# About Maxwell Boltzmann Equation.

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# What is the Maxwell Boltzmann Equation ? I

In $\Omega \subset \mathbf{R}^d_x \times \mathbf{R}^d_v$ Assumption: Interactions between Electrons and Ions only due to electro magnetic forces with external given Magnetic Field $B$.

$$\partial_t f_+ + v \cdot \nabla_x f_+ + (E + v \times B)\nabla_v f_+ = \eta_+ \mathcal{C}_+(f_+)\,,$$
$$St_- \partial_t f_- + v \cdot \nabla_x f_- - (E + v \times B)\nabla_v f_- = \eta_- \mathcal{C}_-(f_-)\,.$$
$$f_\pm(t,x,v) = f_\pm(t,x,v - 2(v \cdot \vec{n})\vec{n}(x)) \quad \text{Specular reflection if } \partial\Omega \neq \emptyset$$
$$-\lambda^2 \Delta\phi = \int_{\mathbf{R}_v} f_+(x,v,t)dv - \int_{\mathbf{R}_v} f_-(x,v,t)dv\,, \partial_{\vec{n}}\phi = 0\,, \text{on } \partial\Omega\,.$$

The two kernels $\mathcal{C}_\pm$ satisfies the standard hypothesis: Conservation of mass momentum and energy and $\mathcal{H}$ theorem: Boltzmann, BGK, Fokker Planck ...Other collision operator.???

# What is the Maxwell Boltzmann Equation ? II

With a Korn type hypothesis on $\Omega$:

$$f_-(x,v,t) \simeq (\frac{\beta(t)}{2\pi})^{\frac{d}{2}} e^{\beta(t)(\frac{|v|^2}{2}-\phi(x,t))}$$

$$\Rightarrow \langle f_-\rangle = e^{\beta(t)\phi(x,t)}\,, \int_\Omega \frac{|v|^2}{2} f_-(x,v,t)\rangle dx = \frac{d}{\beta(t)}\int_\Omega \langle f_-\rangle dx = \frac{m_0 d}{\beta(t)}$$

$$-\lambda^2\Delta\phi + e^{\beta(t)\phi(x,t)} = \langle f_+\rangle \quad \partial_{\vec{n}}\phi = 0 \text{ on } \partial\Omega\,,$$

$$\partial_t f_+ + v\cdot\nabla_x f_+ - \nabla_x\phi\cdot\nabla_v f_+ = \eta_+ C_+(f_+)\,.$$

$\beta(t)$ is the inverse of the temperature. What beta?

# Want I can do, what I want to do

- Try to justify the scaling. From a mathematical poin tof view..Because they work!

- Prove a convergence theorem under an extra regularity assumption

- Give a complete treatement of the so called reduced ions problem.

As final remarks

- Give a Arnold type stability result for the asymptotic solutions.
- Combine with the quasineutral limit.

# Convergence to the Ions problem

Consider in a domain $\Omega$ which satisfies the Korn hypothesis, the problem

$$\partial_t f_+^\epsilon + v\cdot\nabla_x f_+^\epsilon - \nabla_x\phi_\epsilon\cdot\nabla_v f_+^\epsilon = 0$$
$$St_-^\epsilon\partial_t f_-^\epsilon + v\cdot\nabla_x f_-^\epsilon + \nabla_x\phi_\epsilon\cdot\nabla_v f_-^\epsilon = \eta_-(\epsilon)\mathcal{C}_-(f_-^\epsilon)\,,$$
$$-\lambda^2\Delta\phi_\epsilon = \langle f_+^\epsilon\rangle - \langle f_-^\epsilon\rangle$$

With if $\partial\Omega\neq\emptyset$ the following boundary conditions

$$f_\pm^\epsilon(t,x,v) = f_\pm^\epsilon(t,x,v-2(v\cdot\vec{n}))\vec{n}(x) \text{ Specular reflection}$$
$$-\lambda^2\Delta\phi_\epsilon = \langle f_+^\epsilon\rangle - \langle f_-^\epsilon\rangle \quad \partial_n\phi^\epsilon = 0 \text{ Neumann boundary condition}$$

