \cdot \frac{\partial f}{\partial \mathbf{r}_1}(1) + \frac{N-1}{N}\langle \mathbf{F}\rangle(1) \cdot \frac{\partial f}{\partial \mathbf{v}_1}(1) = -\frac{1}{m}\frac{\partial}{\partial \mathbf{v}_1} \cdot \int \mathbf{F}(2\to 1) g(1,2)\, d\mathbf{x}_2, \qquad (6)$$

$$\frac{1}{2}\frac{\partial g}{\partial t}(1,2) + \mathbf{v}_1 \cdot \frac{\partial g}{\partial \mathbf{r}_1}(1,2) + \mathbf{F}(2\to 1) \cdot \frac{\partial g}{\partial \mathbf{v}_1}(1,2) + \langle \mathbf{F}\rangle(1) \cdot \frac{\partial g}{\partial \mathbf{v}_1}(1,2) + \tilde{\mathbf{F}}(2\to 1) \cdot \frac{\partial f}{\partial \mathbf{v}_1}(1) f(2) + \frac{1}{m}\left[\int \mathbf{F}(3\to 1) g(2,3)\, d\mathbf{x}_3\right] \cdot \frac{\partial f}{\partial \mathbf{v}_1}(1) + (1 \leftrightarrow 2) = 0. \qquad (7)$$

Equations (6) and (7) are *exact at the order* $1/N$. They form the right basis to develop the kinetic theory of stellar systems at this order of approximation.

The limit $N \to +\infty$: the Vlasov equation (collisionless regime)

For $N \to +\infty$, we can neglect correlations between stars and we obtain the (mean field) *Vlasov equation* (1938)

$$\boxed{\begin{gathered}\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} + \langle \mathbf{F}\rangle \cdot \frac{\partial f}{\partial \mathbf{v}} = 0, \\ \langle \mathbf{F}\rangle = -\nabla\Phi, \qquad \Delta\Phi = 4\pi G\int f\, d\mathbf{v}.\end{gathered}} \tag{8}$$

The Vlasov equation (introduced by Jeans 1915) describes the *collisionless evolution* of stellar systems $\Rightarrow$ Collisionless Boltzmann equation

Remark : The Vlasov-Poisson equation can experience a process of **violent relaxation** towards a quasistationary state (QSS) (Lynden-Bell 1968)

The order $O(1/N)$: collisional regime

If we neglect *strong collisions* and *collective effects*, the first two equations of the BBGKY hierarchy can be written symbolically as

$$\boxed{\begin{aligned} \frac{\partial f}{\partial t} + \mathcal{V} f = \mathcal{C}[g], \\ \tfrac{\partial g}{\partial t} + \mathcal{L} g = \mathcal{S}[f]. \end{aligned}} \tag{9}$$

The first equation gives the evolution of the one-body distribution function. The l.h.s. corresponds to the (Vlasov) advection term. The r.h.s. takes into account *correlations* (finite N effects, graininess, discreteness effects) between stars that develop due to their interactions. These correlations correspond to encounters (“collisions”).

The generalized Landau equation

Substituting the two-body correlation function in the first equation of the BBGKY hierarchy, we obtain (Gilbert 1968, Severne & Haggerty 1976, Kandrup 1981, Chavanis 2013) :

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}}+\frac{N-1}{N}\langle \mathbf{F}\rangle\cdot\frac{\partial f}{\partial \mathbf{v}}=\frac{\partial}{\partial v^{\mu}}\int_0^{+\infty}d\tau\int d\mathbf{r}_1\,d\mathbf{v}_1\,F^{\mu}(1\rightarrow 0)$$
$$\times G(t,t-\tau)\left[\tilde{F}^{\nu}(1\rightarrow 0)\frac{\partial}{\partial v^{\nu}}+\tilde{F}^{\nu}(0\rightarrow 1)\frac{\partial}{\partial v_1^{\nu}}\right]f(\mathbf{r},\mathbf{v},t)\frac{f}{m}(\mathbf{r}_1,\mathbf{v}_1,t),$$

in which we must move the particles between t and $t-\tau$ according to the mean field trajectories.

- Generalized Fokker-Planck equation : **diffusion** and **friction**
- Temporal intergral of the force auto-correlation function weighted by f (diffusion) or by $\nabla_{\mathbf{v}} f$ (friction) $\Rightarrow$ generalized Kubo formula.

Remark : we have used Bogoliubov's synchronization ansatz (g varies much faster than f) valid at the order $1/N$.

