About the convergence of the kinetic equation for gravitational and Coulomb systems.

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Collisionless Boltzmann (Vlasov) Equation and Modeling of Self-Gravitating Systems and Plasmas. CIRM 30/10 - 3/11/2017.

Collaboration

This is a joint work with

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Motivation

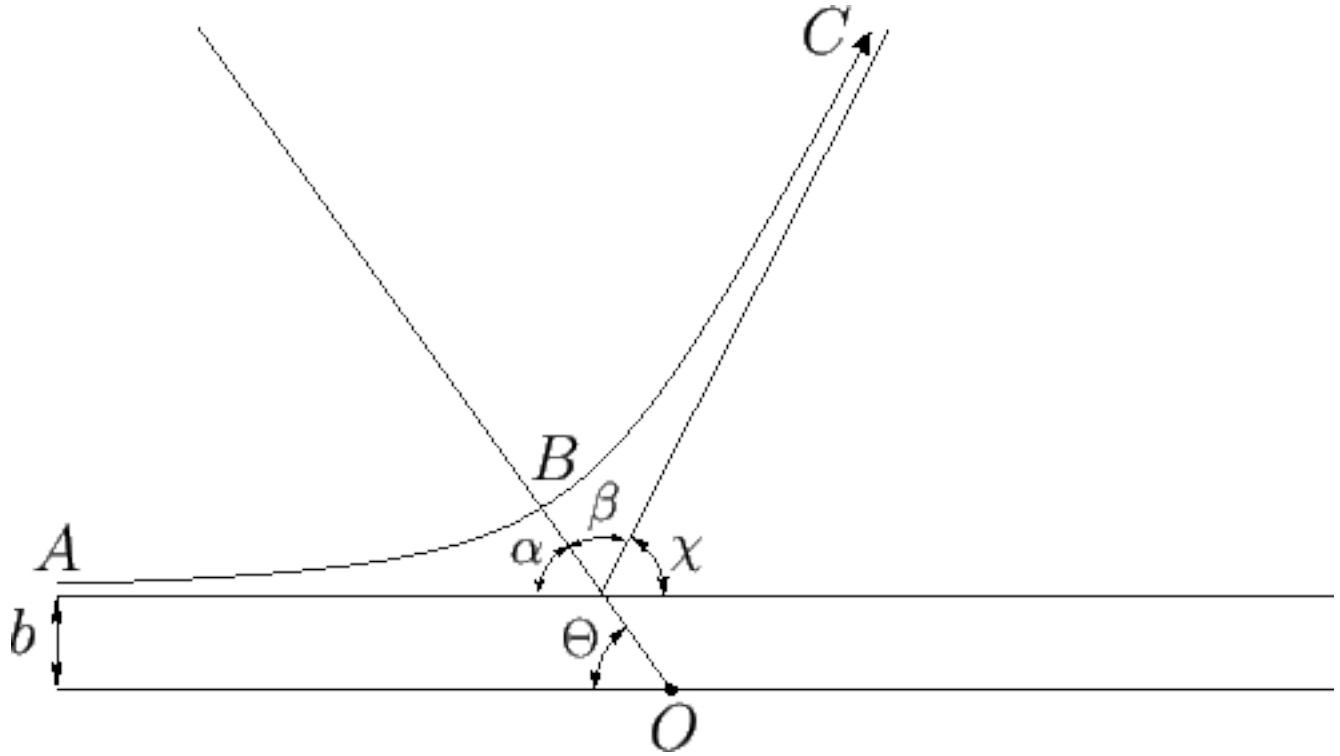
Description of collisional relaxation to equilibrium needs inclusion of collision term into Vlasov equation

$$\frac{\partial}{\partial t} f(\vec{r},\vec{v},t) + \vec{v}.\frac{\partial}{\partial \vec{r}} f(\vec{r},\vec{v},t) = -\frac{1}{m}\int d^3r' \int d^3v' F(\vec{r}-\vec{r}').\frac{\partial}{\partial \vec{v}} f(\vec{r},\vec{v},t) f(\vec{r}',\vec{v}',t) + C[f]$$

However: Collision term $C[f]$ **diverges** for

$$\vec{F}(\vec{r}-\vec{r}') = \gamma \frac{\vec{r}-\vec{r}'}{\left|\vec{r}-\vec{r}'\right|^3}$$

Two-body collision geometry



Center of mass frame

***b*: impact parameter**

***χ*: deflection angle**

Collision terms and their divergences for systems with 1/r interaction potentials.