## Theorem

*Assume that for* $\epsilon \to 0$ *one has* $\liminf (St^{\epsilon}_{-})^{-1}\eta_\epsilon = \infty$, *and* $\limsup \lim_{\epsilon\to 0} \eta_\epsilon < +\infty$ *while for* $t \in (0, T)$ *the distributions* $f^{\epsilon}_{\pm}(x, v, t)$ *remains uniformly bounded in a convenient class (specified below) then:*

$$(f^{\epsilon}_{-}, f^{\epsilon}_{+}) \Rightarrow ((\frac{\beta(t)}{2\pi})^{\frac{d}{2}} e^{-\beta(t)(\frac{|v|^2}{2} - \phi(x,t))}, f_{+}(x, v, t))$$

$$-\lambda^2 \Delta\phi + e^{\beta(t)\phi(x,t)} = \langle f_{+}(t)\rangle \quad \partial_{\vec{n}}\phi = 0 \text{ on } \partial\Omega\,,$$

$$\partial_t f_{+} + v\cdot\nabla_x f_{+} - \nabla_x\phi\cdot\nabla_v f_{+} = \eta_{+} C_{+}(f_{+})\,.$$

*and* $\beta(t)$ *is uniquely determined by the conservation of energy:*

$$\int_\Omega \frac{|v|^2}{2}\langle f_{+}(t)\rangle dx + \frac{m_0 d}{\beta(t)} + \frac{1}{2}\int_\Omega |\nabla_x\phi(x,t)|^2 dx = \mathcal{E}_0\,.$$

# Proof of the "soft theorem" Conservation of Mass

Assume that two kernels $\mathcal{C}_\pm$ satisfies the standard hypothesis: Conservation of mass momentum and energy and $\mathcal{H}$ theorem then for smooth solutions one has with well prepared initial data:

$$\langle \mathcal{C}_\pm(f_\pm)\rangle = 0 \Rightarrow \frac{d}{dt}\int_\Omega \langle f_\pm\rangle(x,t)dx = 0$$

$$\Rightarrow \int_\Omega \langle f_-\rangle(x,t)dx = \int_\Omega \langle f_+\rangle(x,t)dx = M_0$$

And the elliptic problem (with Neumann or periodic boundary data)

$$-\Delta\phi = \langle f_+\rangle - \langle f_-\rangle \quad \partial_{\vec{n}}\phi_{|\partial\Omega} = 0\,, \int_\Omega \phi(x)dx = 0$$

is well posed.

# Proof of the "soft theorem" Conservation of Energy

Use $\langle \mathcal{C}_\pm(f_\pm)\frac{|v|^2}{2}\rangle = 0$ and obtain formally

$$\frac{d}{dt}\Big(\int_\Omega (\langle \frac{|v|^2}{2}(f_- + f_+)\rangle + \frac{1}{2}|\nabla_x \phi(x,t)|^2)dx = 0$$

$$\int_\Omega (\langle \frac{|v|^2}{2}(f_- + f_+)\rangle + \frac{1}{2}|\nabla_x \phi(x,t)|^2)dx = \mathcal{E}_0$$

For the Ions problem with the Boltzmann Maxwell Relation

$$f_-(x,v,t) \simeq \Big(\frac{\beta(t)}{2\pi}\Big)^{\frac{d}{2}} e^{-\beta(t)(\frac{|v|^2}{2}-\phi(x,t))}$$

$$\frac{m_0 d}{\beta(t)} + \frac{1}{2}\int_\Omega |\nabla_x \phi(x,t)|^2 dx = \mathcal{E}_0 - \int_\Omega \frac{|v|^2}{2}\langle f_+\rangle dx\,.$$