The Vlasov-Landau equation

Within the *local approximation*, we can proceed as if the system were spatially homogeneous. The stars have rectilinear orbits ($\mathbf{v}(t-\tau)=\mathbf{v}(t)$ and $\mathbf{r}(t-\tau)=\mathbf{r}(t)-\mathbf{v}(t)\tau$) and the integrals over τ and $\mathbf{r}_1$ can be calculated explicitly. We then find that the evolution of the distribution function is governed by the *Vlasov-Landau equation*

$$\boxed{\begin{gathered}\frac{\partial f}{\partial t}+\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}}+\frac{N-1}{N}\langle \mathbf{F}\rangle\cdot\frac{\partial f}{\partial \mathbf{v}}=\pi(2\pi)^3 m \\ \times\frac{\partial}{\partial v^\mu}\int k^\mu k^\nu \delta(\mathbf{k}\cdot\mathbf{w})\hat{u}^2(k)\left(f_1\frac{\partial f}{\partial v^\nu}-f\frac{\partial f_1}{\partial v_1^\nu}\right)\,d\mathbf{v}_1\,d\mathbf{k},\end{gathered}} \tag{10}$$

where we have noted $\mathbf{w}=\mathbf{v}-\mathbf{v}_1$, $f=f(\mathbf{r},\mathbf{v},t)$, $f_1=f(\mathbf{r},\mathbf{v}_1,t)$, and where $(2\pi)^3\hat{u}(k)=-4\pi G/k^2$ represents the Fourier transform of the gravitational potential. Under this form, we see that the collisional evolution of a stellar system is due to a *condition of resonance*

$$\boxed{\mathbf{k}\cdot\mathbf{v}=\mathbf{k}\cdot\mathbf{v}_1} \tag{11}$$

encapsulated in the δ-function.

The Vlasov-Landau equation

The Vlasov-Landau equation may also be written as

$$\boxed{\begin{gathered}\frac{\partial f}{\partial t}+\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}}+\frac{N-1}{N}\langle\mathbf{F}\rangle\cdot\frac{\partial f}{\partial \mathbf{v}}=\frac{\partial}{\partial v^{\mu}}\int K^{\mu\nu}\left(f_1\frac{\partial f}{\partial v^{\nu}}-f\frac{\partial f_1}{\partial v_1^{\nu}}\right)d\mathbf{v}_1,\\ K^{\mu\nu}=2\pi m G^2\ln\Lambda\,\frac{w^2\delta^{\mu\nu}-w^{\mu}w^{\nu}}{w^3},\end{gathered}} \tag{12}$$

where

$$\boxed{\ln\Lambda=\int\frac{dk}{k}=+\infty} \tag{13}$$

is the **Coulombian logarithm**. It has to be regularized with appropriate cut-offs.

- Plasmas : collective effects (Debye scale $k_{\rm D}$).
- Stellar systems : spatial inhomogeneity (Jeans scale $k_{\rm J}$)

Remark : The r.h.s. of Eq. (12) is the original form of the collision operator given by Landau (1936) for the Coulombian interaction.

The Vlasov-Lenard-Balescu equation

The Vlasov-Lenard-Balescu equation (1960) writes

$$\boxed{\begin{aligned}\frac{\partial f}{\partial t}+\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}}+\frac{N-1}{N}\langle\mathbf{F}\rangle\cdot\frac{\partial f}{\partial \mathbf{v}}=\pi(2\pi)^3 m\\ \times\frac{\partial}{\partial v^{\mu}}\int k^{\mu}k^{\nu}\delta(\mathbf{k}\cdot\mathbf{w})\frac{\hat{u}^2(k)}{|\epsilon(\mathbf{k},\mathbf{k}\cdot\mathbf{v})|^2}\left(f_1\frac{\partial f}{\partial v^{\nu}}-f\frac{\partial f_1}{\partial v_1^{\nu}}\right)\,d\mathbf{v}_1\,d\mathbf{k},\end{aligned}} \quad (14)$$

where

$$\boxed{\epsilon(\mathbf{k},\omega)=1+(2\pi)^3\hat{u}(k)\int\frac{\mathbf{k}\cdot\frac{\partial f}{\partial \mathbf{v}}}{\omega-\mathbf{k}\cdot\mathbf{v}}\,d\mathbf{v}} \quad (15)$$

is the *dielectric function* (appearing in connexion with the linearized Vlasov equation). The Landau equation is recovered for $|\epsilon(\mathbf{k},\mathbf{k}\cdot\mathbf{v})|^2=1$