1. **Boltzmann collision term** (**dilute** systems):

$$C_B[f]=\int d^3v_2\int d\Omega(\chi,\varphi)\,\frac{d\sigma}{d\Omega}\left|\vec{v}_1-\vec{v}_2\right|\left[f(\vec{r}_1,\vec{v}_1{}')f(\vec{r}_1,\vec{v}_2{}')-f(\vec{r}_1,\vec{v}_1)f(\vec{r}_1,\vec{v}_2)\right]$$

with

$$\frac{d\sigma}{d\Omega}=\left(\frac{\gamma}{m\left|\vec{v}_1-\vec{v}_2\right|^2}\right)^2\frac{1}{\sin^4(\chi/2)}.$$

$C_B[f]$ diverges for deflection angle $\chi \rightarrow 0$ ($b \rightarrow \infty$): **weak** interactions
but converges for $\chi \rightarrow \pi$ ($b \rightarrow 0$): **strong** interactions

Origin of the divergence: ***The effective screening of the potential due to the long-range interactions of many other particles with the two colliding particles is not taken into account!***

Collision terms and their divergences for systems with 1/r interaction potentials.

2. Balescu-Lenard collision term (only **weak** interactions):

$$C_{BL} = 8\pi^4 \int d^3v_2 \int d^3k\, \vec{k}.\frac{\partial}{\partial v_1}\delta[\vec{k}.(\vec{v}_1 - \vec{v}_2)]\,\frac{\tilde{V}(k)^2}{\left|\varepsilon(\vec{k},\vec{v}_1)\right|^2}\,\vec{k}.\frac{\partial}{\partial \vec{v}_1} f(\vec{r}_1,\vec{v}_1) f(\vec{r}_1,\vec{v}_2)$$

$$\tilde{V}(k) = Fourier[\gamma / r] = \frac{\gamma}{2\pi^2 k^2}$$

$$\varepsilon(\vec{k},\vec{v}_1) = dielectric\ function = 1 + 8\pi^4 \tilde{V}(k) \int d^3v\, \delta\_[\vec{k}.(\vec{v}_1 - \vec{v})]\, i\vec{k}.\frac{\partial}{\partial \vec{v}} f(\vec{r}_1,\vec{v})$$

$$\delta\_(x) = \delta(x) - iP(\frac{1}{x})$$

$$\tilde{V}_{eff}(\vec{k},\vec{v}_1) = \frac{\tilde{V}(k)}{\left|\varepsilon(\vec{k},\vec{v}_1)\right|}$$: **Effective short-range potential with Debye-like shielding**

Integral over *k* in C_{BL} diverges for *k* → ∞ **(small distances): No strong interactions!!**
But converges for *k* → 0 **(long distances): includes weak interactions during collision**

Divergence opposite to the Boltzmann collision term!

Convergent kinetic equation

Starting from first principles: BBKY hierarchy.

$$\partial_t f(\vec{r}_1,\vec{v}_1;t) = L_1^0 f(\mathbf{1};t) + \int d^3r_2\, d^3v_2 L'_{12} f(\vec{r}_1,\vec{v}_1;t) f(\vec{r}_2,\vec{v}_2;t) + \int d^3r_2\, d^3v_2 L'_{12}\, g_2(\vec{r}_1,\vec{v}_1;\vec{r}_2,\vec{v}_2;t)$$

$$\frac{\partial}{\partial t} g_2(\mathbf{1,2};t) = \left[L_1^0 + L_2^0\right] g_2(\mathbf{1,2};t) + L'_{12}\left[g_2(\mathbf{1,2};t) + f(\mathbf{1};t) f(\mathbf{2};t)\right] + \int d\mathbf{3}\left\{L'_{13} f(\mathbf{1};t) g_2(\mathbf{2,3};t) + L'_{23} f(\mathbf{2};t) g_2(\mathbf{1,3};t) + (L'_{13} + L'_{23})[f(\mathbf{3};t) g_2(\mathbf{1,2};t) + g_3(\mathbf{1,2,3};t)]\right\}$$

$$\frac{\partial}{\partial t} g_3(\mathbf{1,2},3;t) = ...$$

$$i = \mathbf{1,2,3} = (\vec{r}_i,\vec{v}_i)\ ;\ di = d^3r_i\, d^3v_i$$

$$L_i^0 = -\vec{v}_i.\frac{\partial}{\partial \vec{r}_i}\ ;\ i = 1,2$$ **: free motion operator**

$$L'_{12} = -\frac{1}{m}\vec{F}(\vec{r}_1 - \vec{r}_2).\left(\frac{\partial}{\partial \vec{v}_1} - \frac{\partial}{\partial \vec{v}_2}\right)$$ **: interaction operator**

