# The arrow of time in the collapse of collisionless self-gravitating systems

## Non-validity of Vlasov-Poisson in violent relaxation


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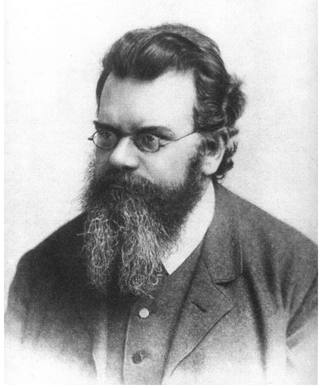

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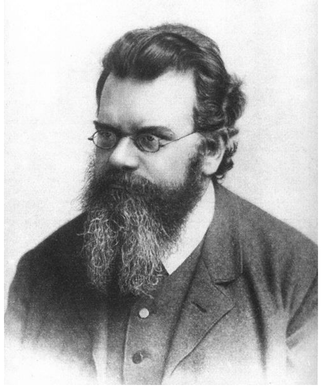

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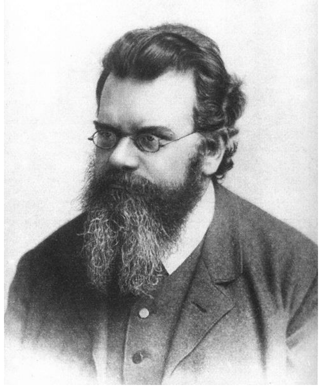

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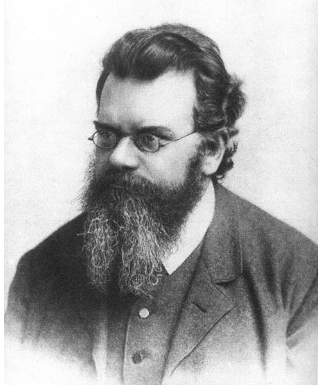

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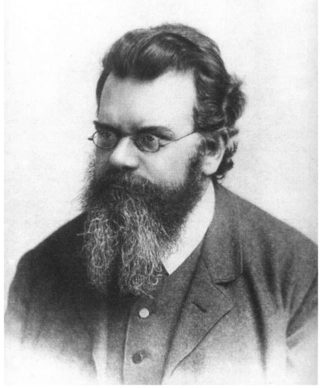

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- One particle ~ whole system: Mechanics + Statistics

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$N \approx 10^6$ stars

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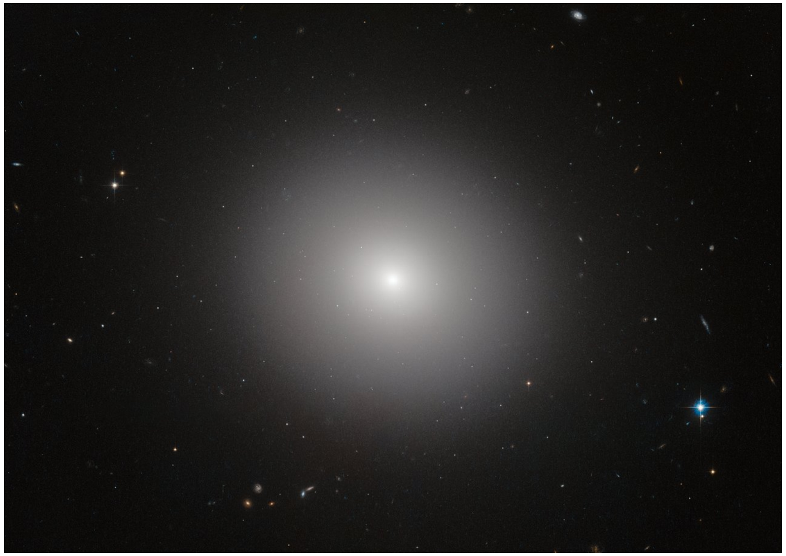

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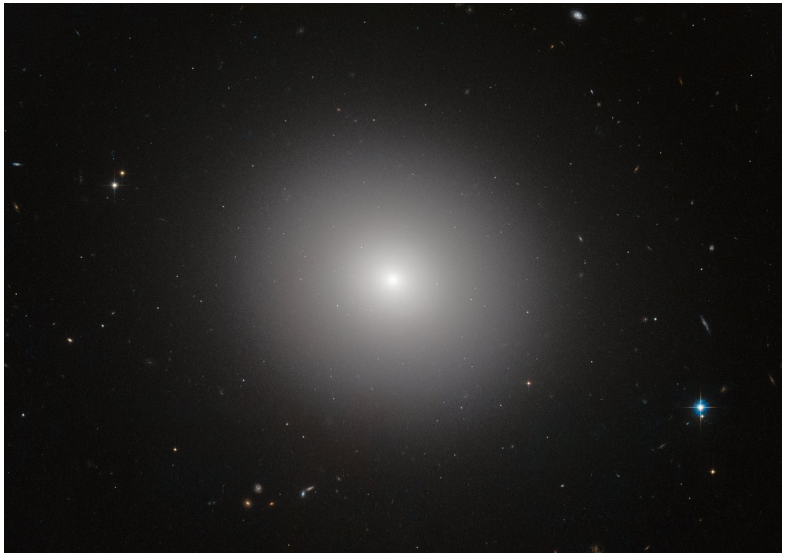

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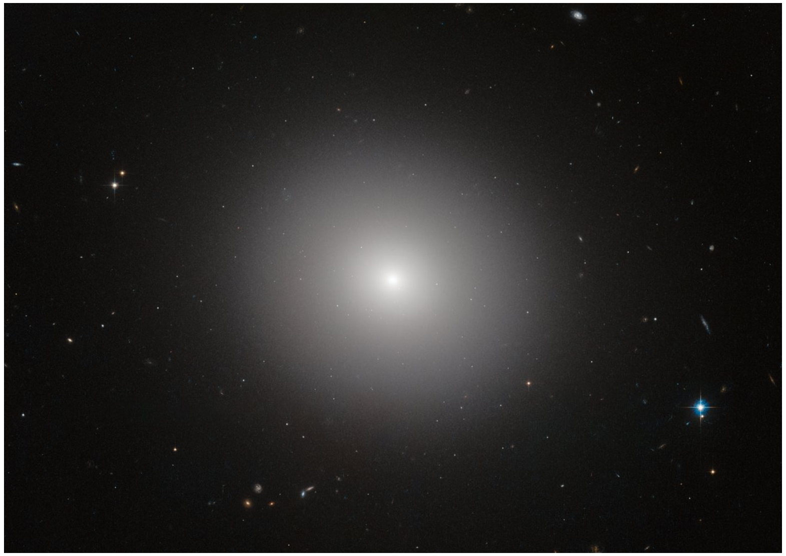

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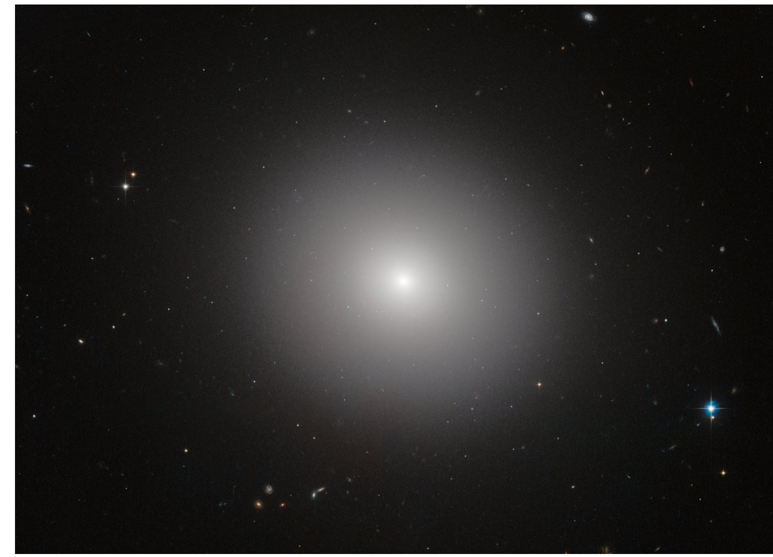
Binney & tremaine 2008



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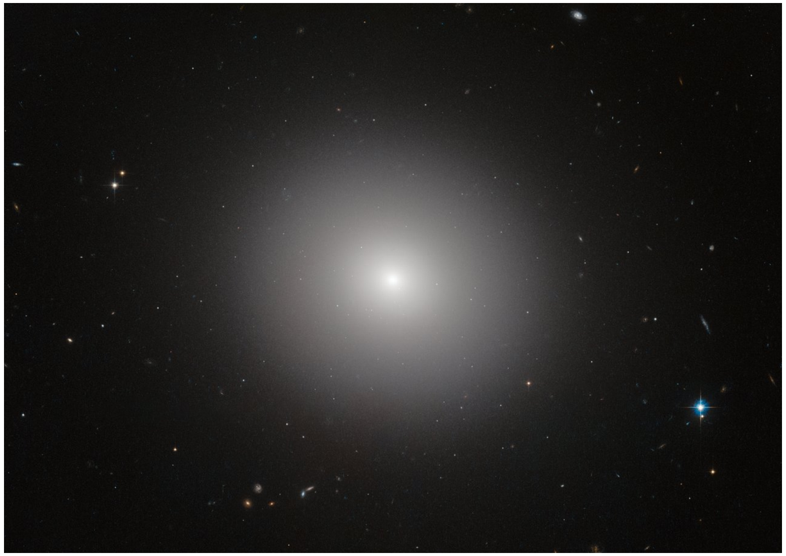

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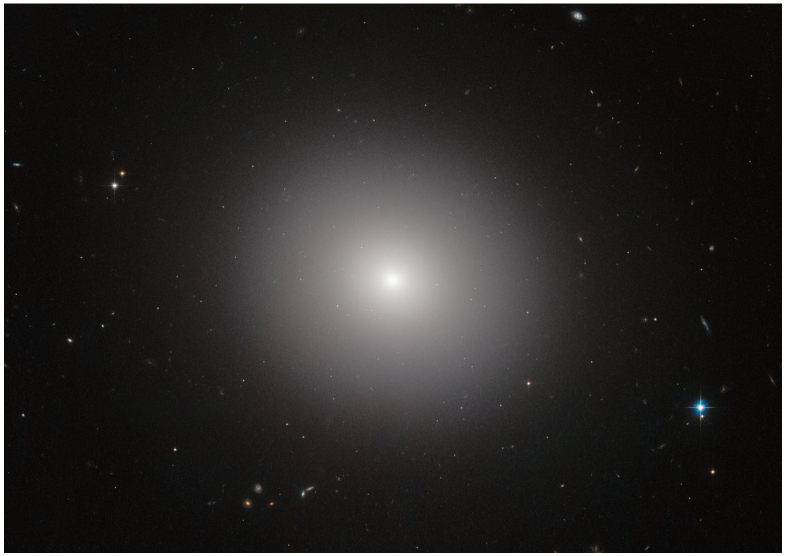

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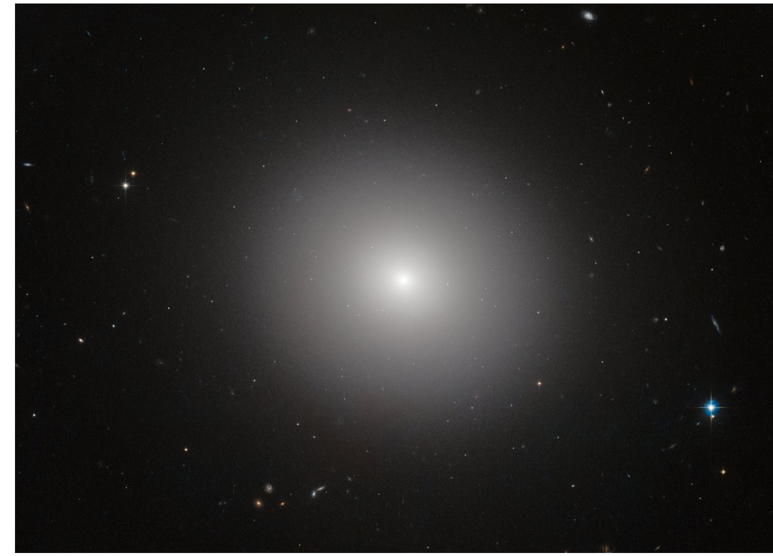