# Proof of the "soft theorem" Stationnary solution

**Theorem**

*Under a Korn Hypothesis any local Maxwellian*

$$f_-(x,v)=\rho(x)\Big(\frac{\beta(x)}{2\pi}\Big)^{\frac{d}{2}}e^{-\beta(x)\big(\frac{|v-u(x)|^2}{2}}$$

*solution of the equation:*

$$\{\mathcal{E},f_-\}=v\cdot\nabla_x f_- +\nabla_x\phi\cdot\nabla_v f_- =0$$

*is of the following form:*

$$f_-(x,v)=\Big(\frac{\beta}{2\pi}\Big)^{\frac{d}{2}}e^{-\beta\big(\frac{|v|^2}{2}-\phi(x)\big)}$$

For any smooth vector $u : \Omega \mapsto \mathbb{R}^d$ with $u \cdot n = 0$ one has

$$\|\frac{\nabla u + \nabla u^t}{2}\|_{L^2(\Omega)} \geq \overline{K}(\Omega)\|\nabla u\|^2_{L^2(\Omega)} \tag{1}$$

is generally valid.
It holds for the flat torus $\mathbb{T}^d$ and in dimension 2 and 3 if $\Omega$ has no axis of symmetry.

*Cf. Proposition 13. of Desvillettes and Villani. Invent. Math. 159 (2005) and Desvillettes and Villani ESAIM: Control, Optimisation and Calculus of Variations June 2002, Vol. 8.*

# Proof II

$f_-$ is a Maxwellian

$$f_-(x,v) = \rho(x)(\frac{\beta(x)}{2\pi})^{\frac{d}{2}} e^{-\beta(x)(\frac{|v-u(x)|^2}{2})}$$

$$\nabla_x f_- = f_-\left( (\frac{\nabla_x \rho(x)}{\rho(x)} - \frac{d}{2}\frac{\nabla_x \beta}{\beta}) + \beta(x)\nabla_x u \cdot (v-u(x)) \right.$$

$$\left. - \frac{|v-u(x)|^2}{2}\frac{\nabla_x\beta}{\beta} \right).$$

$$\nabla_v f_-(x,v,t) = -\beta(v-u))f_-$$

$$\Rightarrow (B\times v)\nabla_v f_- = -\beta B\times v\cdot(v-u)f_- = -\beta(B\times u)(v-u)f_-$$

# Proof III Identification of the Terms of Order $3, 2, 1, 0$ in $(v-u)$.

Order 3 $(v-u)|v-u|^2 \quad \Rightarrow \nabla_x \beta = 0$ ,
Order 2 $(v-u)\nabla_x u\cdot(v-u) \quad \Rightarrow \frac{\nabla_x u + \nabla_x u^t}{2} = 0$ With Korn inequality
$\Rightarrow \nabla u = 0$ ,
Order 1 $(v-u) \Rightarrow \quad \Rightarrow (\frac{\nabla_x \rho(x,)}{\rho(x)} - \beta(\nabla\phi + B\wedge u)) = 0$
Order 0 $u\cdot\nabla_x \log\rho(x) = 0$ .
With $\nabla\beta = 0$ and $u = 0$ ($u$ constant and tangent to the boundary) only remain the equations:

$$\nabla\log(\rho(x)) = \beta(\phi(x) \Rightarrow f_-(x,v) = (\frac{\beta}{2\pi})^{\frac{d}{2}} e^{-\beta(\frac{|v|^2}{2} - \phi(x))}$$

with $\phi$ changed into $\phi + (\log \mathrm{constante})/\beta$

# Determination of $\beta$ through energy conservation

**Theorem**

*Let $\Omega \subset \mathbb{R}^d_x$ be a bounded domain and $\mathcal{E} > 0$. Fix a nonnegative ion density $I(x) \in L^2(\Omega)$ with finite mass $m_0$. Then, there exists a unique solution $(\beta, \phi)$ to the following elliptic problem:*

$$-\Delta\phi + e^{\beta\phi} = I(x), \qquad \frac{\partial\phi}{\partial n}_{|\partial\Omega} = 0 \tag{2}$$

