Weakly unstable stationary states of Vlasov equation

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Vlasov equation

Vlasov equation for $f(t,x,v)$, phase space density:

$$\partial_t f + v\nabla_x f - \nabla_x \phi \nabla_v f = 0$$
$$\phi(x) = \int f(t,y,v) V(x-y) dy\ dv.$$

$V(x) =$ interaction potential.

Goal: a qualitative study (stationary state, stability, instabilities, **bifurcations**, asymptotic behavior...)

Neighborhood of a stationary state

- Vlasov equations have many stationary states
  $\rightarrow$ a selection principle?
- Linear and non linear stability analysis, dynamics in the vicinity of a stable stationary state (Landau damping): old, rich and lively subject...
- Question in this talk: what happens close to a weakly unstable stationary state?
  "weakly" = one (or several) eigenvalues with a small positive real part
  $\rightarrow$ hope for a perturbative approach; a bifurcation theory question.

Example 1: homogeneous background

• Example 1: 1D, interaction potential $V(x) = 1 - \cos x$, $\Omega =]-\pi, \pi]$, periodic boundary conditions.
$F_\beta(v) \propto e^{-\beta v^2/2}$, stable for $\beta \leq 2$, unstable for $\beta > 2$.

Movie: homogeneous background

$\rightarrow$ complex dynamics leading to saturation (cat's eye pattern)

Examples 2 and 3: non homogeneous background

- Example 2: 1D, $V(x) = 1 - \cos x$, non homogeneous case

$$F_\mu(x, v) \propto \frac{1}{1 + e^{\beta(v^2/2 - M_\mu \cos x - \mu)}} \, , \; M_\mu = \iint \cos x F_\mu(x, v) dx \, dv.$$

The family F_μ undergoes a bifurcation for a certain μ_c.

Movie 1: Perturbation $+\epsilon$ Movie 2: Perturbation $-\epsilon$

$\rightarrow$Very different non linear dynamics from Ex.1.

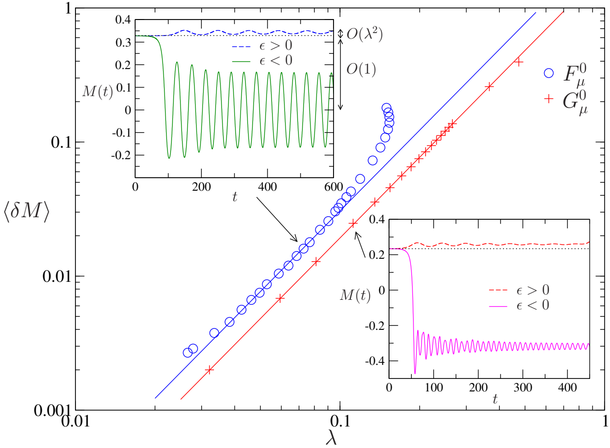
- Example 3: $\Omega = \mathbb{R}^3$, $V(x) = -\frac{C}{|x|}$. Radial Orbit Instability.

$\rightarrow$Similarities with Ex. 2.

Common features Ex. 2 and 3: appearance of a new stationary state close to the reference one; dependence on initial condition. **Explanation?**

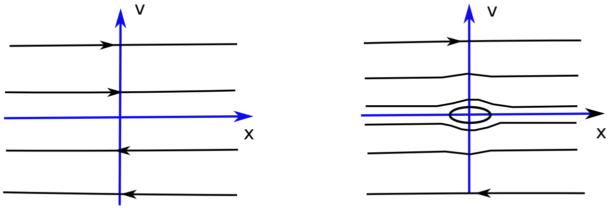
Perturbation potential vs Time for example 2

$\delta M \simeq$ norm of the perturbation ($-\cos x$ interaction).