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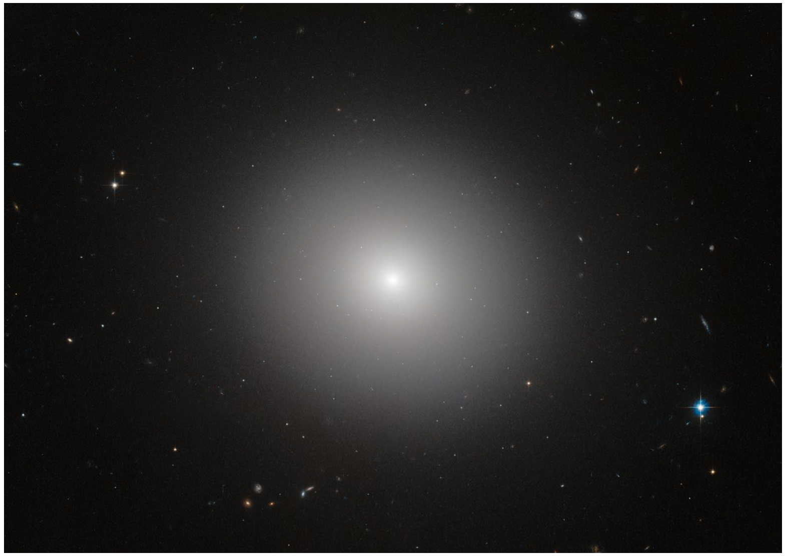

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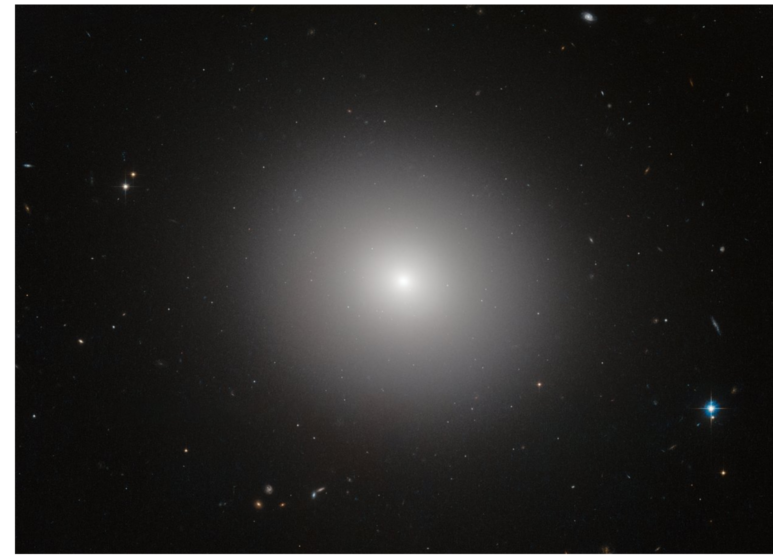

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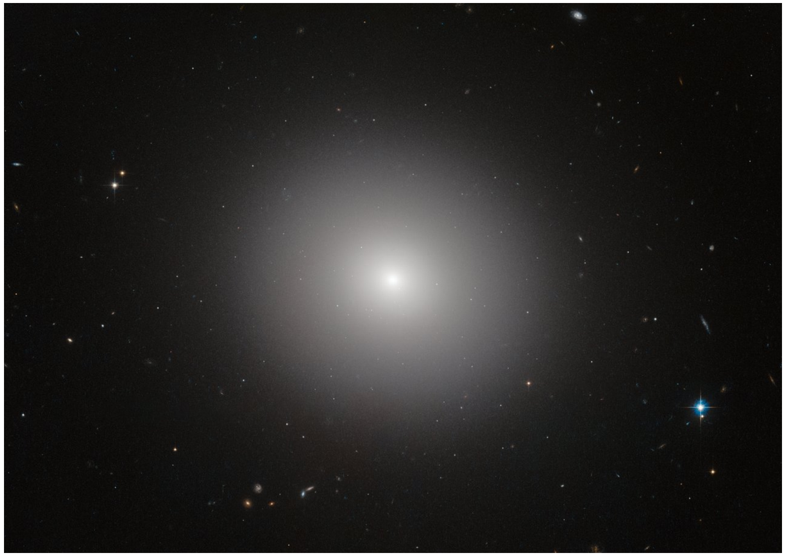

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- Violent relaxation in $\approx \tau_{cr}$ Lynden-Bell 1967; King 1962; Hénon 1964

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- N-body simulation → Estimate S → Is it conserved?

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BeS, Siqueira-Pedra, Sodré, Duarte, Lima 2017

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Sharma & Steinmetz 2006

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Nearest Neighbor and Variable Kernel: Metric-dependent

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$\hat{S} \xrightarrow[N\rightarrow\infty]{} S$ Beirlant et al 1997

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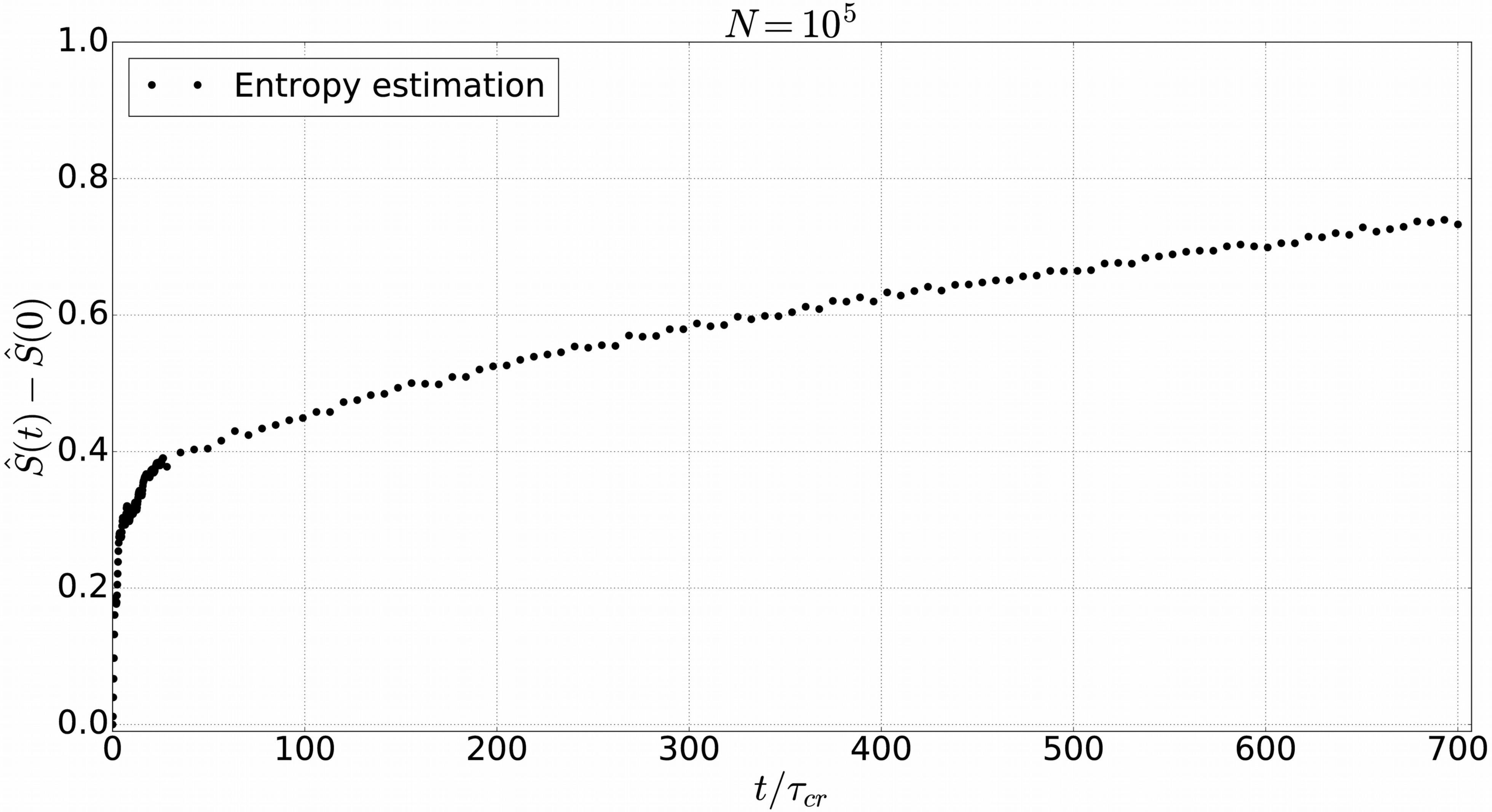
# Overview



$N = 10^5$
Entropy estimation
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
700
$\hat{S}(t) - \hat{S}(0)$
$t/\tau_{cr}$

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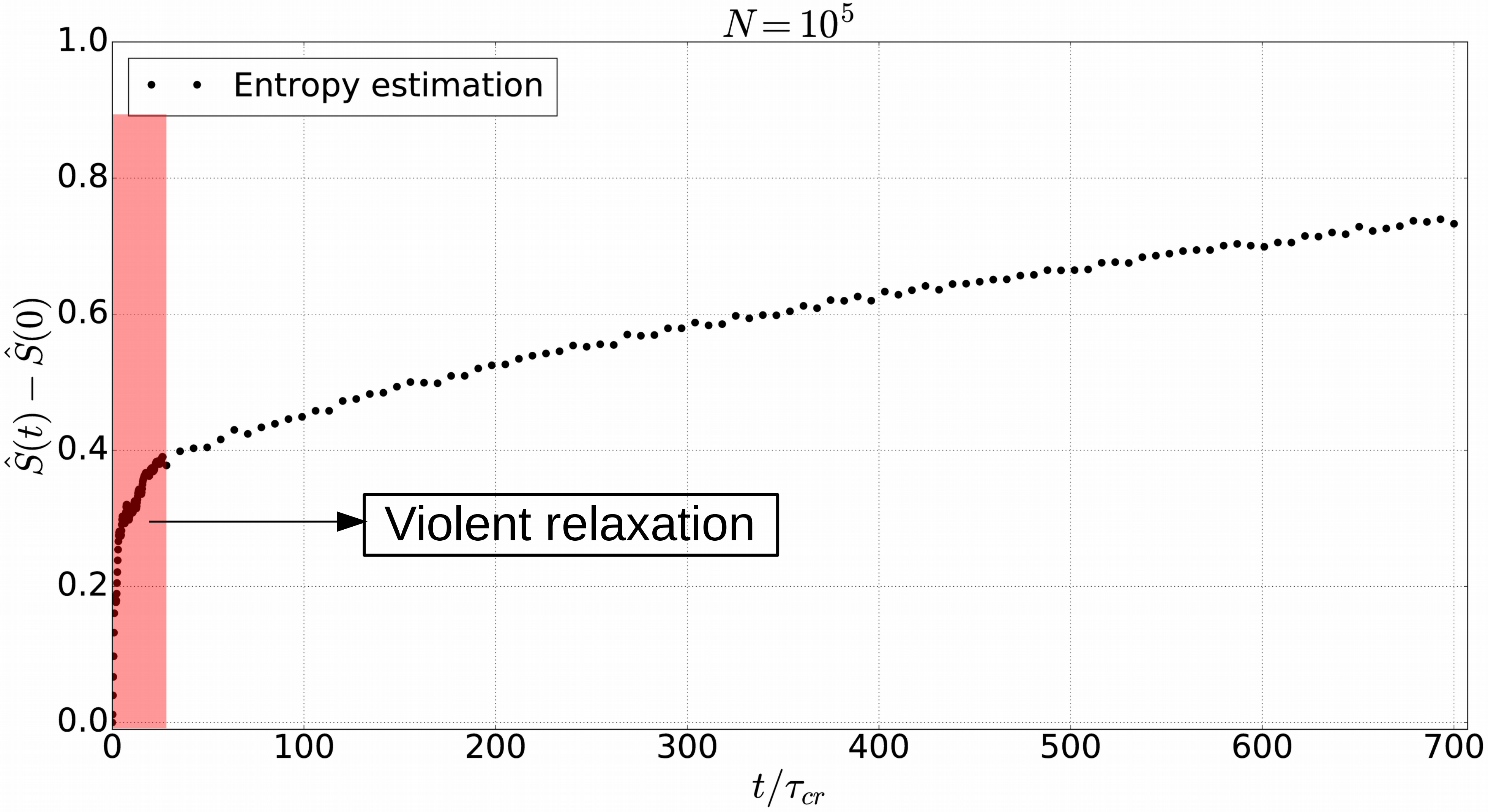