*together with the mass and energy constraints*

$$\frac{m_0 d}{\beta} + \frac{1}{2}\int_\Omega |\nabla_x\phi(x,t)|^2 dx = \mathcal{E}$$

*Below Sentis direct proof. Idea The energy is an increasing function of the temperature.*

For each fixed $\beta > 0$, the elliptic problem has a unique solution $\phi^\beta \in H^2(\Omega)$ with total mass $m_0$ and "energy"

$$\mathcal{E}(\beta) = \frac{m_0 d}{\beta} + \frac{1}{2}\int_\Omega |\nabla_x \phi^\beta(x,t)|^2 dx$$

and derivative solution of

$$-e^{-\beta\phi^\beta}\Delta\partial_\beta\phi^\beta + \beta\partial_\beta\phi^\beta = -\phi^\beta\,, \qquad \frac{\partial\partial_\beta\phi^\beta}{\partial n}_{|\partial\Omega} = 0$$

Hence

$$\partial_\beta\mathcal{E}(\beta) = -\frac{m_0 d}{\beta^2} + \int_\Omega \nabla_x\phi^\beta\nabla_x\partial_\beta\phi^\beta dx = -\frac{m_0 d}{\beta^2} - \int_\Omega \phi^\beta\Delta_x\partial_\beta\phi^\beta dx$$

$$= -\frac{m_0 d}{\beta^2} - \int_\Omega e^{-\beta\phi^\beta}(\Delta_x\partial_\beta\phi^\beta)^2 dx + \beta\int_\Omega \partial_\beta\phi^\beta\Delta_x\partial_\beta\phi^\beta dx$$

$$= -\Big(\frac{m_0 d}{\beta^2} + \int_\Omega e^{-\beta\phi^\beta}(\Delta_x\partial_\beta\phi^\beta)^2 dx + \beta|\nabla_x\partial_\beta\phi^\beta|^2 dx\Big) \leq 0$$

# End of determination of $\beta$

For the existence

1 One has $\mathcal{E}(\beta) > (m_0 d)\beta^{-2} \Rightarrow \lim_{\beta\to 0}\mathcal{E}(\beta) = \infty$

2 From the elliptic equation for $\phi^\beta$,

$$\int_\Omega |\nabla\phi^\beta|^2 dx = \int_\Omega \left(I(x)\phi^\beta(x) - e^{\beta\phi^\beta}\phi^\beta\right)dx$$

$$\leq \int_{\{\phi^\beta\geq 0\}} (I(x)\phi^\beta(x) - e^{\beta\phi^\beta}\phi^\beta)dx - \frac{1}{\beta}\int_{\{\phi^\beta\leq 0\}} e^{\beta\phi^\beta}\beta\phi^\beta dx.$$

$x \geq 0 \Rightarrow e^x \geq x \Rightarrow$

$$\int_{\{\phi^\beta\geq 0\}} (I(x)\phi^\beta(x) - e^{\beta\phi^\beta}\phi^\beta)dx \leq \|I\|_{L^2}\|\phi^\beta\|_{L^2} - \beta\|\phi^\beta\|^2_{L^2} \leq \frac{1}{2\beta}\|I\|^2_{L^2}$$

$x \leq 0 \Rightarrow xe^x \leq e^{-1} \Rightarrow$

$$\frac{1}{\beta}\int_{\{\phi^\beta\leq 0\}} e^{\beta\phi^\beta}\beta\phi^\beta dx \leq \frac{|\Omega|e^{-1}}{\beta}.$$

Therefore $\lim_{\beta\to\infty}\mathcal{E}(\beta) = 0$ and existence of $\beta$ follows.

