Angular moment models for plasma physics

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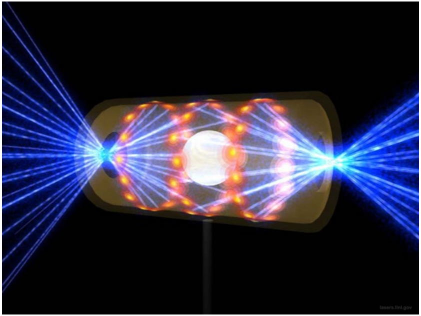
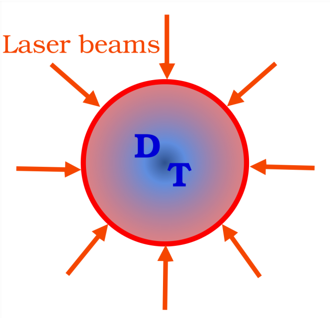
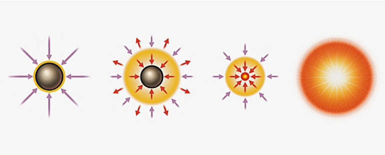


Physical context

Context : Inertial Confinement Fusion-Direct attack





Aim : Description of electrons motion, fixed ions

Models

Kinetic description : electrons dynamics

Electron distribution function : $f(t, \mathbf{x}, \mathbf{v})$, fixed ions

$\hookrightarrow$ Resolution of Fokker-Planck-Landau equation

$$\underbrace{\frac{\partial f}{\partial t} + \mathbf{v}.\nabla_{\mathbf{x}} f}_{advection\ \ term} + \underbrace{\frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}).\nabla_{\mathbf{v}} f}_{force\ \ \ term} = \underbrace{C_{e,e}(f,f) + C_{e,i}(f),}_{collisional\ \ \ terms}$$

C_{ee} : electron/electron collision operator
C_{ei} : electron/ion collision operator
Precise but time consuming

One solution : Fluid models : (Euler-Lorentz, ...).
Pb : Not precize for out of equilibrium regimes.

Good compromise : Angular moments models

Moments model

$\hookrightarrow$ Angular moments extraction : $v = \zeta\, \Omega$ with $\zeta = |v|$.

$$f_0(\zeta) = \zeta^2 \int_{S_2} f(v) d\Omega, \quad f_1(\zeta) = \zeta^2 \int_{S_2} f(v)\Omega d\Omega, \quad f_2(\zeta) = \zeta^2 \int_{S_2} f(v)\Omega \otimes \Omega d\Omega.$$

Moments methods

$$f(x, \epsilon, \Omega) \qquad \leftrightarrow \qquad f_i(x, \epsilon) \quad \text{for} \quad 0 \leq i \leq N$$

Pb of the closure

Moments extraction on the kinetic equation $\Rightarrow$ moments system

$$\partial_t \overline{f} + \zeta \partial_x \widetilde{f} = \overline{Q}$$

$$\overline{f} = \begin{pmatrix} f^0 \\ \vdots \\ f^N \end{pmatrix}, \widetilde{f} = \begin{pmatrix} f^1 \\ \vdots \\ f^{N+1} \end{pmatrix}, \quad \overline{Q} = \begin{pmatrix} Q^0 \\ \vdots \\ Q^N \end{pmatrix}.$$

Pb : Non closed system : $N+1$ equations for $N+2$ unknowns

$\Rightarrow$ Define a closure : $f \approx f_R$

Construction principle of the closure

Moments

$$\boldsymbol{f} = (f_0, \ldots, f_{Card(\mathbf{m})})$$

Construction principle of the closure

Moments $\rightarrow$ Ansatz

$$\boldsymbol{f} = (f_0, \ldots, f_{Card(\mathbf{m})}) \rightarrow f_R(\Omega)$$

Ansatz f_R in

$$\mathcal{C}(\boldsymbol{f}) = \left\{ g \quad , \quad \int_{S^2} \mathbf{m}(\Omega) g(\Omega) d\Omega = \boldsymbol{f} \right\} \quad ,$$

Construction principle of the closure

Moments → Ansatz → Closure

$$\boldsymbol{f} = (f_0, \dots, f_{Card(\mathbf{m})}) \rightarrow \quad f_R(\Omega) \rightarrow \quad F \approx \int_{S^2} \Omega \otimes \mathbf{m}(\Omega) f_R(\Omega) d\Omega$$