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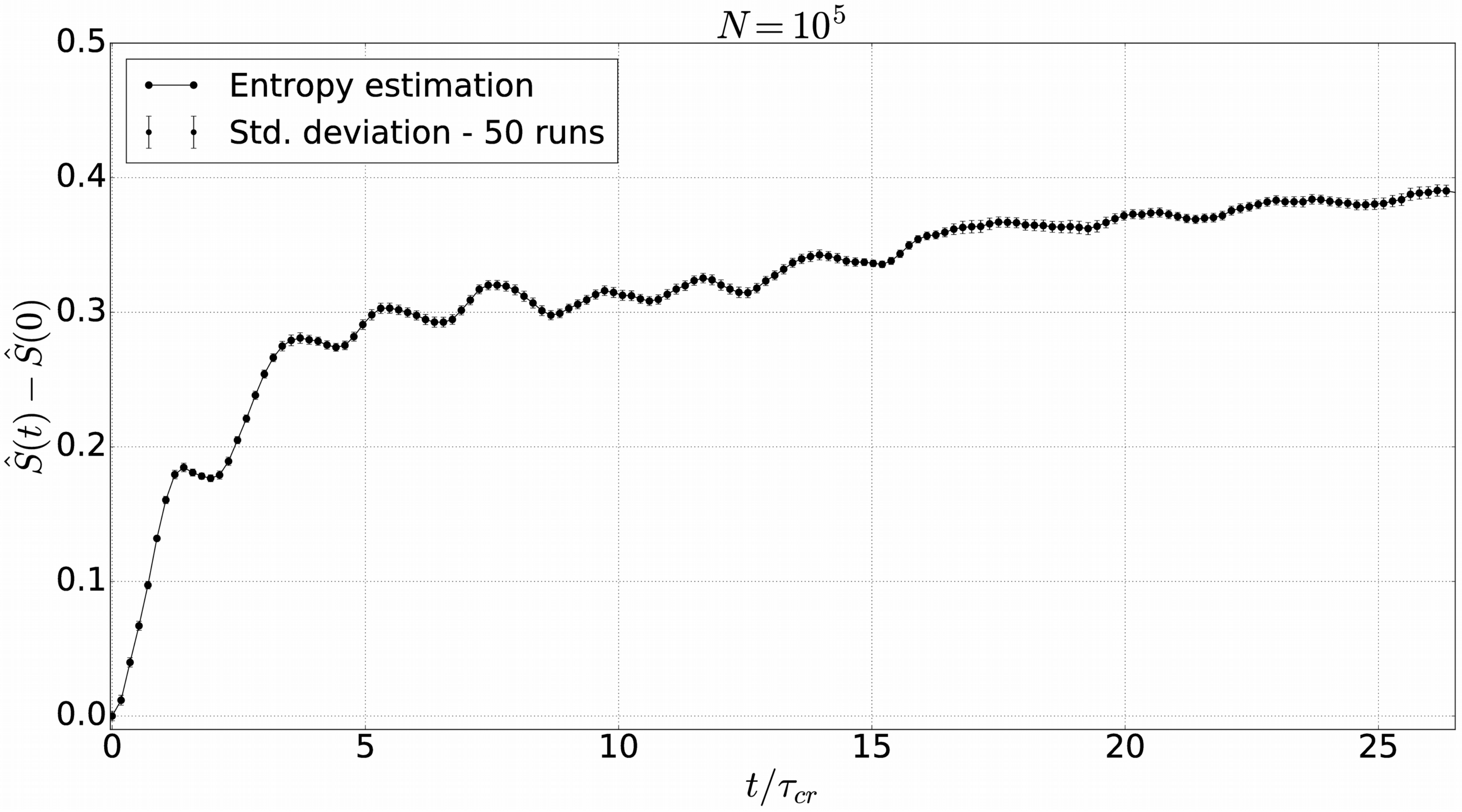
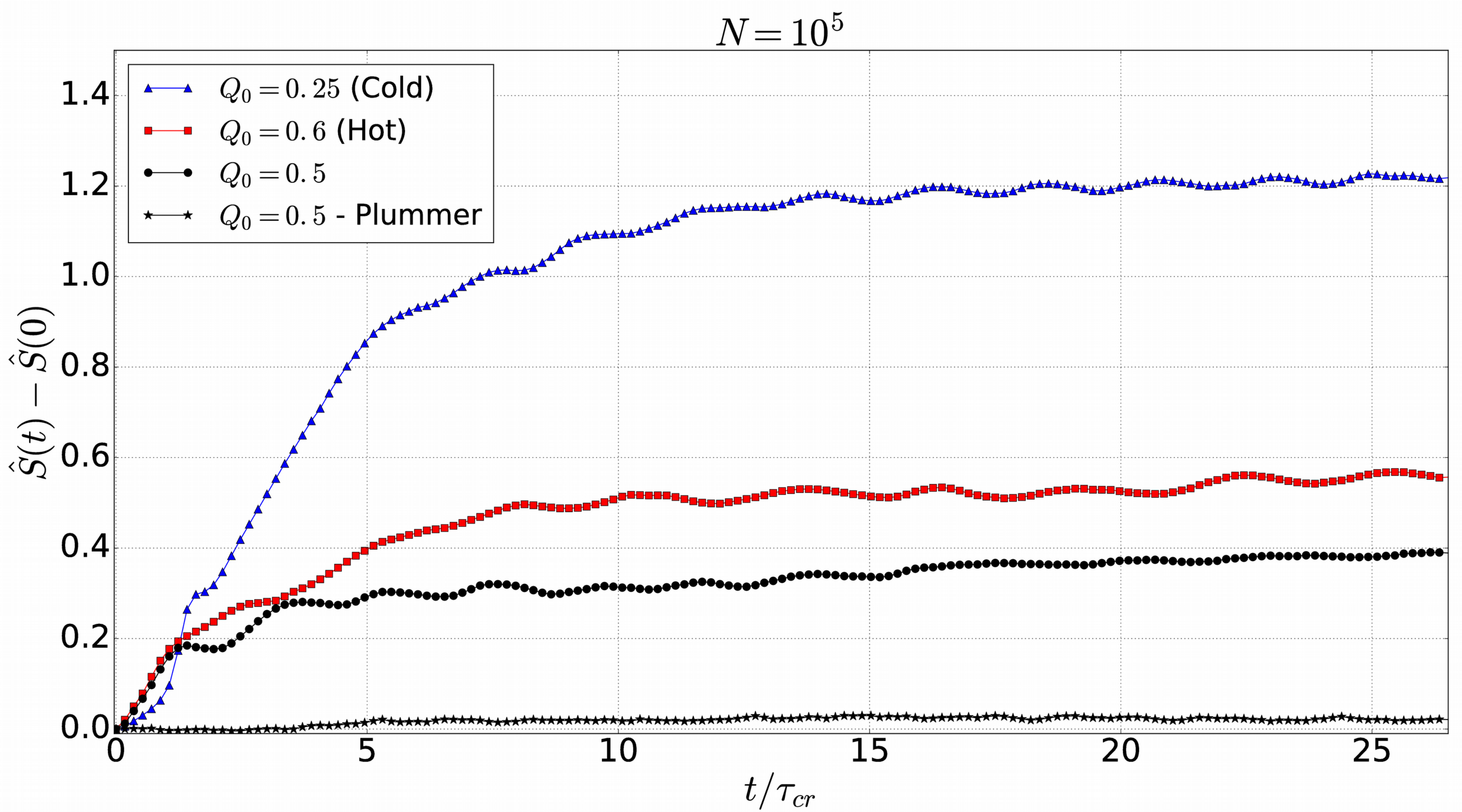


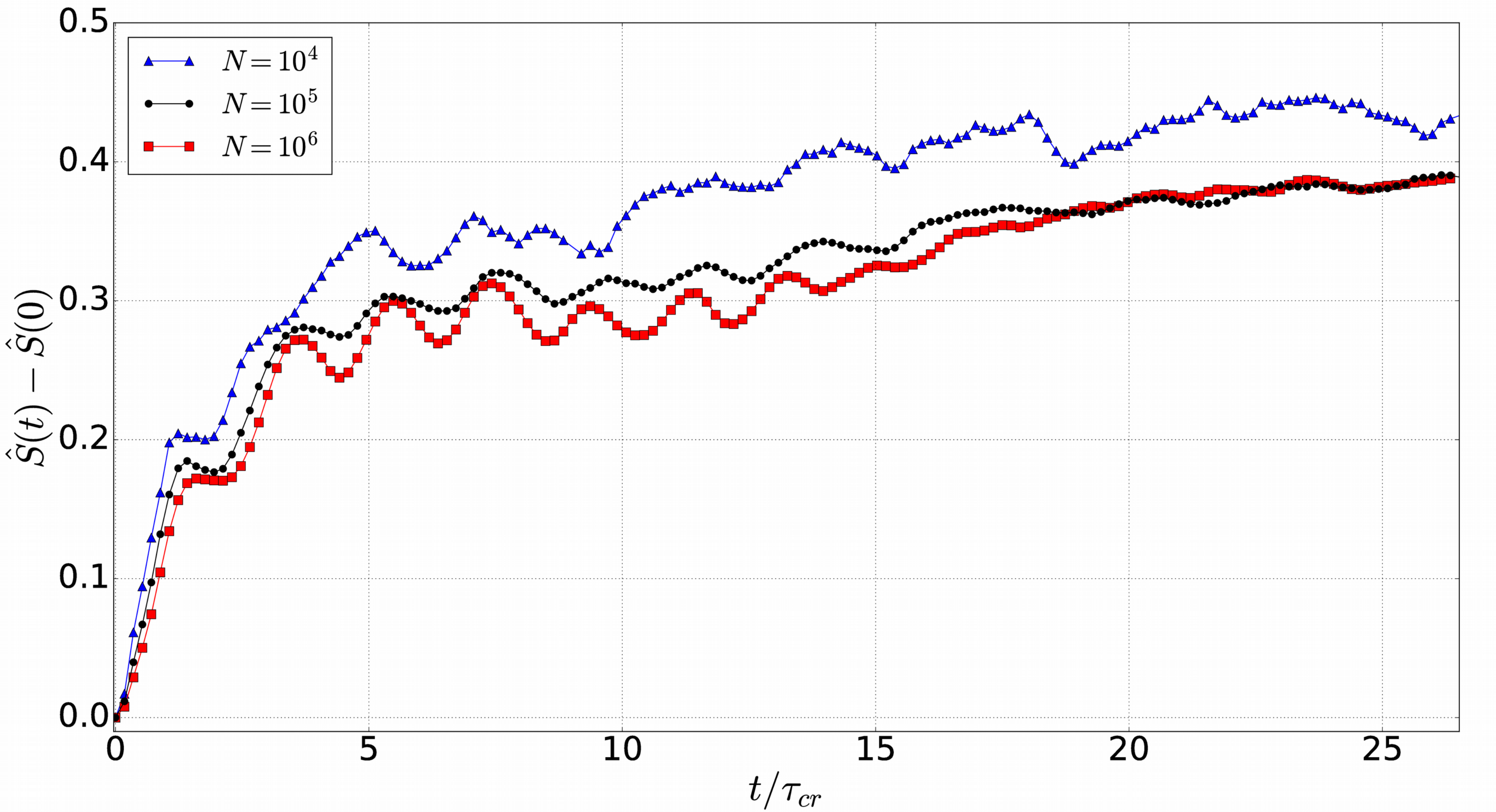

S increases → Vlasov not valid in violent relaxation

# Varying Initial Conditions




– Farther from equilibrium → larger entropy production
– Self-consistent model → entropy is conserved

# N dependence




– Hints of convergence
– Not due to collisional relaxation

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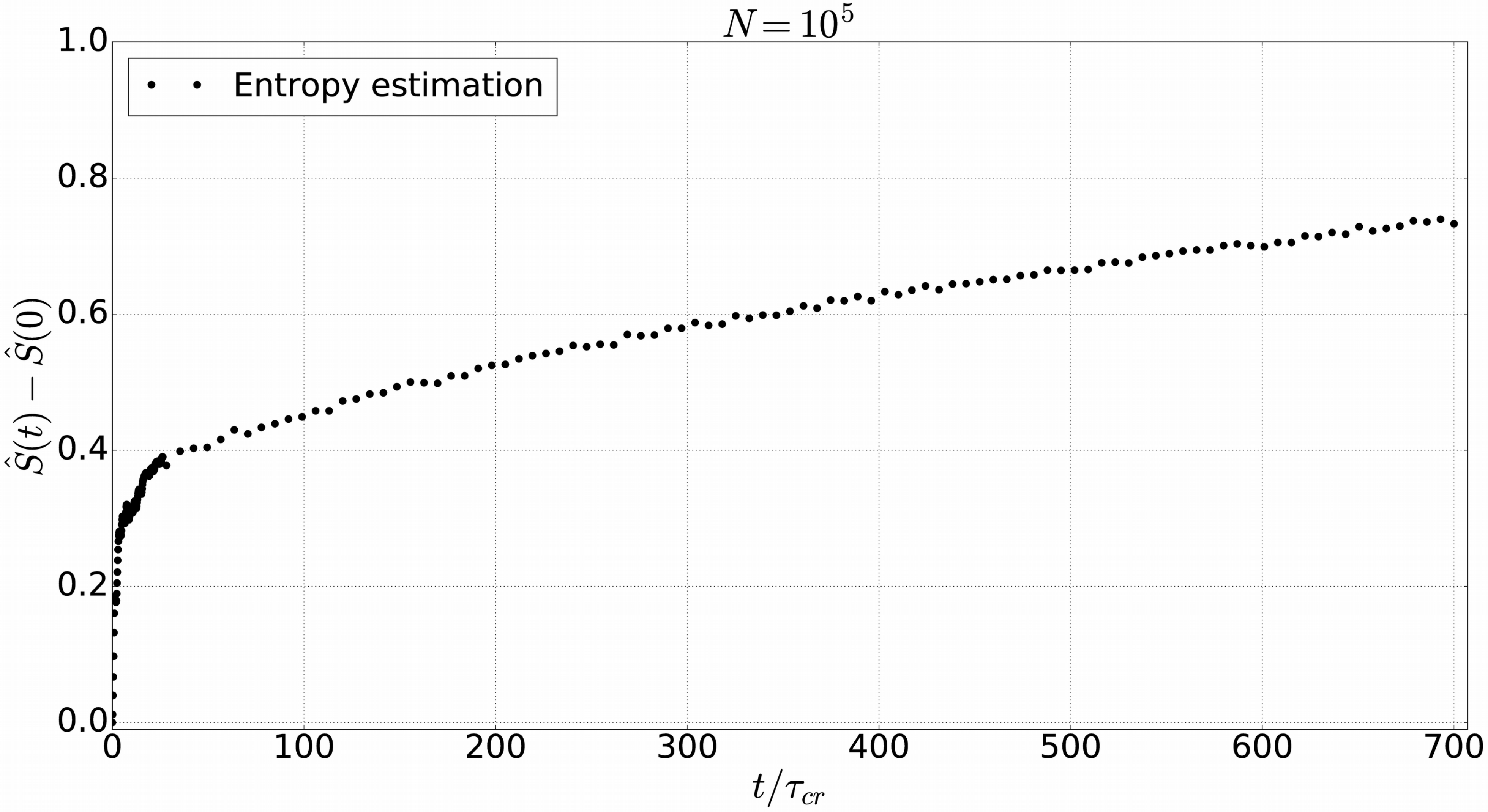