Dressed potential of interaction :

$$\boxed{\hat{u}_{\rm bare}(k)=\hat{u}(k)\Rightarrow\hat{u}_{\rm dressed}(k)=\frac{\hat{u}(k)}{|\epsilon(\mathbf{k},\mathbf{k}\cdot\mathbf{v})|}} \quad (16)$$

Debye-Hückel approximation

• Debye-Hückel approximation for **plasmas** : static screening $|\epsilon(\mathbf{k},0)|$ + Maxwell distribution :

$$\boxed{\begin{array}{c}(2\pi)^3\hat{u}_{\rm DH}(k)=\dfrac{4\pi e^2}{m^2}\dfrac{1}{k^2+k_{\rm D}^2}\\ \ln\Lambda=\int_0 \frac{k^3}{(k^2+k_{\rm D}^2)^2}\,dk\end{array}} \quad (17)$$

No large scale divergence anymore ! Collective effects (Debye shielding) solve the large scale divergence.

• Debye-Hückel approximation for **stellar systems** :

$$\boxed{\begin{array}{c}(2\pi)^3\hat{u}_{\rm DH}(k)=-4\pi G\dfrac{1}{k^2-k_{\rm J}^2}\\ \ln\Lambda=\int_{k_J} \frac{k^3}{(k^2-k_{\rm J}^2)^2}\,dk\end{array}} \quad (18)$$

Even worse divergence at large scales ! (Jeans instability). If we want to take collective effects into account, we must also account for spatial inhomogeneity. Collective effects (anti-shielding) can *boost* the relaxation (see Fouvry *et al.* 2015).

The inhomogeneous Landau equation

The generalized Landau equation can be simplified by making an *adiabatic approximation* and using *angle-action variables* : $f \simeq f(\mathbf{J}, t)$
If we neglect collective effects we get the inhomogeneous Landau equation (Polyachenko & Shukhman 1982, Luciani & Pellat 1987, Chavanis 2007, 2010) :

$$\boxed{\begin{aligned}\frac{\partial f}{\partial t} = \pi(2\pi)^d m \frac{\partial}{\partial \mathbf{J}} \cdot \sum_{\mathbf{k},\mathbf{k}'} \int d\mathbf{J}'\, \mathbf{k}\, |A_{\mathbf{k},\mathbf{k}'}(\mathbf{J},\mathbf{J}')|^2 \delta(\mathbf{k}\cdot\Omega - \mathbf{k}'\cdot\Omega') \\ \times \left(\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}} - \mathbf{k}'\cdot\frac{\partial}{\partial \mathbf{J}'}\right) f(\mathbf{J},t) f(\mathbf{J}',t),\end{aligned}} \quad (19)$$

where $\Omega = \Omega(\mathbf{J}, t)$ and $\Omega' = \Omega(\mathbf{J}', t)$ are the pulsations of the orbits and $A_{\mathbf{k},\mathbf{k}'}(\mathbf{J}, \mathbf{J}')$ is the Fourier transform of the potential of interaction written in angle-action variables. The *condition of resonance* (distant encounters) is

$$\boxed{\mathbf{k}\cdot\Omega = \mathbf{k}'\cdot\Omega'} \quad (20)$$

see Lynden-Bell & Kalnajs (1972), Tremaine & Weinberg (1984), Rauch & Tremaine (1996)...

The inhomogeneous Lenard-Balescu equation

If we take collective effects into account we get the inhomogeneous Lenard-Balescu equation (Heyvaerts 2010, Chavanis 2012) :

$$\frac{\partial f}{\partial t}=\pi(2\pi)^d m\frac{\partial}{\partial \mathbf{J}}\cdot\sum_{\mathbf{k},\mathbf{k}'}\int d\mathbf{J}'\,\mathbf{k}\,\frac{1}{|D_{\mathbf{k},\mathbf{k}'}(\mathbf{J},\mathbf{J}',\mathbf{k}\cdot\Omega)|^2}\delta(\mathbf{k}\cdot\Omega-\mathbf{k}'\cdot\Omega')$$
$$\times\left(\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}}-\mathbf{k}'\cdot\frac{\partial}{\partial \mathbf{J}'}\right)f(\mathbf{J},t)f(\mathbf{J}',t),$$

where $1/D_{\mathbf{k},\mathbf{k}'}(\mathbf{J},\mathbf{J}',\omega)=\psi^{(\alpha)}_{\mathbf{k}}(\mathbf{J})\epsilon^{-1}_{\alpha\beta}(\omega)\psi^{(\beta)*}_{\mathbf{k}'}(\mathbf{J}')$ with the “dielectric” function