Ansatz f_R in

$$\mathcal{C}(\boldsymbol{f}) \quad = \quad \left\{ g \quad , \quad \int_{S^2} \mathbf{m}(\Omega) g(\Omega) d\Omega = \boldsymbol{f} \right\} \quad ,$$

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Ansatz f_R in

$$\mathcal{C}(\boldsymbol{f}) = \left\{ g \geq 0, \quad \int_{S^2} \mathbf{m}(\Omega) g(\Omega) d\Omega = \boldsymbol{f} \right\} \neq \emptyset,$$

Construction principle of the closure

Moments → Ansatz → Closure

$$\boldsymbol{f} = (f_0, \dots, f_{Card(\mathbf{m})}) \rightarrow \quad f_R(\Omega) \rightarrow \quad F \approx \int_{S^2} \Omega \otimes \mathbf{m}(\Omega) f_R(\Omega) d\Omega$$

Ansatz f_R in

$$\mathcal{C}(\boldsymbol{f}) \quad = \quad \left\{ g \geq 0, \quad \int_{S^2} \mathbf{m}(\Omega) g(\Omega) d\Omega = \boldsymbol{f} \right\} \neq \emptyset,$$

Realizability domain

$$\mathcal{R}_{\mathbf{m}} \quad = \quad \Big\{ \boldsymbol{f} \in \mathbb{R}^{Card(\mathbf{m})}, \quad \text{s.t.}$$
$$\exists g \in L^1(S^2), \quad g \geq 0, \quad \boldsymbol{f} = \int_{S^2} \mathbf{m}(\Omega)\, g(\Omega) d\Omega \Big\}$$

M_1 closure

Moments	$\rightarrow$	Ansatz	$\rightarrow$	Closure
(f_0, f_1)	$\rightarrow$	$f_{M_1}(\Omega)$	$\rightarrow$	$f_2 \approx \int_{S^2} \Omega \otimes \Omega f_{M_1}(\Omega) d\Omega$

Ansatz ψ_{M_1} in

$$\mathcal{C}_1 = \left\{ g \geq 0, \int_{S^2} g \, d\Omega = f_0, \int_{S^2} \Omega \, g \, d\Omega = f_1 \right\} \neq \emptyset \text{ if } \boldsymbol{f} \in \mathcal{R}_{\mathbf{m}},$$

Choice of the ansatz [1, 2]

$$f_{M_1} = \underset{g \in \mathcal{C}_1}{\operatorname{argmin}} (\mathcal{H}(g)) \quad \Rightarrow \quad f_{M_1} = \exp\left(S + V.\Omega\right)$$

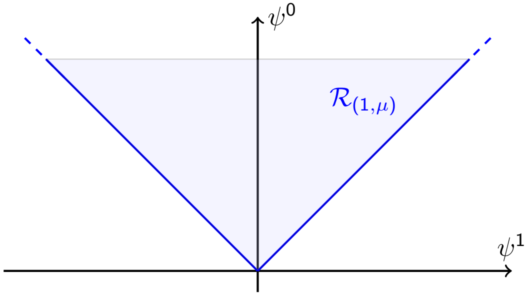
1. Minerbo, *J. Quant. Spect. Rad. Transfer*, 1977
2. Levermore, *J. Stat. Phys.*, 1995

Realizability domain[2,3] (First order moments)

Proposition

$\mathbf{m}(\Omega) = (1, \Omega)$, *then* $\mathcal{R}_{\mathbf{m}} = \left\{ \boldsymbol{\psi} \in \mathbb{R}^4, \quad s.t. \quad |\psi^1| < \psi^0 \right\} \cup \{0_{\mathbb{R}^4}\}$

If $\quad |\psi^1| = \psi^0, \quad$ *then* $\quad \gamma = \psi^0 \delta \left(\Omega - \frac{\psi^1}{\psi^0} \right)$



2. Kershaw, 1976
3. Akhiezer, 1962

M_1 closure

Advantages :

- Symmetric hyperbolic[5]
- Realizable
- Entropy decay[5]

Numerical cost :