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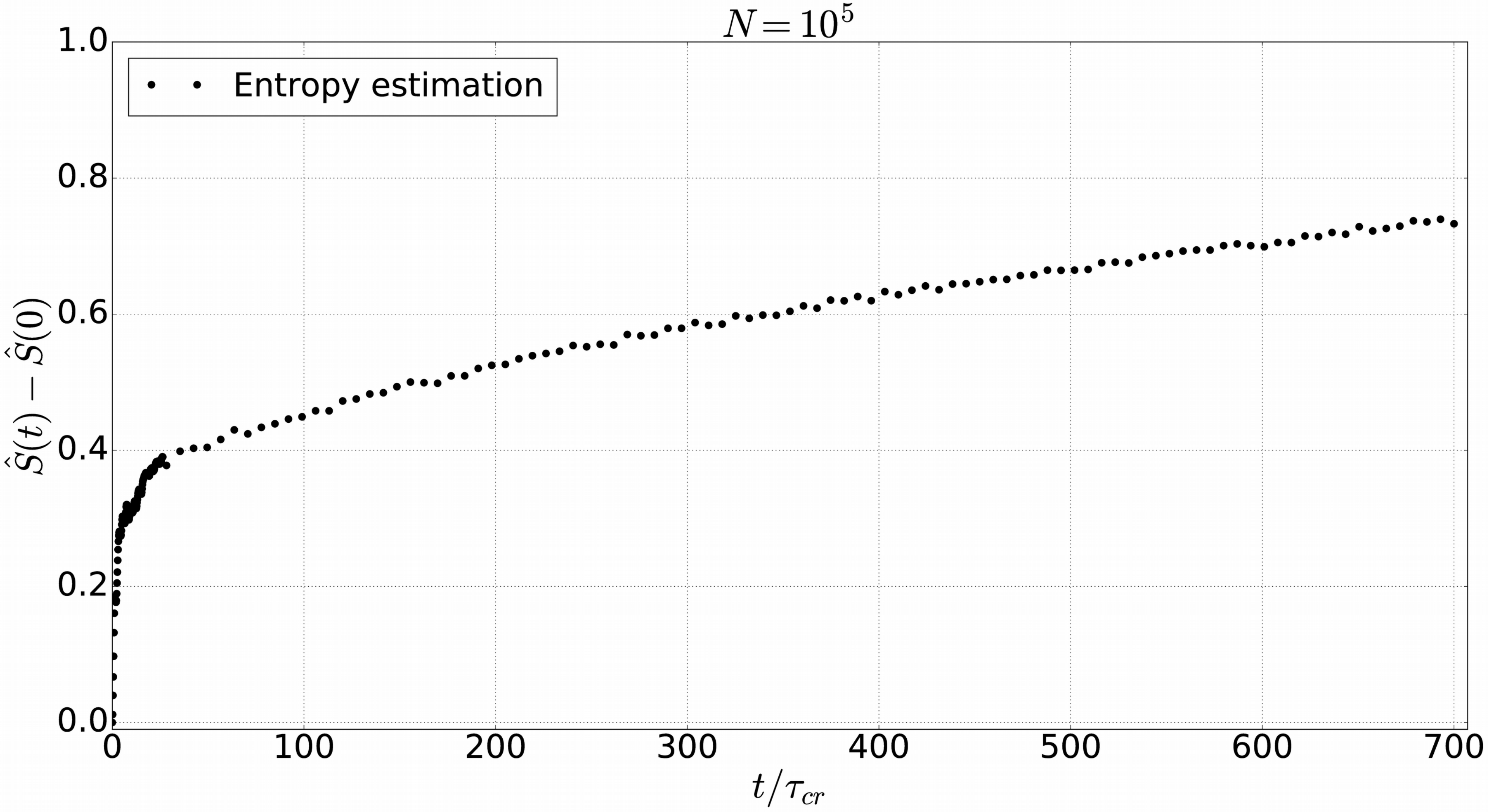


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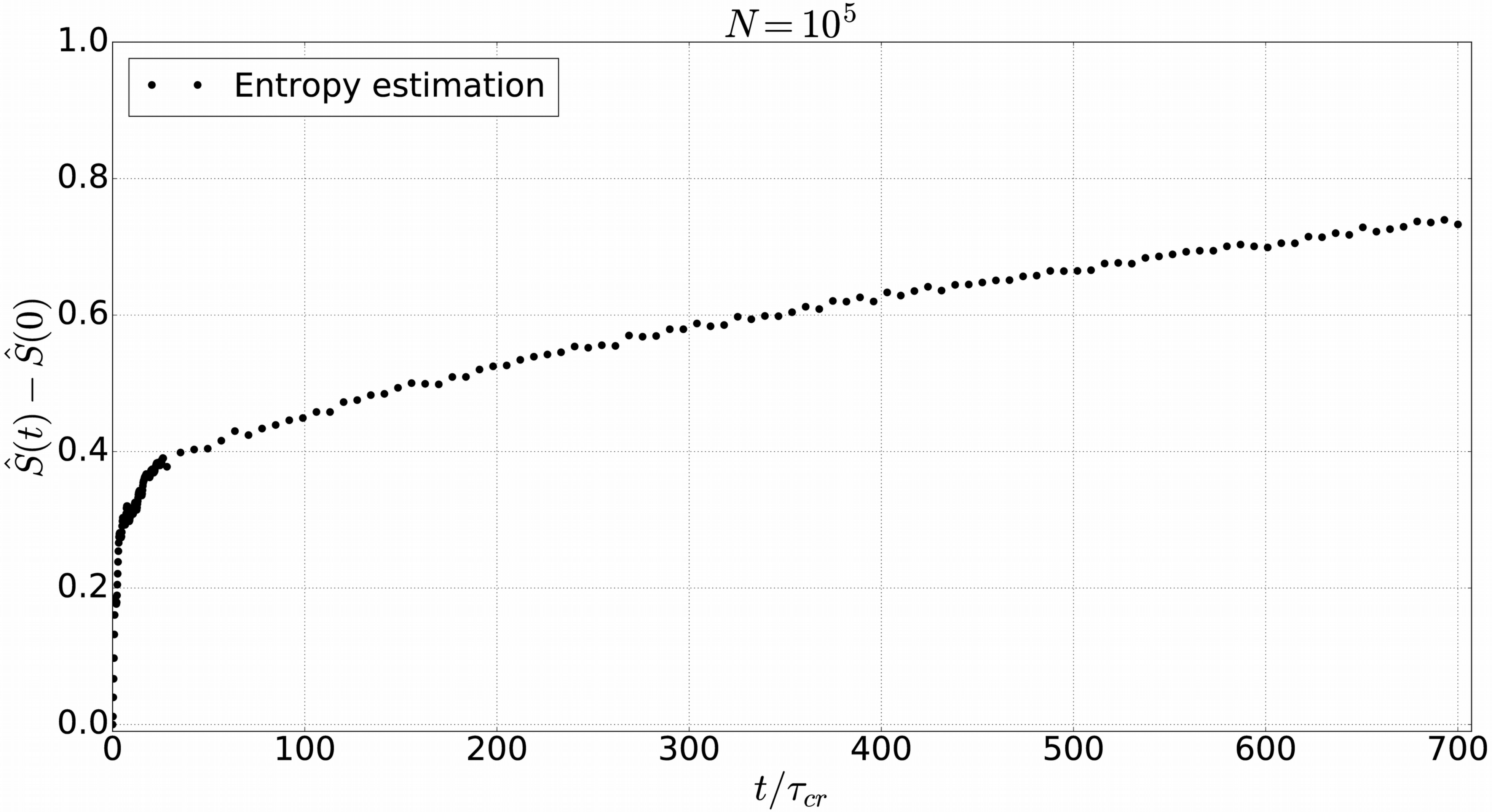


– Collisional relaxation?

– Are these estimators reliable?

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$$\left.\begin{aligned} \langle\Delta E\rangle &\propto \ln\Lambda(I_0 - I_{1/2}) \\ \langle(\Delta E)^2\rangle &\propto \ln\Lambda(I_0 + I_{3/2}) \end{aligned}\right\}$$

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(Coulomb Logarithm)

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Vasiliev 2017

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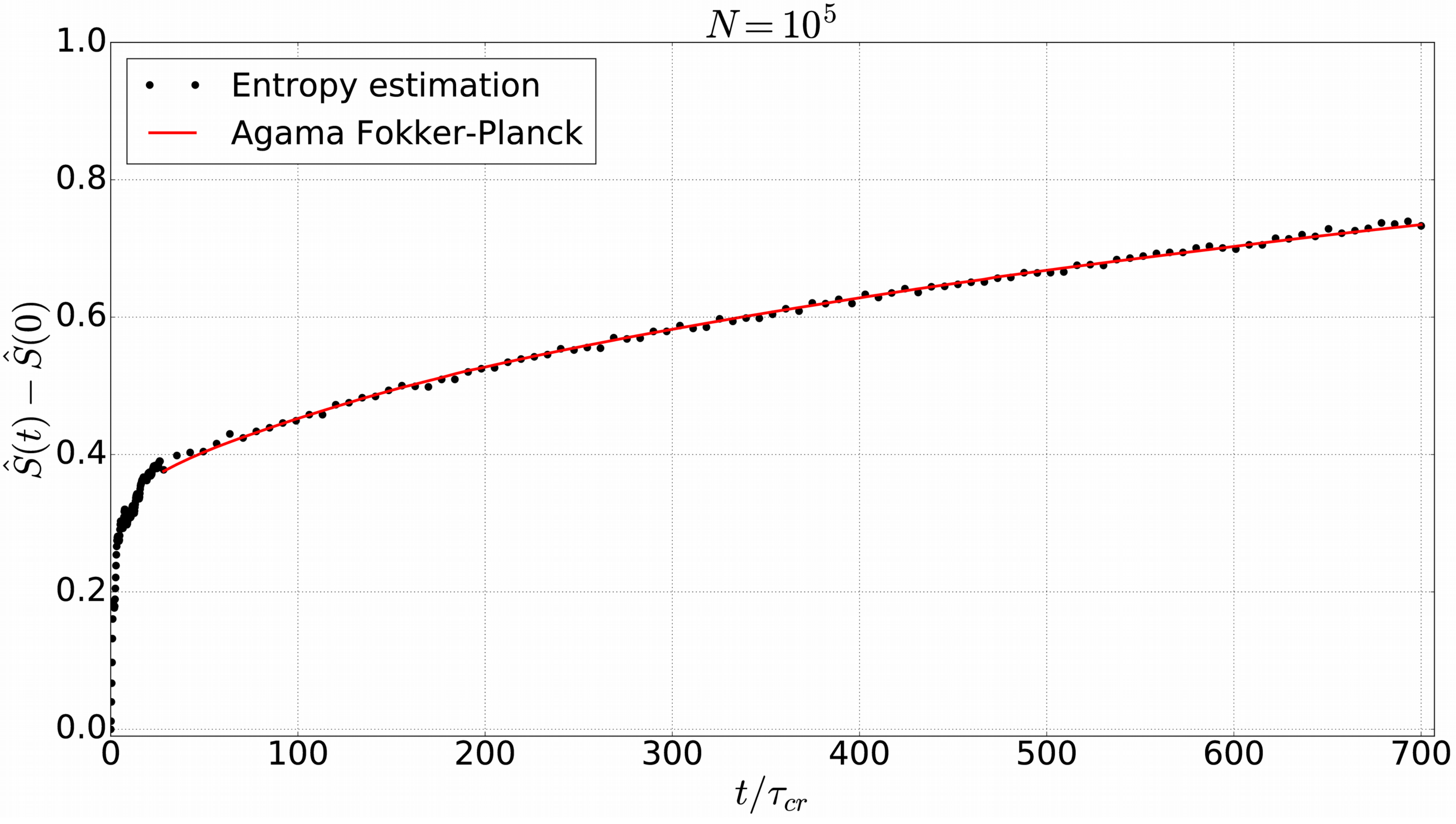