# Uniform regularity assumptions: URA

In the ions equation there is no non linear relaxation term the solution remain $f_+^\epsilon$ remain bounded in any $L^p(\Omega \times \mathbb{R}_x^d)$ . The energy remains bounded. Therefore the only hypothesis needed is the convergence of this energy

For almost every $t \in (0, T)$ $\lim_{\epsilon \to 0} \int_\Omega \langle \frac{|v|^2}{2} f_+^\epsilon(t) \rangle dx \to \int_\Omega \langle \frac{|v|^2}{2} \overline{f}_+^\epsilon(t) \rangle dx$

For the electrons because the role of the collision operator is essential the situation is more subtle and one has to assume the convergence of $f_-^\epsilon$ almost everywhere, and the uniform integrability ( a strong assumption?):

$$f_-^\epsilon \leq Ce^{-\delta |v|^m}$$

# End of proof of the soft theorem

With $\eta_-(\epsilon) > 0$ and bounded, $\lim (St_-^\epsilon)^{-1}\eta_-^\epsilon \to \infty$ start from

$$\partial_t f_+^\epsilon + v\cdot\nabla_x f_+^\epsilon - \nabla_x\phi_\epsilon\cdot\nabla_v f_+^\epsilon = 0 \tag{3a}$$

$$St_-^\epsilon \partial_t f_-^\epsilon + v\cdot\nabla_x f_-^\epsilon + \nabla_x\phi_\epsilon\cdot\nabla_v f_-^\epsilon = \eta_-^\epsilon \mathcal{C}_-(f_-^\epsilon) \tag{3b}$$

$$-\lambda^2\Delta\phi^\epsilon = \langle f_+^\epsilon(t)\rangle - \langle f_-^\epsilon\rangle \tag{3c}$$

the corresponding standard estimates and the Uniform regularity assumption.

$$\int_\Omega \langle f_+^\epsilon(t)\rangle dx = \int_\Omega \langle f_-^\epsilon(t)\rangle dx = m_0\,; \tag{4a}$$

$$0 \le f_+^\epsilon(x,v,t) = \sup_{x,v} f_+^\epsilon(x,v,0) \tag{4b}$$

$$\int_\Omega \langle \frac{|v|^2}{2} f_+^\epsilon(t)\rangle + \langle \frac{|v|^2}{2} f_-^\epsilon(t)\rangle + \frac{1}{2}\int_\Omega |\nabla_x\phi^\epsilon|^2 dx = \mathcal{E}_0 \tag{4c}$$

which imply in particular the uniform bound:

$$\|\phi^\epsilon(t)\|_{W^{2,\frac{d+2}{2}}}, \|\partial_t\phi^\epsilon(t)\|_{W^{1,\frac{d+2}{2}}} \le C_0$$

Then one deduces that $(f_-^\epsilon, f_+^\epsilon, \phi^\epsilon)$ converge (in a weak sense ) to $(\overline{f}^\epsilon_-, \overline{f}^\epsilon_+, \overline{\phi}^\epsilon)$ and that $(\overline{f}^\epsilon_+, \overline{\phi}^\epsilon)$ are solution of the system:

$$\partial_t \overline{f}^\epsilon_+ + v\nabla_x \overline{f}^\epsilon_+ - \nabla_v(\nabla_x \overline{\phi}^\epsilon \overline{f}^\epsilon_+) = 0 \tag{5}$$

$$-\lambda^2 \Delta \overline{\phi}_\epsilon = \langle \overline{f}^\epsilon_+ \rangle - \langle \overline{f}^\epsilon_- \rangle \tag{6}$$

Multiply (3b ) by $\log f_-^\epsilon$ and integrate over $\Omega \times \mathbf{R}^d_v \times (0, T)$ to obtain:

$$\begin{aligned} 0 &\leq -\int_0^T \int_\Omega \langle \mathcal{C}_-(f_-^\epsilon) \log f_-^\epsilon \rangle dxdt \\ &\leq \frac{St_-^\epsilon}{\eta_-^\epsilon} \Big( \int_\Omega \langle f_-^\epsilon \log f_-^\epsilon \rangle (x, 0) dx - \int_\Omega \langle f_-^\epsilon \log f_-^\epsilon \rangle (x, T) dx \Big) \to 0 \end{aligned} \tag{7}$$

Then with the uniform regularity hypothesis and the H theorem $\overline{f}^\epsilon_-$ is a local Maxwellian $M_{\rho,u,\theta}$ .