- minimization problem
- numerical quadrature

Alternative computation :

$$\psi^2 = \psi^0 \left(\frac{1-\chi}{2} Id + \frac{3\chi - 1}{2} \frac{\psi^1 \otimes \psi^1}{|\psi^1|^2} \right),$$

where χ depends only on $\frac{|\psi^1|}{\psi^0}$

5. Levermore, *J. Stat. Phys.*, 1995

Notations

Spherical coordinates : $\mu = \cos(\theta)$, $\theta \in [0, \pi]$

$$\Omega = \begin{pmatrix} \mu \\ \sqrt{1-\mu^2}\cos\varphi \\ \sqrt{1-\mu^2}\sin\varphi \end{pmatrix}, \quad v = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \zeta\Omega = \begin{pmatrix} \zeta\mu \\ \zeta\sqrt{1-\mu^2}\cos\varphi \\ \zeta\sqrt{1-\mu^2}\sin\varphi \end{pmatrix}.$$

Simplification : one angular dimension : $\mu \in [-1, 1]$.

N first moments

$$f^i = 2\pi\zeta^2 \int_{-1}^{1} f(\zeta,\mu)\mu^i d\mu = \zeta^2 \langle f\mu^i \rangle,\ i \in \{0, N\},$$

$$F^i = \frac{f^i}{\zeta^2},\ \langle \Psi \rangle = 2\pi \int_{-1}^{1} \Psi(\mu)\, d\mu$$

F^0 : isotropic part of f

Approximation of the collision operator

↪ C_{ee} non-linear : complicated angular moment extraction

↪ Approximated collisional operators : [5]

$$C_{ee}(f,f) \approx Q^0_{ee} = C_{ee}(F_0, F_0)$$

Problem : No conservation of the realisibility domain [6].

$$\text{Moment extraction : } \partial_t f + v\partial_x f = Q^0_{ee} \Rightarrow \begin{cases} \partial_t f^0 = Q^0_{ee} \\ \partial_t f^1 = 0 \end{cases}$$

Initial conditions

$$f^0(t=0) = \frac{1}{3}\chi_{[0,3]}(\zeta), \quad f^1(t=0) = \frac{1}{4}\chi_{[0,3]}(\zeta)$$

Asymptotic state for f^0 and f^1

$$f^0 = \sqrt{\frac{2}{\pi}} \exp\left(\frac{-\zeta^2}{2}\right)\zeta^2 , \quad f^1(t=0) = \frac{1}{4}\chi_{[0,3]}(\zeta)$$

5. Berezin, Khudick, Pekker, 1987, Buet, Cordier, 1998
6. J. Mallet, S. Brull and B. Dubroca. 2013.

Asymptotic state for Q^0_{ee}

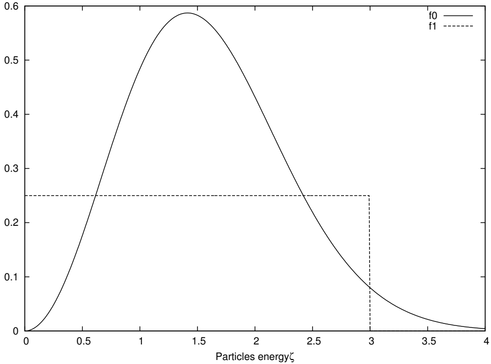


FIGURE: f^0 *et* f^1 *in funtion of* ζ *for the stationary state,* $\zeta_{\max} = 4$*.*

Conclusion : Non conservation of the realisability domain by Q^0_{ee}.

Approximation of the collision operator

Equilibrium state of C_{ei} : set of isotropic functions $f = f(|\mathbf{v}|)$.

Equilibrium state of C_{ee} : Maxwellian distribution

	Isotropisation		Maxwellisation	
$f(\mathbf{v})$	$\rightarrow$	$f(\|\mathbf{v}\|)$	$\rightarrow$	$f(\mathbf{v}) = e^{-\frac{m\mathbf{v}^2}{2k_B T}}$
	ν_{ei}		ν_{ee}	

$\Rightarrow$ Linearisation of C_{ee} around the isotropic state.

$$Q_{e,e}(f) = \frac{1}{\zeta^2}\partial_\zeta\Big(\zeta\int_0^{+\infty}\tilde{J}(\zeta,\zeta')\Big[F^0(\zeta')\frac{1}{\zeta}\partial_\zeta f(\zeta) - f(\zeta)\frac{1}{\zeta'}\partial_{\zeta'}F^0(\zeta')\Big]\zeta'^2 d\zeta'\Big),$$

$$\tilde{J}(\zeta,\zeta') = \frac{2}{3}inf(\frac{1}{\zeta^3},\frac{1}{\zeta'^3})\zeta'^2\zeta^2.$$