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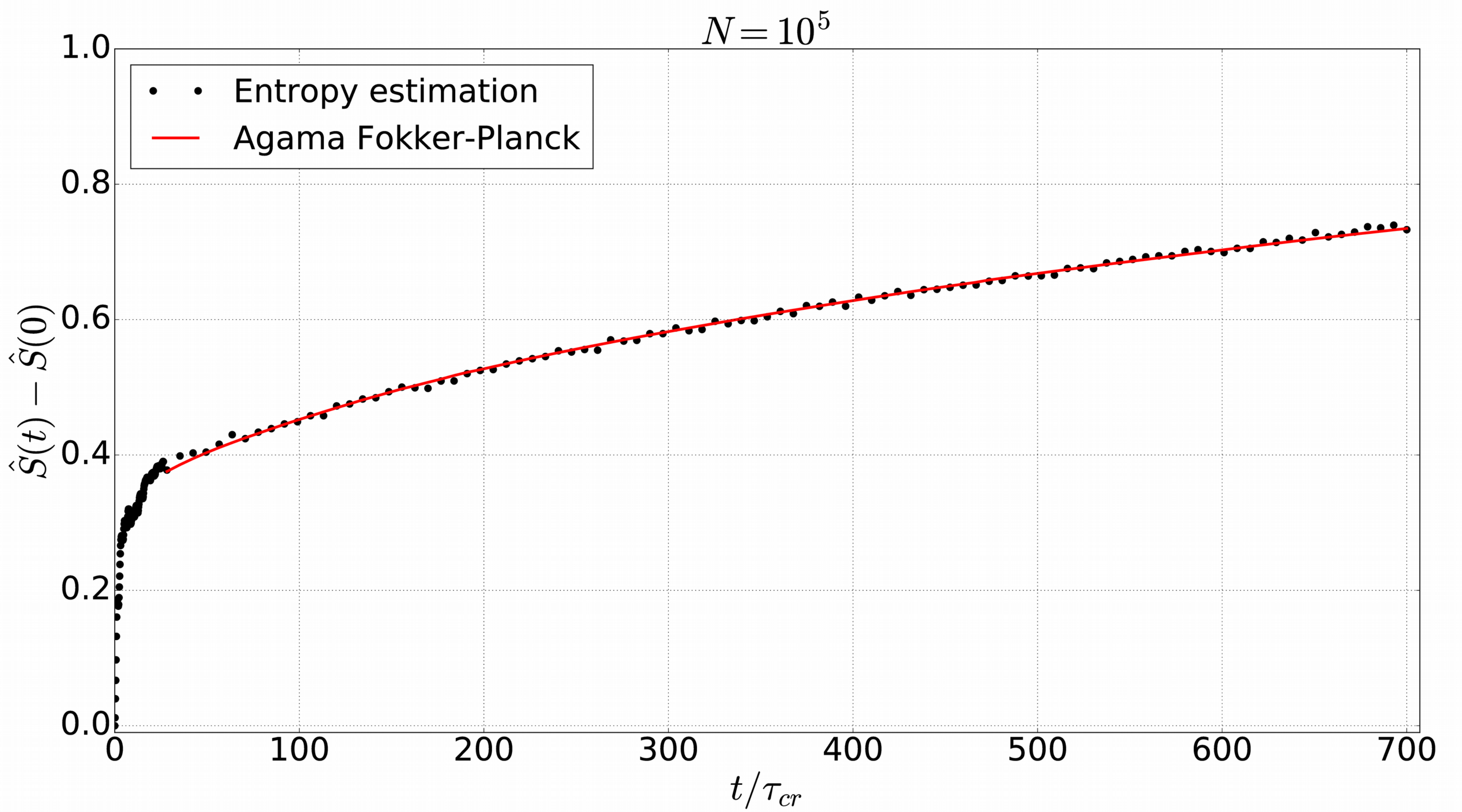


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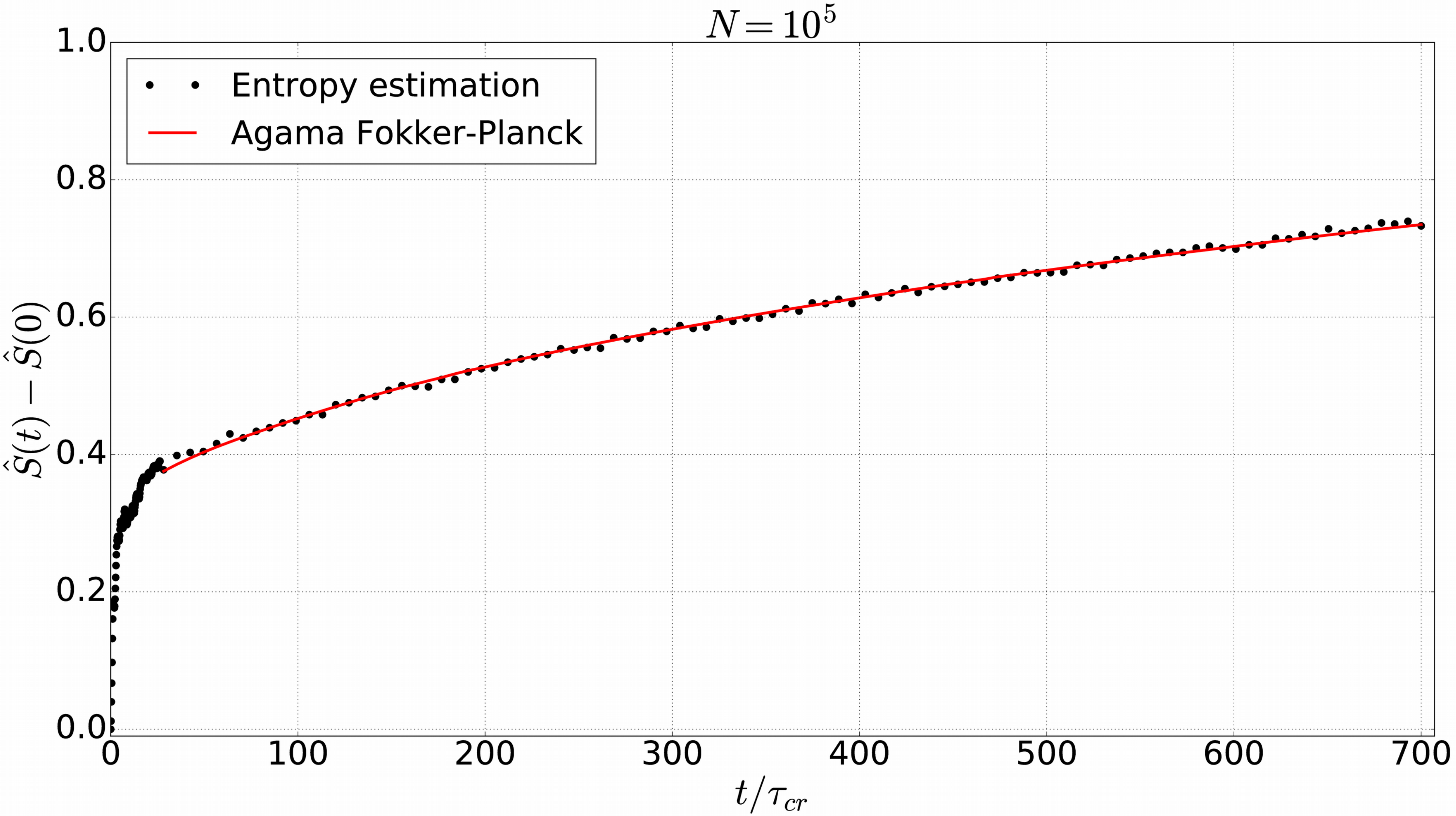
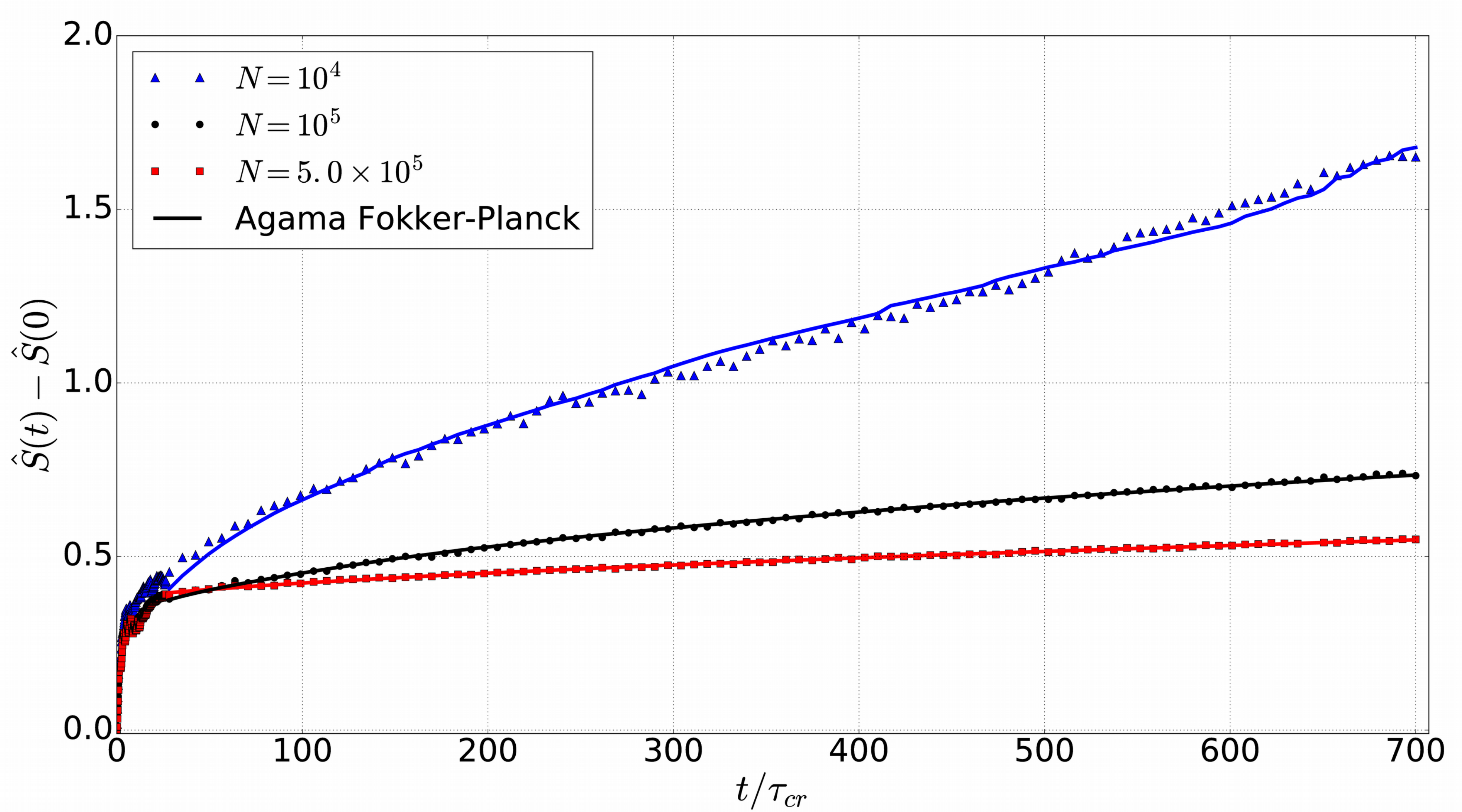