Vlasov equation: $$\partial_t f(\vec{r}_1,\vec{v}_1;t) = L_1^0 f(\vec{r}_1,\vec{v}_1;t) + \int d^3r_2\, d^3v_2 L'_{12} f(\vec{r}_1,\vec{v}_1;t) f(\vec{r}_2,\vec{v}_2;t)$$

Collision term stems from: $$C = \int d^3r_2\, d^3v_2 L'_{12}\, g_2(\vec{r}_1,\vec{v}_1;\vec{r}_2,\vec{v}_2;t)$$

Convergent kinetic equation

Split C into two terms: $C = C_1 + C_2$

$$C_1 = \int_{S_1} d^3 r_2 \int d^3 v_2 L'_{12}\, g_2(\vec{r}_1, \vec{v}_1; \vec{r}_2, \vec{v}_2; t)$$

$$C_2 = \int_{\mathbb{R}^3 \setminus S_1} d^3 r_2 \int d^3 v_2 L'_{12}\, g_2(\vec{r}_1, \vec{v}_1; \vec{r}_2, \vec{v}_2; t)$$

S_1: Small spherical volume of radius L centered at particle 1.

L : Landau length such that $\dfrac{\gamma}{L} = E_{kin}$

The second BBGKY equation must be solved for $g_2(\vec{r}_1, \vec{v}_1; \vec{r}_2, \vec{v}_2; t)$ with different approximations in the case of C_1 or C_2 !

Convergent kinetic equation

Assumption that system is globally weakly coupled:

Γ = (Average potential energy/average kinetic energy) << 1

a. For C_1 : -In small sphere S_1, interactions are dominant over free motion
-probability of third particle being in sphere S_1 negligible (dilute gas).

→ **keep only second term in 2d BBGKY equation.**

b. For C_2 : -Interactions are weak between particle 2 located in $R^3 \backslash S_1$ and particle 1 in S_1.
-Interactions between particles 1, 2 and a third particle 3 play an important role.
-But 3-particle correlation g_3 can be neglected (Γ << 1) .

→ **all terms kept in 2d BBGKY equation except term with g_3**
: leads to *Balescu-Lenard* collision term C_{BL} with natural cut-off at L

Convergent kinetic equation

3D systems with potential *V*(*r*) = γ/*r* :

$$C_1 = -\frac{1}{5}\left(\frac{2\pi|\gamma|}{m}\right)^{3/2}\int_0^t \frac{d\tau}{\sqrt{\tau}}\, n(\vec{r}_1;t-\tau)\left(-\Delta_{\vec{v}_1}\right)^{3/4} f(\vec{r}_1,\vec{v}_1;t-\tau)$$

$$\left(-\Delta_{\vec{v}_1}\right)^{3/4} e^{i\vec{\zeta}_1\cdot\vec{v}_1} \equiv (\vec{\zeta}.\vec{\zeta}_1)^{3/4} e^{i\vec{\zeta}_1\cdot\vec{v}_1} = \zeta_1^{3/2} e^{i\vec{\zeta}_1\cdot\vec{v}_1}$$

$$n(\vec{r}_1;t) \equiv \int d^3v\, f(\vec{r}_1,\vec{v};t)$$

Kinetic equation

$$\frac{\partial}{\partial t} f(\vec{r},\vec{v},t) + \vec{v}.\frac{\partial}{\partial \vec{r}} f(\vec{r},\vec{v},t) = Vlasov + C_1 + C_{BL}$$

Convergent kinetic equation

Remarks about C_1:

C_1 is nonlinear in $f(r,v,t)$. It is non-Markovian due to a fractional iterated integral in time.

C_1 is the first term of a convergent series expansion whose terms are corrections in **even integer** powers of the Laplacian Δ_v.

The small parameter of this expansion corresponds to a short-time expansion.

So, for larger times, the abnormal non-markovian diffusion in velocity-space due to the new term is progressively transformed into a normal diffusion.

Kinetic equation for homogeneous system

For homogeneous systems: Vlasov and free motion terms vanish.

Time-scale of new term compared to relaxation time:

$$t_{new} / t_{rel} = \Gamma^3 \ll 1$$

only the new term is acting for short times:

$$D_t^{3/2} \varphi(\vec{v};t) = -\frac{n\pi^2}{5} \left(\frac{2|\gamma|}{m} \right)^{3/2} (-\Delta_{\vec{v}})^{3/4} \varphi(\vec{v};t)$$

Solution:

$$\varphi(\vec{v};t) = \int \frac{d^3\zeta}{(2\pi)^3} e^{i\vec{\zeta}\cdot\vec{v}} \tilde{\varphi}(\vec{\zeta};0) E_{3/2}\left(-\frac{n\pi^2}{5} \left[\frac{2|\gamma|}{m} \right]^{3/2} \zeta^{3/2} t^{3/2} \right)$$

$E_{3/2}(z)$: Mittag-Leffler function of index *3/2*.