$$\epsilon_{\alpha\beta}(\omega)=\delta_{\alpha\beta}-(2\pi)^d\sum_{\mathbf{k}}\int d\mathbf{J}\,\frac{\mathbf{k}\cdot\frac{\partial f}{\partial \mathbf{J}}}{\omega-\mathbf{k}\cdot\Omega}\psi^{(\alpha)*}_{\mathbf{k}}(\mathbf{J})\psi^{(\beta)}_{\mathbf{k}}(\mathbf{J}) \qquad (21)$$

expressible in terms of the elements of a *biorthogonal basis* (Kalnajs 1976, Weinberg 1986, 1989). The inhomogeneous Lenard-Balescu equation is the most refined kinetic equation of stellar systems : exact at the order $1/N$, spatial inhomogeneity, collective effects.

The test particle approach : Fokker-Planck equation

- Test particle : $f(\mathbf{J},t) \Rightarrow P(\mathbf{J},t)$
- Field particles : $f(\mathbf{J}',t) \Rightarrow f(\mathbf{J}')$

This procedure transforms an integrodifferential equation (LB) into a differential (FP) equation (Chavanis 2012, Heyvaerts *et al.* 2017) :

$$\boxed{\begin{aligned}\frac{\partial P}{\partial t} = \pi(2\pi)^d m \frac{\partial}{\partial \mathbf{J}} \cdot \sum_{\mathbf{k},\mathbf{k}'} \int d\mathbf{J}'\, \mathbf{k}\, \frac{1}{|D_{\mathbf{k},\mathbf{k}'}(\mathbf{J},\mathbf{J}',\mathbf{k}\cdot\Omega)|^2} \delta(\mathbf{k}\cdot\Omega - \mathbf{k}'\cdot\Omega') \\ \times \left(\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}} - \mathbf{k}'\cdot\frac{\partial}{\partial \mathbf{J}'}\right) P(\mathbf{J},t) f(\mathbf{J}').\end{aligned}} \tag{22}$$

Friction by polarization

The Lenard-Balescu equation may be rewritten in the form of a "modified" Fokker-Planck equation

$$\boxed{\frac{\partial P}{\partial t} = \frac{\partial}{\partial J_i}\left(D_{ij}\frac{\partial P}{\partial J_j} - PF_i^{pol}\right),} \tag{23}$$

where the diffusion coefficient is "sandwiched" between the ∂_i and ∂_j derivatives.

- Diffusion tensor :

$$D_{ij} = \pi(2\pi)^d m \sum_{\mathbf{k},\mathbf{k}'}\int d\mathbf{J}'\, k_i k_j \frac{1}{|D_{\mathbf{k},\mathbf{k}'}(\mathbf{J},\mathbf{J}',\mathbf{k}'\cdot\Omega')|^2}\delta(\mathbf{k}\cdot\Omega - \mathbf{k}'\cdot\Omega')f(\mathbf{J}'),$$

- Friction by polarization :

$$\mathbf{F}_{\rm pol} = \pi(2\pi)^d m \sum_{\mathbf{k},\mathbf{k}'}\int d\mathbf{J}'\,\mathbf{k}\frac{1}{|D_{\mathbf{k},\mathbf{k}'}(\mathbf{J},\mathbf{J}',\mathbf{k}\cdot\Omega)|^2}\delta(\mathbf{k}\cdot\Omega - \mathbf{k}'\cdot\Omega')\left(\mathbf{k}'\cdot\frac{\partial f'}{\partial \mathbf{J}'}\right).$$

Linear response theory.

True friction

General form of the Fokker-Planck equation :

$$\frac{\partial P}{\partial t} = \frac{\partial^2}{\partial J_i \partial J_j}\left(D_{ij}P\right) - \frac{\partial}{\partial J_i}\left(PF_i^{\rm fric}\right), \qquad D_{ij} = \frac{1}{2}\left\langle \frac{\Delta J_i \Delta J_j}{\Delta t}\right\rangle, \qquad \mathbf{F}_{\rm fric} = \left\langle \frac{\Delta \mathbf{J}}{\Delta t}\right\rangle. \tag{24}$$

The total friction is

$$F_i^{\rm fric} = F_i^{\rm pol} + \frac{\partial D_{ij}}{\partial J_j}, \tag{25}$$

- True friction :

$$\mathbf{F}_{\rm fric} = \pi(2\pi)^d m \sum_{\mathbf{k},\mathbf{k}'} \int d\mathbf{J}'\, f(\mathbf{J}') \left(\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}} - \mathbf{k}'\cdot\frac{\partial}{\partial \mathbf{J}'}\right) \times \frac{1}{|D_{\mathbf{k},\mathbf{k}'}(\mathbf{J},\mathbf{J}',\mathbf{k}\cdot\Omega)|^2}\delta(\mathbf{k}\cdot\Omega - \mathbf{k}'\cdot\Omega'). \tag{26}$$