Properties of the operator $Q(f) = Q_{ee}(f) + Q_{ei}(f)$

- Preserves mass and energy
- Dissipates entropy

The M_1-Ampère model

Simplified electrostatic case ($B = 0$) with one space dimension ($x \in \mathbb{R}$)

Scaling used for collisional processes

$$\tilde{t} = \nu_{e,i} t, \qquad \tilde{x} = x/\lambda_{e,i}, \qquad \tilde{v} = v/v_{th}.$$

Dimensionless parameter

$$\alpha = \frac{\lambda_{De}}{\lambda_{ei}} \qquad \alpha \to 0 \quad \Rightarrow \text{ quasi-neutral regime}$$

$$(S_\alpha) \begin{cases} \partial_t f_0 + \nabla_x.(\zeta f_1) + \partial_\zeta \left(\dfrac{qE}{m} f_1 \right) = Q_0(f_0), \\ \partial_t f_1 + \nabla_x.(\zeta f_2) + \partial_\zeta \left(\dfrac{qE}{m} f_2 \right) - \dfrac{qE}{m\zeta}(f_0 - f_2) = Q_1(f_1) + Q_0(f_1), \\ \dfrac{\partial E}{\partial t} = -\dfrac{j}{\alpha^2}, \end{cases}$$

with Maxwell-Poisson satisfied at the initial time.

$S_\alpha \to S_0$: singular limit $\Rightarrow$ AP scheme.

Present case : $\alpha = 4.10^{-4}$

Batishchev[7] test case

Relaxation of a localised temperature profile.

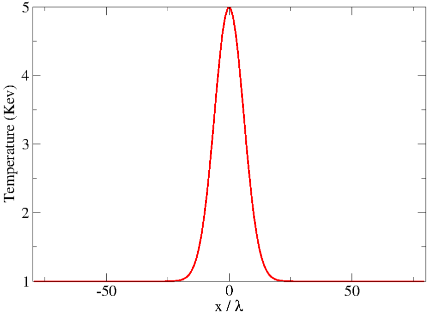
$\hookrightarrow$ Study of the non-local heat transport.

Initialy :

$$T(0,x) = T_0 + T_1 exp(-\frac{x^2}{(\delta L)^2})$$

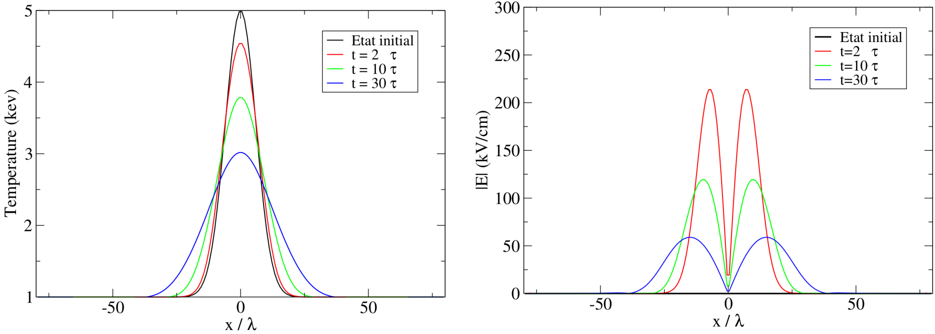
T_0= 1Kev, T_1 = 4Kev, $\frac{\lambda_{ei}}{\delta L} = 0.01$

$$\begin{cases} f_0 = \sqrt{\frac{2}{\pi}}\Big(\frac{1}{T(x)}\Big)^{\frac{3}{2}} exp\Big(-\frac{\zeta^2}{2T(x)}\Big) \\ f_1 = 0 \end{cases}$$