Bifurcation from a homogeneous stationary state

Old problem in plasma physics, with an interesting history (O'Neil, Crawford, Del-Castillo-Negrete, Balmforth et al.......).
Main feature / difficulty = resonance:

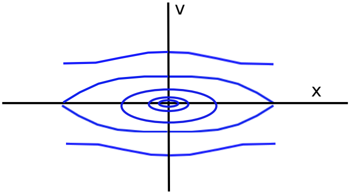


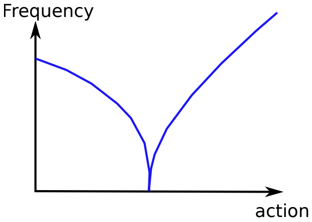
Left: homogeneous stationary state
Right: perturbed homogeneous stationary state $\rightarrow$ resonance
Messages: strong non linear effects, naive computations diverge; universal dynamics close to threshold ("Single Wave Model").

NON homogeneous stationary state

Question: what about the resonances?
Simple example: inhomogeneous stationary state = pendulum dynamics:

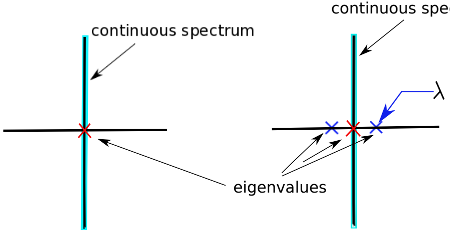
$$\partial_t f + v\partial_x f - M\sin(x-\varphi)\partial_v f = 0\ ,\ \ Me^{i\varphi} = \iint e^{ix} f(x,v)dx\ dv.$$





Real eigenvalue (zero frequency), few particles with zero velocity
$\rightarrow$ weak resonance.

Linear analysis



Before bifurcation *After bifurcation*

Structure of the linearized operator at the bifurcation, restricted to $E = \mathrm{Vect}(\psi_0, \psi_1, \psi_2)$:

$$L = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Goal: build a local invariant manifold + a reduced dynamics on it.

Non linear analysis, order 2

$$f(x, v, t) = F_{\mu_c}(x, v) + g(x, v, t)$$

Representing the perturbation:

$$g = A_0(t)\psi_0 + A_1(t)\psi_1 + A_2(t)\psi_2 + H[A_0, A_1, A_2]$$

Reduced dynamics for the A_i, quadratic order (no divergence here!):

$$\begin{aligned}
\dot{A}_0 &= A_1 + \lambda^2 b A_1 + \alpha_{01} A_0 A_1 \\
\dot{A}_1 &= (1 + \lambda^2 c) A_2 + \lambda^2 a A_0 + \beta_{00} A_0^2 + \beta_{02} A_0 A_2 \\
\dot{A}_2 &= \gamma_{01} A_0 A_1,
\end{aligned}$$

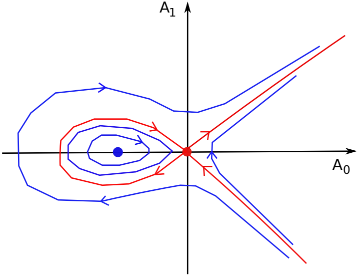
All coefficients have explicit (but complicated) expressions.

Analysis of the reduced dynamics

Conserved quantity:

$$G = A_2 - \frac{\gamma_{01}}{\alpha_{01}} A_0 + \frac{\gamma_{01}(1+\lambda^2 b)}{\alpha_{01}^2} \ln\left(1 + \frac{\alpha_{01}}{1+\lambda^2 b} A_0\right).$$

Typical initial conditions: close to $(0,0,0) \to G \simeq 0$.



New fixed point: at distance $O(\lambda^2)$ from $(0,0,0)$.

Conclusion: main messages

- Bifurcation from a non homogeneous stationary solutions: may be very different from the homogeneous case. In particular, much **weaker resonances**.
- There seems to be **some universality** for this new type of bifurcation (linearized structure, new fixed point "close" to the reference stationary state, dependence on the initial condition...)
- **A 3D reduced dynamics** has been obtained, which reproduces qualitatively very well the observations.
- I have no theorem, but worse than that: it is not quite clear what could a theorem be $\rightarrow$ many questions! Might be easier than in the homogeneous case...

Comparison Vlasov/reduced dynamics

- Purple = potential perturbation, Vlasov Simulations; initial conditions = slightly unstable
- Green = potential perturbation, reduced model.