$D_t^{3/2}$: Riemann-Liouville fractional time derivative of order *3/2*.

Kinetic equation for homogeneous systems

- For most initial velocity distributions, the velocity distribution obtained at time *t* has a **long algebraic tail in $1/v^{5/2}$**.

- This algebraic tail is truncated at very large v due to the growth with time of the diffusive corrections to the new term.

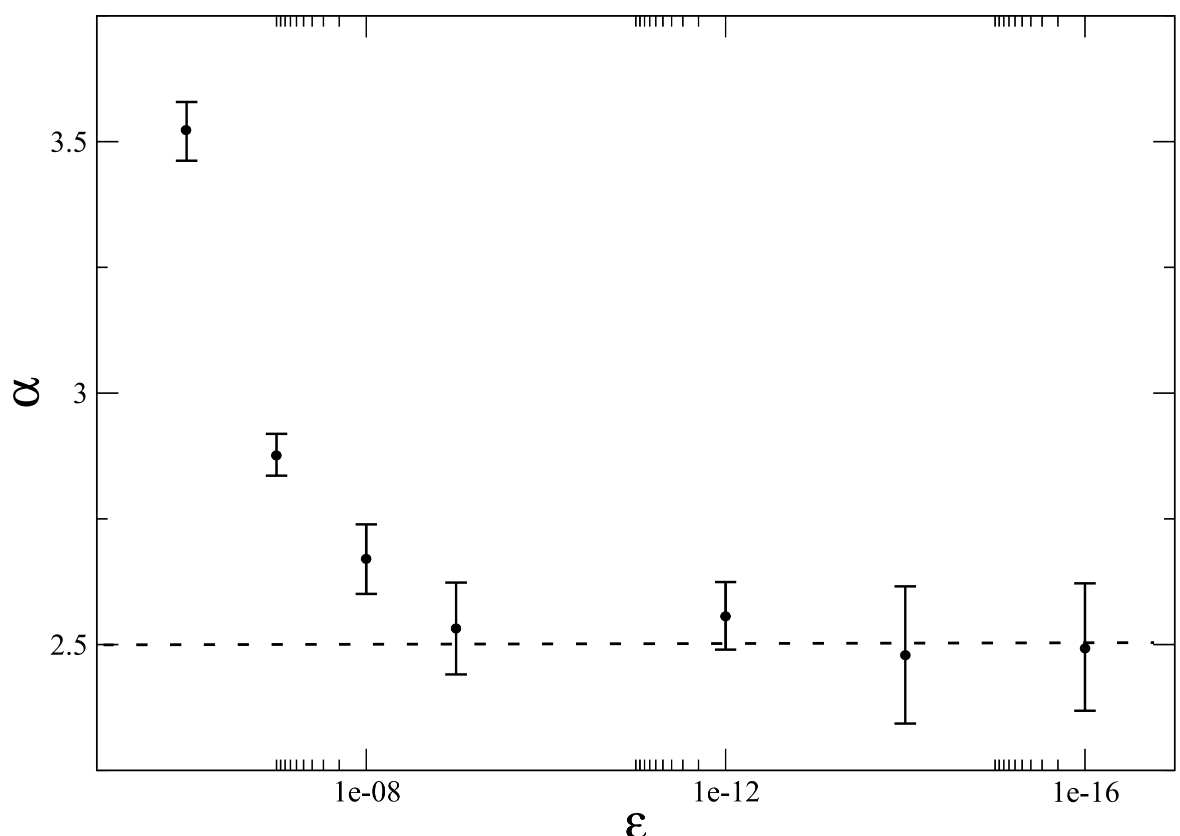
Simulations

Previous result should hold for short times after a homogenous initial condition.

Molecular dynamics with 4th order symplectic integrator for a 3D system of 131,072 particles with attractive regularized potential:

$$-|\gamma|/\sqrt{r^2+\varepsilon^2}$$

Initial condition: homogeneous in a sphere. Algebraic tail: $1/v^{\alpha}$



Other results and conclusion

-Generalisation to potentials in $1/r^s$.

-Derivation from quantum BBGKY hierarchy with Wigner reduced distributions: leads to same new term in the classical approximation plus quantum corrections.

-New term introduces irreversibility (abnormal diffusion in velocity space).

-New term is 1/N with respect to Vlasov term.

-For 1-D systems such as the Ring model: collision term vanishes at order 1/N but not the new term: See next talk by Tarcisio M. Rocha Filho.

The End

THANK YOU!