Joint work by P. Bechouche (U. Granada)

① Relativistic Vlasov-Maxwell Eq.

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + (E + v\times B)\nabla_p f = 0 & f = f(t,x,p) \quad t\in\mathbb{R},\ x\in\mathbb{R}^3,\ p\in\mathbb{R}^3 \\ \partial_t E - \nabla\times B = -J, \quad \partial_t B + \nabla\times E = 0, \ \nabla\cdot E = \rho, \ \nabla\cdot B = 0 \\ \rho = \int f\,dp, \quad j = \int v f\,dp \quad + \text{ initial Conditions.} \end{cases}$$

② Diperna-Lions 89: Existence of weak solution

$f \in L^\infty(0,\infty, L^1\cap L^\infty(\mathbb{R}^6_{xp}))$, $E, B \in L^\infty(0,\infty; L^2(\mathbb{R}^3_x))$

Open Questions: Regularity? Uniqueness?

③ Bouchut-Golse-Pallard 2004 (BGP-04).

. $E_0, B_0 \in H^1_x$, $f_0 \in L^1 \cap L^\infty$

Macroscopic Kinetic Energy (MKE) $= \int \sqrt{1+|p|^2}\, f\,dp \in L^q(\mathbb{R}_+ \times \mathbb{R}^3_x)$, $q\in]\frac{3}{2}, 2]$

(Natural energy bound: $MKE \in L^\infty_t L^1_x$

Then $E, B \in H^s_{loc}(\mathbb{R}^+_t \times \mathbb{R}^3_x)$ $\quad s < \dfrac{2q-3}{2q+3/2}$ $\quad q=2 \quad s<\dfrac{2}{11}$

Question of BGP: is this regularity optimal?

Ideas of proof: Non-resonant smoothing effect (Glassey 1986):

A smoothing effect on (E,B) from particles such that $|v|<1$ (or $|p|\le R<\infty$)

3 ingredients

. Under $|v|<1$, a well chosen combination of free streaming operator and wave operator lead to elliptic operator in space and time ⇒ Gain of elliptic Regularity

. Kinetic formulation of Maxwell eq. scalar wave operator for a generalized potential depending on time, space and momentum.

. Control in L^2 norm of the (E,B) which are created from large momentum ($|p|>R$) particles thank to the L^q control of MKE.

N.B. P. Bechouche Under the same assumptions of BGP-04 we have

$E, B \in H^s_{loc}(\mathbb{R}^{*,+}_t \times \mathbb{R}^3_x)$ with $s < \frac{2q-3}{2q-3+\ell(q)}$, $\ell(q) = \frac{19}{6} + q\left(-1+\sqrt{1-\frac{7}{6q}+\frac{41}{q^2}}\right) \leq \frac{10}{3}$ $q \in]\frac{3}{2}, 2]$

$q = 2 \Rightarrow s = \frac{6}{14+\sqrt{142}} > \frac{3}{11} > \frac{2}{11}$

Proof : Use a Fourier approach $\Phi = (E, B)$

- $\partial_t^2 \Phi_k - \Delta \Phi_k = J_k(f)$

  Use Fourier Integral operator: (FIO)

  $$\Phi_k(t,x) = FIO(f)$$

- $\Phi_k = \Phi^0_k + \Phi^{<R}_k + \Phi^{>R}_k$

  (initial) $(|p| \leq R)$ $(|p| > R)$

1. $\|\Phi^0_k\| < C \|\Phi_0\|_{H^1_{loc}(\mathbb{R}^3)}$
2. $\|\Phi^{<R}\|_{H^1_{loc}(\mathbb{R}^*_t \times \mathbb{R}^3_x)} \leq C R^{10/3}$
   - Time Integration by part $\Rightarrow$ symbol (Green fonction in fourier ξ-space dual to x-space) is divided by $\frac{1}{|\xi| D^\pm}$ where $D^\pm = 1 \pm \frac{v \cdot \xi}{|\xi|}$ is the resonant factor coming from the free streaming advection.

     $|v| < 1$ and $\frac{1}{|\xi| D^\pm} \Rightarrow$ gain 1 derivative in x-space.
   - fine estimate of p-integral involving $\frac{1}{D^\pm}$ to minimize the polynomial power in R
   - Standard estimates of FIO
3. $\Phi^{>R} = \Phi^{>R}_1 + \Phi^{>R}_2$

$\Phi^{>R}_1$ satisfies a wave equation (WE) with source term $J^{>R}_1 \in L^\infty(0,\infty, L^2(\mathbb{R}^3_x))$

$\Rightarrow \|\Phi^{>R}_1\|_{H^1(\mathbb{R}^*_t \times \mathbb{R}^3_x)} \leq C R^2$

$\Phi^{>R}_2$ satisfies a WE with source term $J^{>R}_2 \in L^\infty(0,\infty, H^{-1}(\mathbb{R}^3_x))$ + L^q-control of MKE $= \|\Phi^{>R}_2\|_{L^2} \leq C R^{3-2p}$

4. End of Proof: interpolation inequality + boostraping argument on Regularity to improve estimate of $\|\Phi^{<R}_1\|_{H^1_{loc}}$