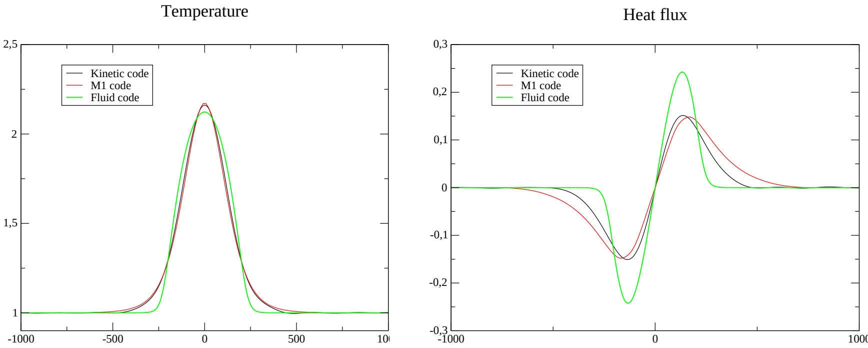
7. O.V Batishchev & al Physics of Plasmas (2002).

Results



Temperature (left)-Electric field (right)

Comparison



Kinetic : [A.V. Bobylev, I.F. Potapenko, JCP, 2013] , 2 days, 80 process
M1-AP : 3 minutes, 1 process.

Compution of transport plasma coefficient

- Plasma transport coefficients :
  Electric current density : j, Electron heat flux : q

$$j = \sigma E^* + \alpha \nabla_x T_e, \qquad q = -\alpha T_e E^* - \chi \nabla_x T_e,$$

with

$$E^* = E + (1/(en_e))\nabla_x (n_e T_e)$$

σ : electrical conductivity

α : thermoelectric coefficient

χ : thermal conductivity

Heat conductivity : κ

$$\kappa = \chi - \alpha^2 \frac{T_e}{\sigma}$$

Transport plasma coefficients

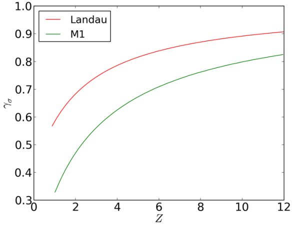


FIGURE: σ in function of Z

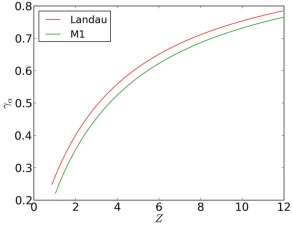


FIGURE: α in function of Z

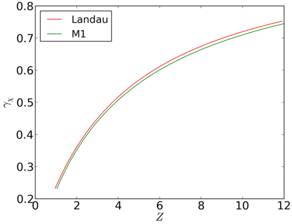


FIGURE: χ in function of Z

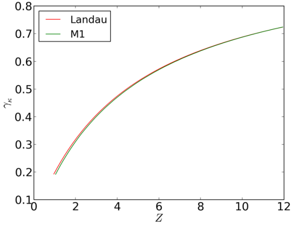


FIGURE: κ in function of Z

2D test case : double electron beam

Phd thesis of Teddy Pichard : context of radiotherapy

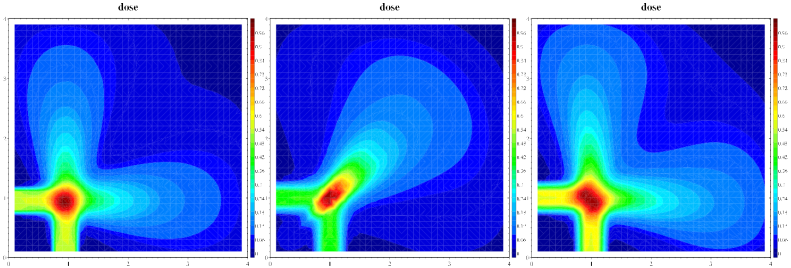


FIGURE: Dose deposition (Monte-Carlo, left), M_1 (middle), M_2 (right) solvers

Monte-Carlo : 14 h ; M_1 : 5 min ; M_2 : 14 min

Conclusion

<u>Conclusions</u>

- Introduction of angular moment models obtained from an entropy minimisation principle.
- Derivation of a new approached electron/electron collision operator satisfying : H theorem, preservation of the realisability domain.
- Coulpling with Maxwell, implementation and tests close to the quasi-neutral regime
- Precise transport plasma coefficients

<u>Related results</u>

- Derivation of an entropic scheme
- Long time behaviour $\Rightarrow$ Asymptotic preserving scheme
- Limit of the M_1 and M_2 model for collisionless plasma applications.
- Motion of ions (hydrodynamic description) $\Rightarrow$ moving frame

Thank you