– Collisional relaxation? Yes

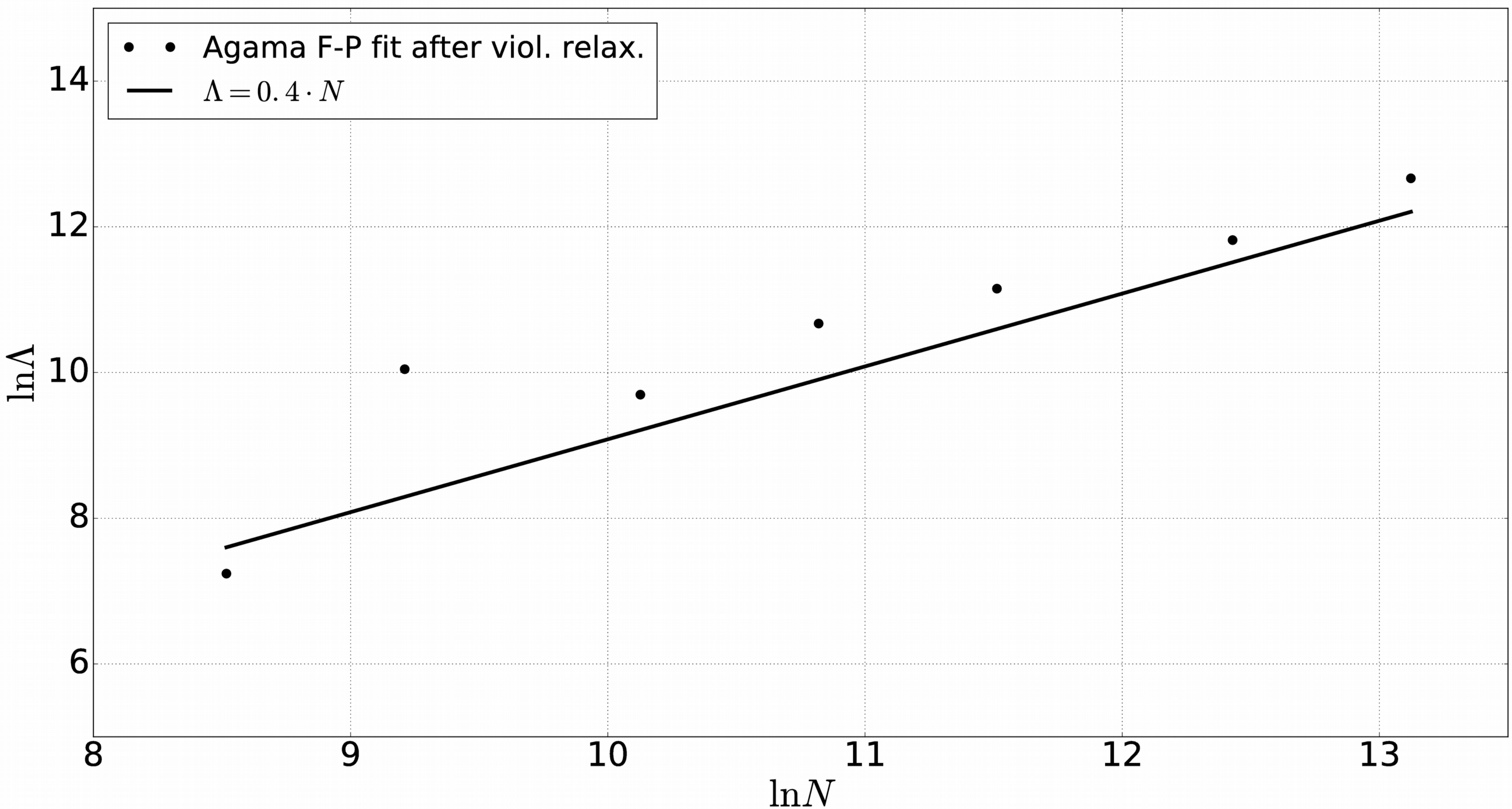
– Are these estimators reliable? Yes

# N dependence



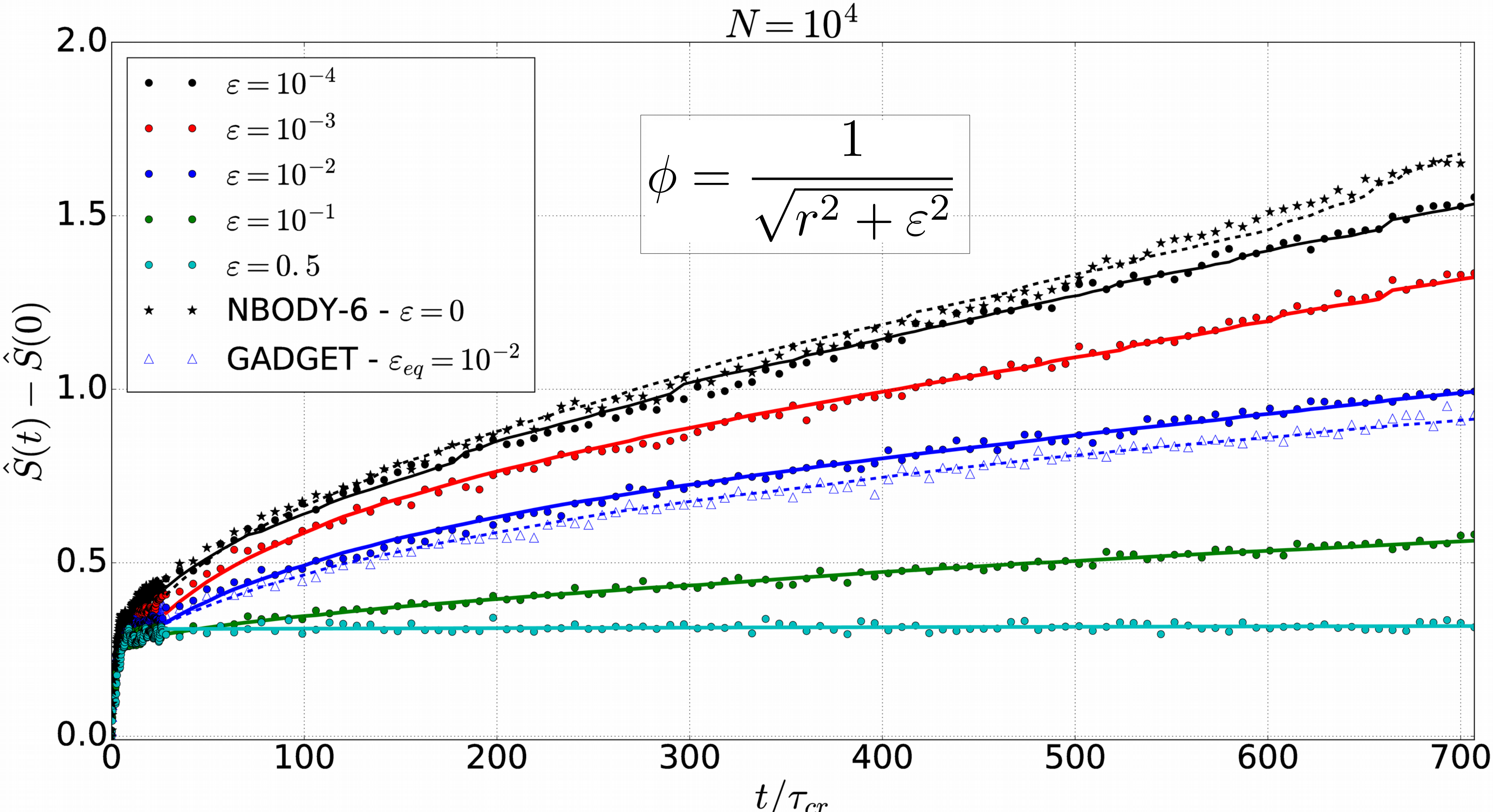

Collisionless for large N

# Coulomb logarithm




Agreement with theoretical expectation

# Varying softening length




– Collisional relaxation suppression for $\varepsilon \gtrsim 10^{-4} \approx R/N \approx b_{90}$

– GADGET-2 as collisional as NBODY-2

Hernquist & Barnes 1990
Sellwood 2015

# Constraining the Galactic potential

with Monica Valluri

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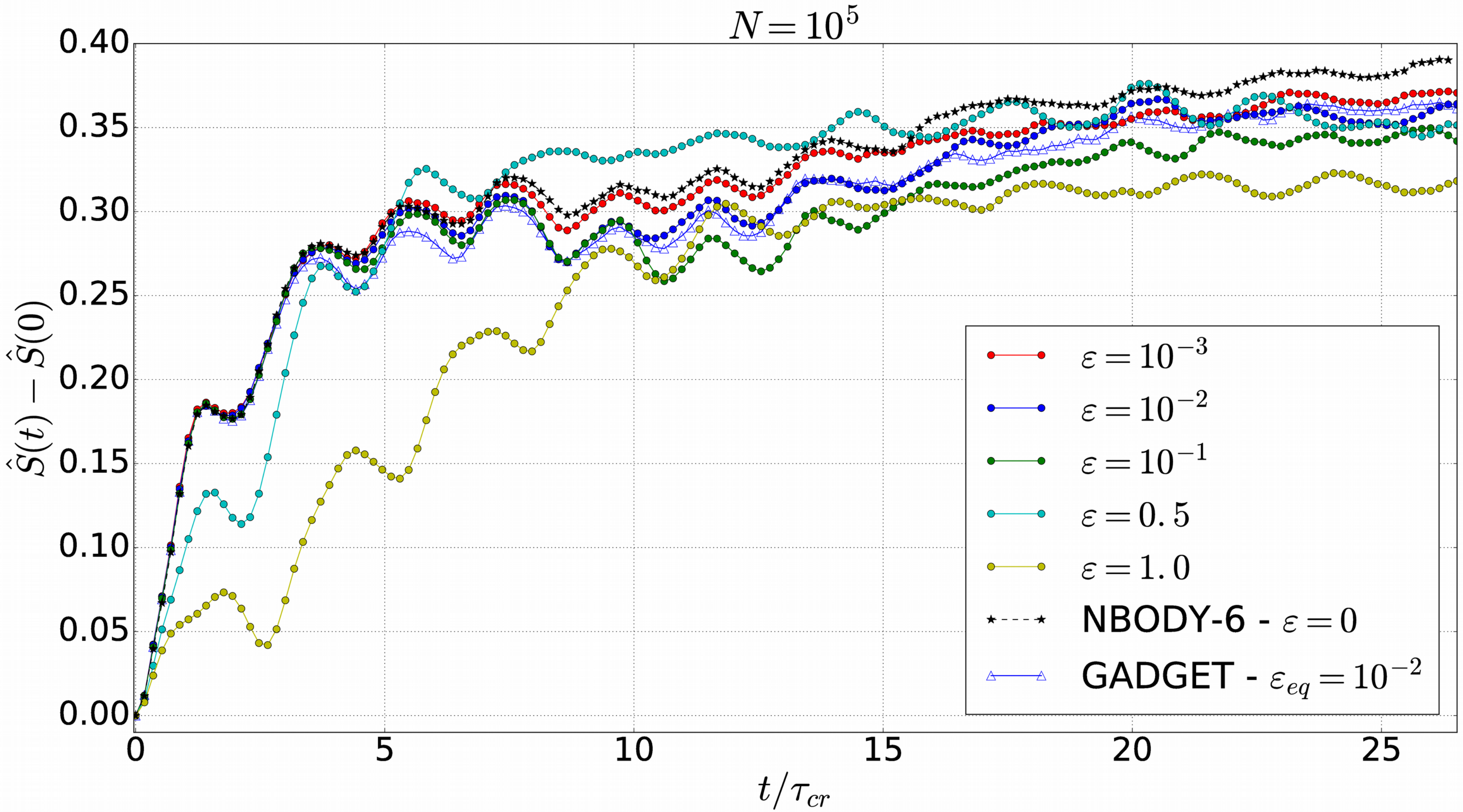
- Possible application to Gaia data

# Summary

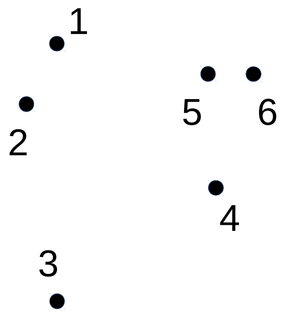
- Violent relaxation:
  - Entropy increase (**macroscopic** irreversibility)
  - $\rightarrow$ Non-validity of Vlasov-Poisson
  - Theoretical alternative?
- Long-term evolution:
  - Collisional relaxation $(R/N \lesssim b \lesssim R)$
  - Agreement with theory
- Possible applications:
  - Testing other theoretical transport equations
  - Constraining Milky Way potential

# Different N-body codes

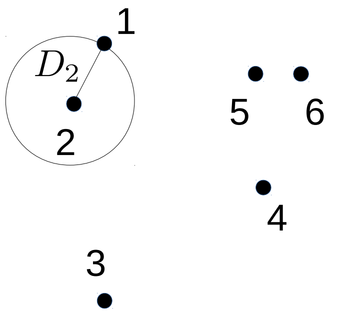



– Same entropy evolution
– Suppression only for $\varepsilon > \bar{d} \approx 0.02$

| NN estimator | Kernel estimator |
| --- | --- |



# NN estimator




$$D_2 = \sqrt{(\vec{x}_2 - \vec{x}_1)^2 + (\vec{v}_2 - \vec{v}_1)^2}$$

# Kernel estimator

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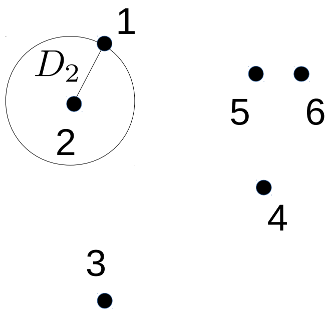