Letting $\epsilon \to 0$ in the equation

$$St_-^\epsilon \partial_t f_-^\epsilon + v \cdot \nabla_x f_-^\epsilon + \nabla_x \phi_\epsilon^\epsilon \cdot \nabla_v f_-^\epsilon = \eta_-^\epsilon \mathcal{C}_-(f_-^\epsilon) \tag{8}$$

one obtains:

$$\begin{aligned} & v \cdot \nabla_x M_{\rho,u,\theta} + \nabla_x \phi_\epsilon \cdot \nabla_v M_{\rho,u,\theta} \\ & = \lim_{\epsilon \to 0} v \cdot \nabla_x \overline{f}_-^\epsilon + \nabla_x \overline{\phi}_\epsilon \cdot \nabla_v \overline{f}_-^\epsilon \\ & = \lim_{\epsilon \to 0} (\eta_-^\epsilon \mathcal{C}_-(f_-^\epsilon) - St_-^\epsilon \partial_t f_-^\epsilon) = 0\,. \end{aligned} \tag{9}$$

Under the uniform regularity hypothesis the last term of (9) converges to 0 (at least in a weak sense) and this implies that

$$M_{\rho,u,\theta} = \left(\frac{\beta(t)}{2\pi}\right)^{\frac{d}{2}} e^{-\beta(t)(\frac{|v|^2}{2} - \phi(x,t))}, \tag{10}$$

Eventually taking the limit in the energy conservation equation:

$$\int_\Omega \langle \frac{|v|^2}{2} f_+^\epsilon(t)\rangle + \langle \frac{|v|^2}{2} f_-^\epsilon(t)\rangle + \frac{1}{2}\int_\Omega |\nabla_x \phi^\epsilon|^2 dx = \mathcal{E}_0 \qquad (11)$$

one reaches the reduced Ions system:

$$\begin{aligned} &\frac{m_0 d}{\beta(t)} + \frac{1}{2}\int_\Omega |\nabla_x \overline{\phi}^\epsilon(x,t)|^2 dx = \mathcal{E}_0 - \int_\Omega \frac{|v|^2}{2} \langle \overline{f}_+^\epsilon \rangle dx \\ &-\lambda^2 \Delta \overline{\phi}^\epsilon + e^{\beta(t)\overline{\phi}^\epsilon} = \langle \overline{f}_+^\epsilon \rangle \\ &\mathrm{St}_+ \partial_t \overline{f}_+^\epsilon + v \cdot \nabla_x \overline{f}_+^\epsilon - \nabla_x \overline{\phi}^\epsilon \cdot \nabla_v \overline{f}_+^\epsilon = 0\,. \end{aligned} \qquad (12)$$

# About Unconditionnal Statements I

- The biggest issue is the “stability " of the solution of the electron equation:

$$St_-^\epsilon \partial_t f_-^\epsilon + v \cdot \nabla_x f_-^\epsilon + \nabla_x \phi_\epsilon \cdot \nabla_v f_-^\epsilon = \eta_-(\epsilon)\mathcal{C}_-(f_-^\epsilon) \qquad (13)$$

The simultaneous presence of a Boltzmann type collision term that will relax to equilibrium and of large time asymptotic with the relation

$$\lim {St_-^\epsilon}^{-1}\eta_-(\epsilon) = \infty$$

seems compulsory and in agreement with the physical scalings.
Even for the genuine Boltzmann equation (no $\phi$ ) there is no general results and perturbations near an absolute maxwellian, because they involve the large time behaviour would handle only the trivial case $\phi = 0$.
Moreover in the above derivation the conservation of energy is a crucial factor.