For homogeneous systems, we recover the results of Hubbard (1961)

The thermal bath approximation

- Thermal bath (Boltzmann distribution) :

$$f(\mathbf{J}) = Ae^{-\beta m H(\mathbf{J})}, \qquad \partial H/\partial \mathbf{J} = \Omega(\mathbf{J}). \tag{27}$$

- Einstein relation :

$$F_i^{\rm pol} = -\beta m D_{ij}(\mathbf{J})\Omega_j(\mathbf{J}). \tag{28}$$

The Einstein relation is satisfied by $\mathbf{F}_{\rm pol}$, not by $\mathbf{F}_{\rm fric}$ (!)

- Fokker-Planck equation :

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial J_i}\left[D_{ij}(\mathbf{J})\left(\frac{\partial P}{\partial J_j} + \beta m P \Omega_j(\mathbf{J})\right)\right]. \tag{29}$$

D_{ij} is anisotropic and depends on $\mathbf{J}$.

Remark : For homogeneous systems with collective effects neglected, we recover the seminal results of Chandrasekhar (1943) and Rosenbluth *et al.* (1957) with $\mathbf{F}_{\rm fric} = 2\mathbf{F}_{\rm pol} = \frac{m+m_{\rm f}}{m}\mathbf{F}_{\rm pol}$.

References

- General formalism :

J. Heyvaerts, J.B. Fouvry, P.H. Chavanis, C. Pichon *Dressed diffusion and friction coefficients in inhomogeneous multicomponent self-gravitating systems,* Month. Not. Royal Astron. Soc. **469**, 4193 (2017)
and references therein.

- Astrophysical applications :

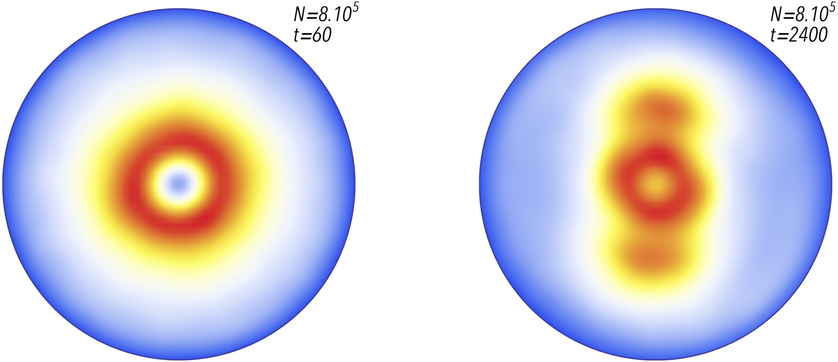
See the papers of Fouvry *et al.* (2015-2017) and Sridhar & Touma (2016-2017).

Applications

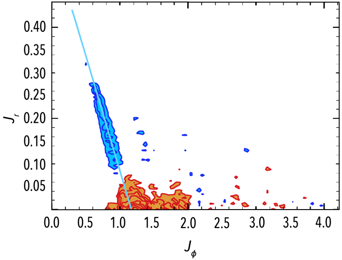
Solve the gravitational Lenard-Balescu equation with angle-action variables in simplified geometries : stellar disks, globular clusters...

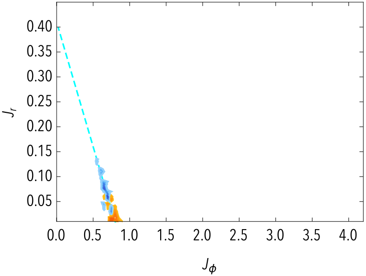
Joined work with Jean-Baptiste Fouvry and Christophe Pichon.

Out of equilibrium phase transition induced by finite N effects



Secular formation of a ridge in action space





Left : result of direct N-body simulations (Sellwood 2012).
Right : prediction of the Lenard-Balescu equation (Fouvry, Pichon, Magorrian, and Chavanis 2015).

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P.H. Chavanis, *Kinetic theory of spatially inhomogeneous stellar systems without collective effects,* Astron. Astrophys. **556**, A93 (2013)

J.B. Fouvry, C. Pichon, J. Magorrian, P.H. Chavanis, *Secular diffusion in discrete self-gravitating tepid discs,* Astron. Astrophys. **584**, A129 (2015)