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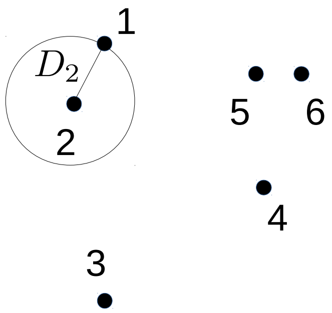


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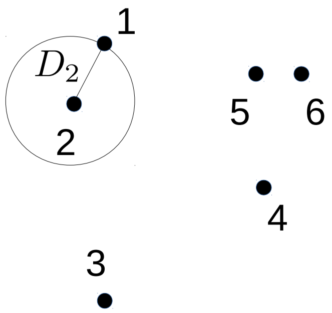


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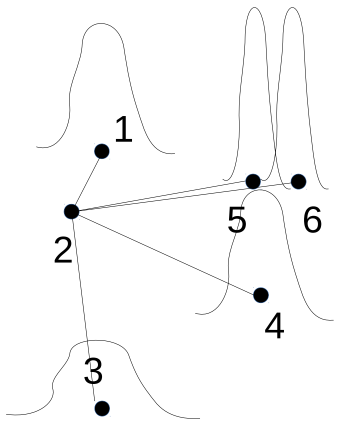

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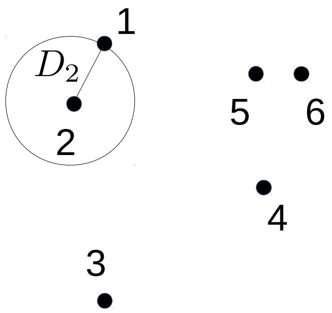


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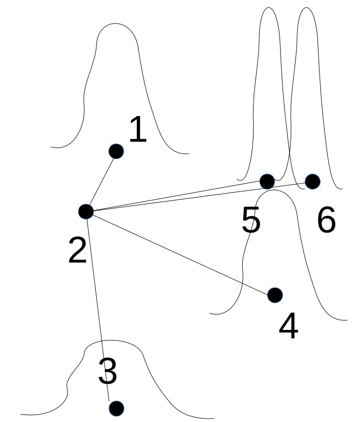


$$\hat{f}_i = \frac{A}{N}\sum_{j=1}^{N} \frac{1}{h_j^6} K\left(\frac{D_{ij}}{h_j}\right)$$

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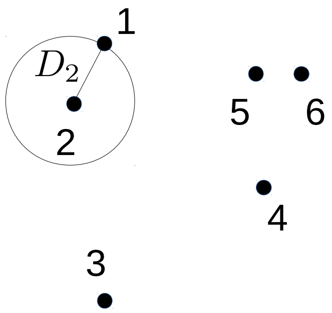


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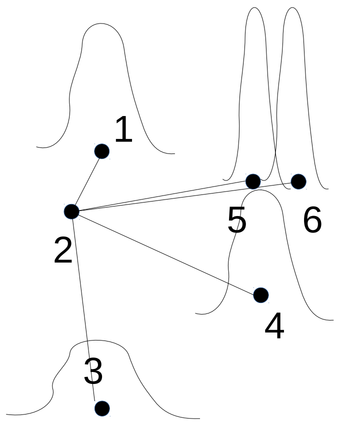


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Heavy tails; Hall, Morton (1993)

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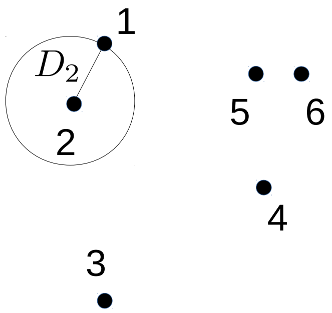


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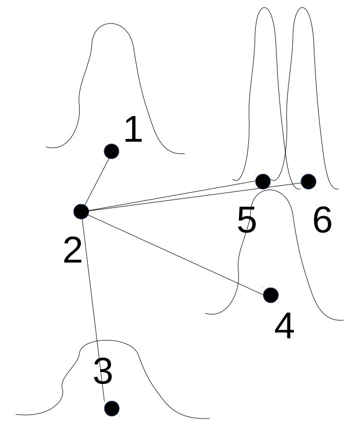


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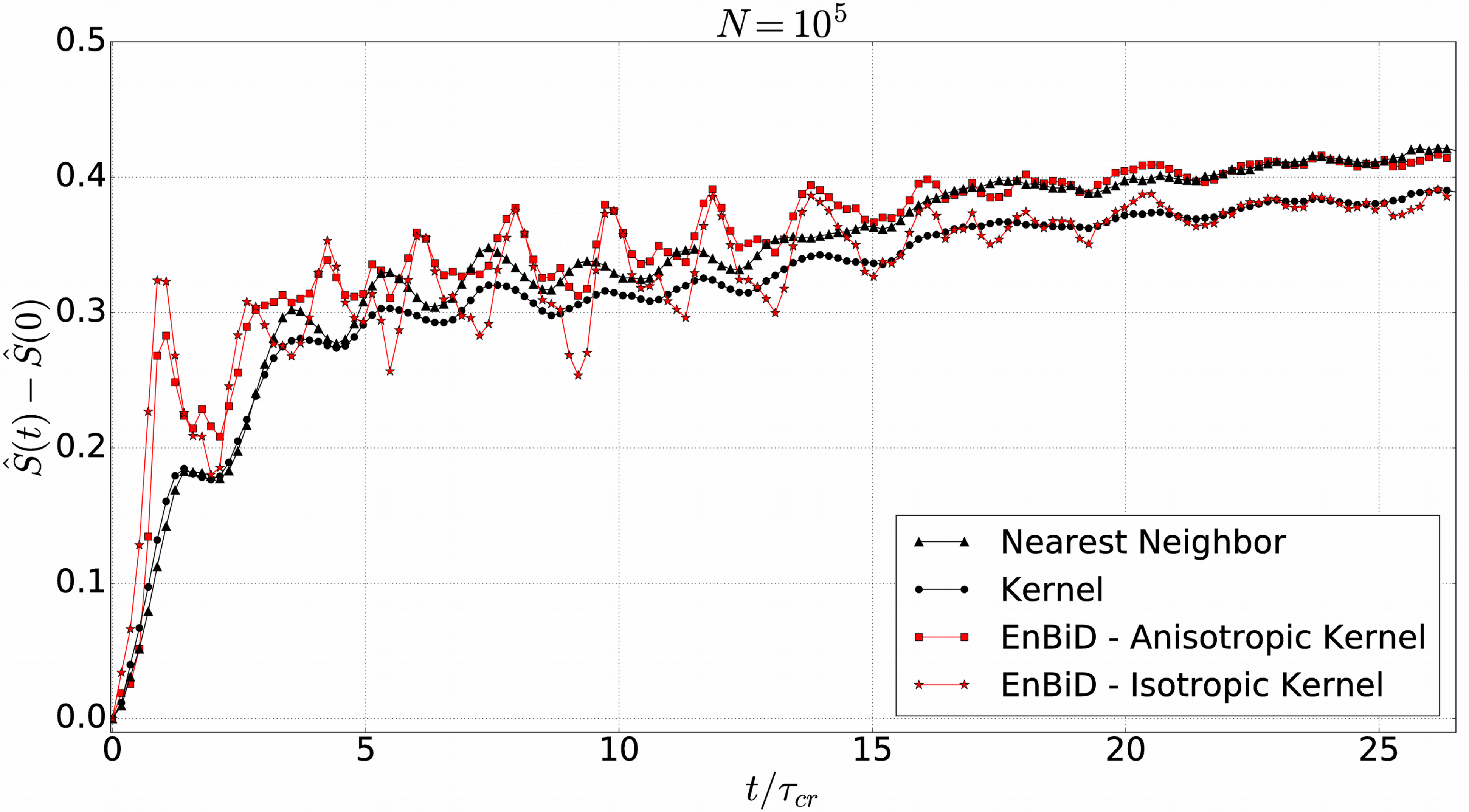

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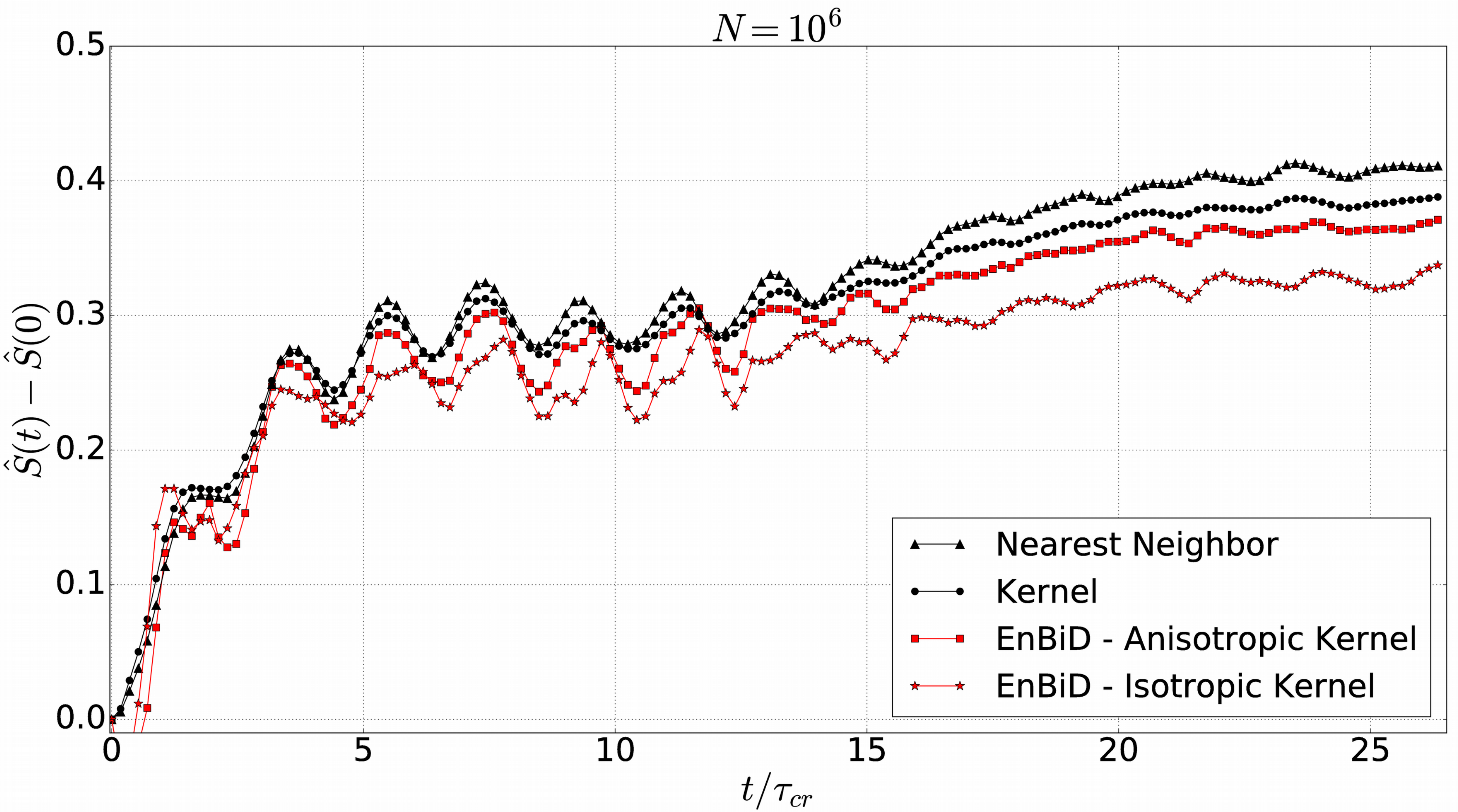
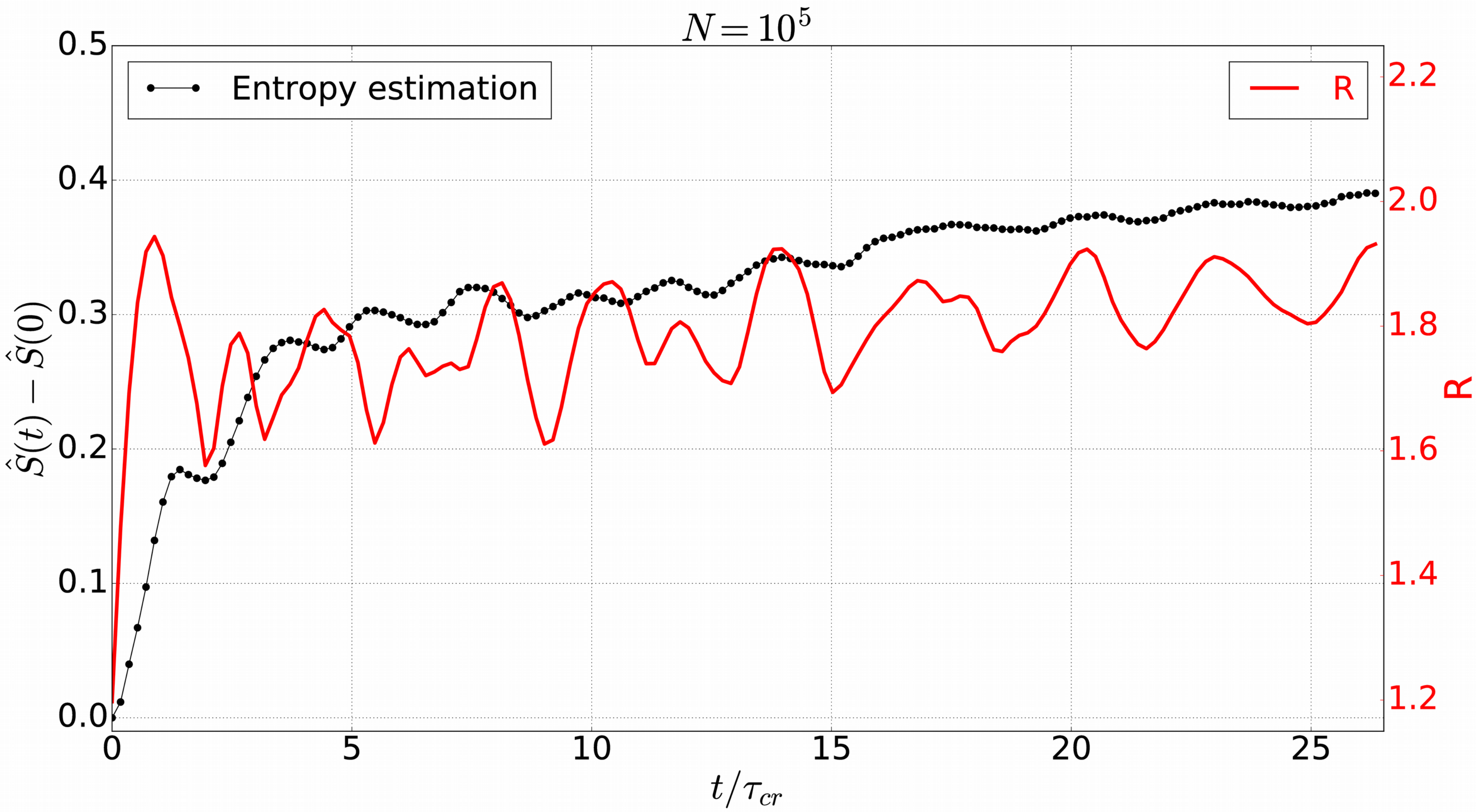
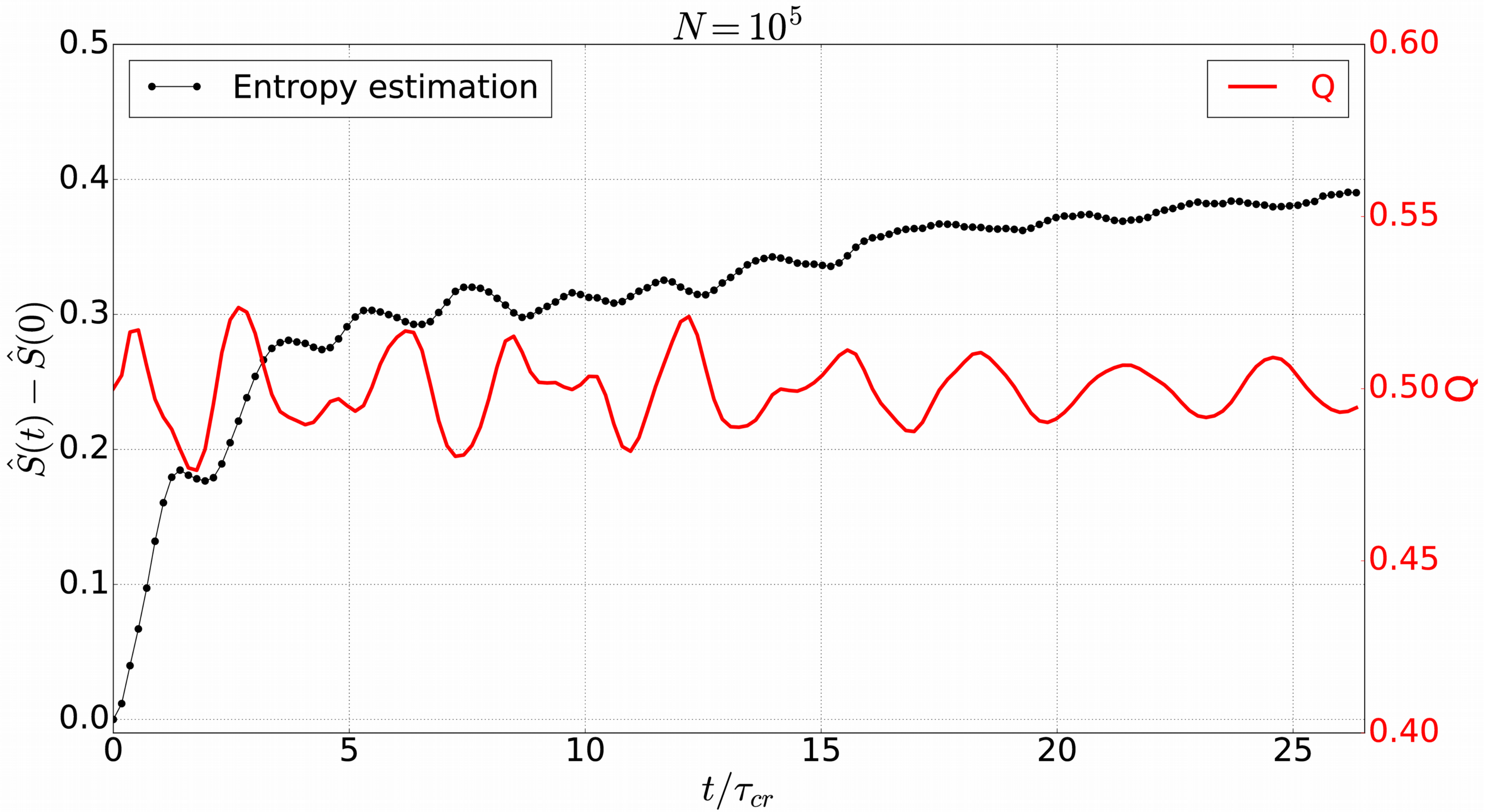



Faster convergence of NN and Kernel



$N = 10^5$
Entropy estimation
R
0.5
0.4
0.3
0.2
0.1
0.0
2.2
2.0
1.8
1.6
1.4
1.2
0
5
10
15
20
25
$\hat{S}(t) - \hat{S}(0)$
R
$t/\tau_{cr}$



$N = 10^5$
Entropy estimation
Q
0.5
0.4
0.3
0.2
0.1
0.0
$\hat{S}(t) - \hat{S}(0)$
0.60
0.55
0.50
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$$S \equiv -\int f \ln f d^3\vec{x} d^3\vec{v} \quad \rightarrow \frac{dS}{dt} = -\int (1 + \ln f)\Gamma[f] d^3\vec{x} d^3\vec{v}$$

$$\frac{d\hat{S}}{dt} = -\frac{1}{N}\sum_{i=1}^{N}\frac{(1+\ln \hat{f}_i)}{\hat{f}_i}\Gamma[\hat{f}_i]$$

$$\hat{S}(t+\Delta t) = \hat{S}(t) + a \cdot \frac{d\hat{S}}{dt}(t)\Delta t \qquad \boxed{a = \ln \Lambda}$$

Agama: Smooth $\phi(r)$ $g(E)$ $f(E)$

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# Long-term evolution

- Collisional relaxation?
- Orbit-averaged Fokker-Planck
  - Weak encounters, static potential, $f = f(E)$

$$S \equiv -\int f \ln f d^3\vec{x} d^3\vec{v} \quad \rightarrow \frac{dS}{dt} = -\int (1 + \ln f)\Gamma[f] d^3\vec{x} d^3\vec{v}$$

$$\frac{d\hat{S}}{dt} = -\frac{1}{N} \sum_{i=1}^{N} \frac{(1 + \ln \hat{f}_i)}{\hat{f}_i} \Gamma[\hat{f}_i]$$

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Agama: Smooth $\phi(r)$ $g(E)$ $f(E)$ $\frac{df}{dE}$ $\frac{d^2 f}{dE^2}$

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# Long-term evolution

- Collisional relaxation?
- (Orbit-averaged) Fokker-Planck $\frac{df}{dt} = \Gamma_{FP}[f]$
  - Weak encounters $(b \gtrsim b_{90})$, static potential, $f = f(E)$

$$S \equiv -\int f \ln f d^3\vec{x} d^3\vec{v} \quad \rightarrow \frac{dS}{dt} = -\int (1 + \ln f)\Gamma[f] d^3\vec{x} d^3\vec{v}$$

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Agama: Smooth $\phi(r)$ $g(E)$ $f(E)$ $\frac{df}{dE}$ $\frac{d^2 f}{dE^2}$ $\langle \Delta E \rangle$ $\langle (\Delta E)^2 \rangle$

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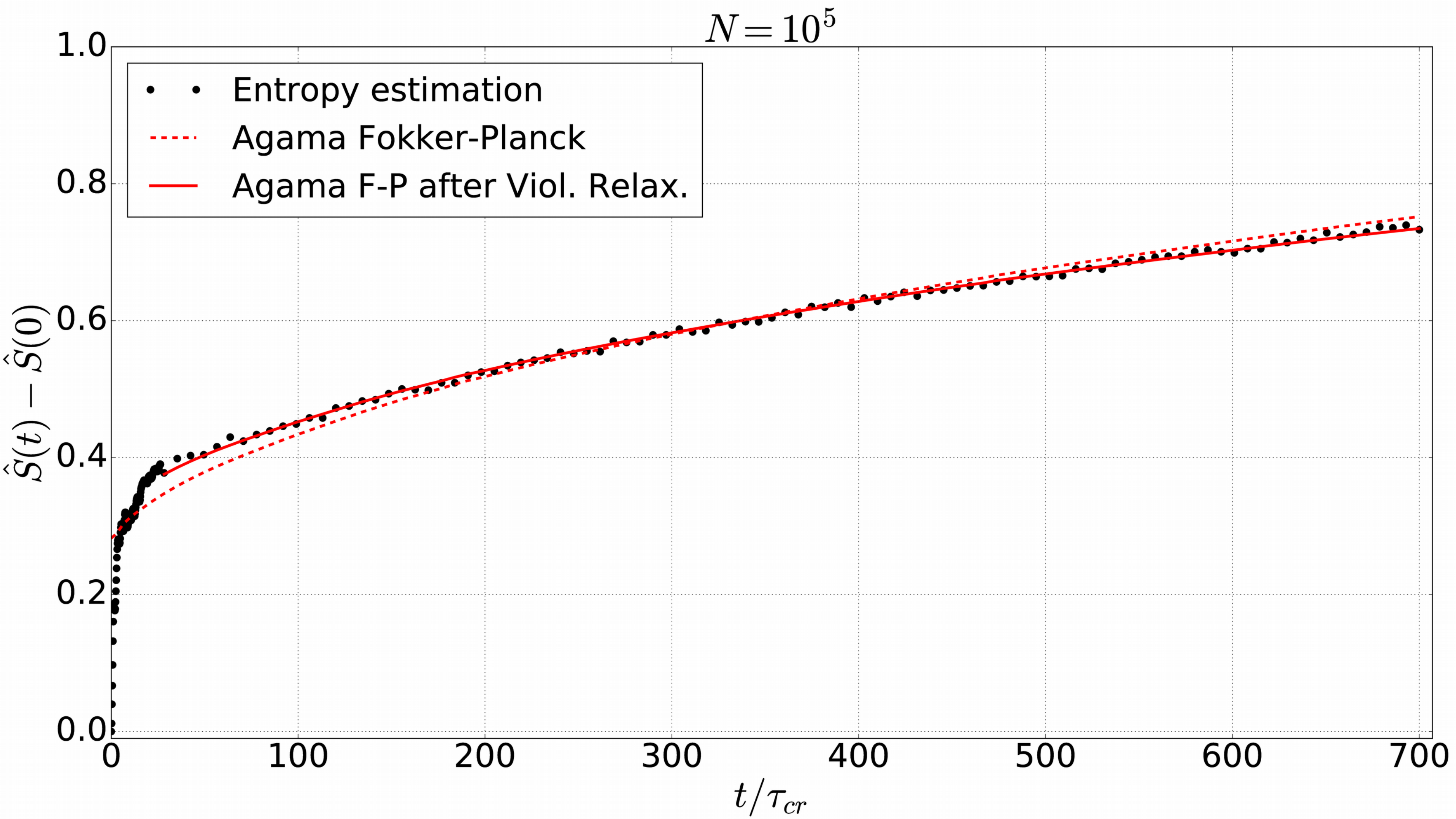
# Long-term evolution




Collisional relaxation explains long-term evolution