# About Unconditionnal Statements II

At the present stage two type of results are available.

- The existence and uniqueness result for the reduced ions system. At variance with the oldest simple construction of weak solution of Vlasov equation here the compulsory conservation of energy requires a construction preserving some regularity.

- An Arnold stability both for the reduced ions problem and for the full system result because it is robust under weak limit (like the standard weak strong stability of compressible or incompressible Euler equation).

# Well posedness of the reduced ions problem

**Theorem (Existence of weak solutions)**

*Assume that initial data $f_{0,+} \in L^1 \cap L^\infty$, compactly supported in $v$. There exists a time $T > 0$ so that weak solutions $(f_+, \phi, \beta)$ to the ion problem so that $f_+$ remains compactly supported in $v$ and satisfies the estimates:*

$$f_+ \in C(0,T; L^1 \cap L^\infty(\Omega \times \mathbb{R}^d)), \qquad \rho_+ \in C(0,T; L^1 \cap L^\infty(\Omega)),$$

*the electric field $E = -\nabla\phi \in C(0,T; L^\infty(\Omega))$, and $\beta \in L^\infty([0,T])$. The solution can be extended globally in time for $d = 1, 2, 3$.*

*The non trivial part is the fact that the solutions remains of compact support in $v$. A priori estimates are given then the rest of the proof involves a classical iteration*

$$\frac{d}{dt}\iint_\Omega \frac{|v|^2}{2} f_+(x,v,t)\, dvdx = \int_\Omega \phi\nabla\cdot\langle v f_+\rangle dx = -\int_\Omega \phi\partial_t\langle f_+\rangle$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\phi|^2\, dx = -\int_\Omega \phi\Delta\phi_t = \int_\Omega \phi\partial_t\rho_+ - \int_\Omega \phi\partial_t e^{\beta\phi} \Rightarrow$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\langle \frac{|v|^2}{2} f_+(x,v,t)\rangle + |\nabla\phi|^2\, dx) = -\int_\Omega \phi\partial_t e^{\beta\phi} = -\frac{1}{\beta}\int_\Omega \beta\phi\partial_t e^{\beta\phi}$$

$$= -\frac{1}{\beta}\partial_t\int_\Omega (\beta\phi - 1)e^{\beta\phi}\, dx = -\frac{1}{\beta}\partial_t\int_\Omega \beta\phi e^{\beta\phi}\, dx$$

$$-\frac{m_0 d}{\beta^2}\partial_t\beta(t) + \frac{1}{2}\frac{d}{dt}\int_\Omega (\langle \frac{|v|^2}{2} f_+(x,v,t)\rangle + |\nabla\phi|^2\, dx) = 0$$

$$\partial_t(m_o d\log\beta(t) + \int_\Omega \beta\phi e^{\beta\phi}\, dx) = 0\,!!!$$

## II More Estimates on the elliptic problem 1

The Vlasov dynamic preserves the $L^\infty$ norm of $f_+(x,v,t)$ and with

$$\frac{m_0 d}{\beta(t)} + \int_\Omega \frac{|v|^2}{2}\langle f_+\rangle dx + \frac{1}{2}\int_\Omega |\nabla_x \phi(x,t)|^2 dx = \mathcal{E}_0$$

$$\text{one has} \quad \sup_{t\geq 0} \|\rho_+(\cdot,t)\|_{L^{\frac{d+2}{d}}(\Omega)} \leq C \text{ and } \beta(t) \geq \frac{m_0 d}{\mathcal{E}_0}$$

Multiplying by $e^{(p-1)\beta(t)\phi}$ $(-\Delta\phi + e^{\beta(t)\phi} = \langle f_+(t)\rangle)$ one obtains:

$$(p-1)\beta(t)\int_\Omega e^{(p-1)\beta(t)\phi}|\nabla\phi|^2\, dx + \int_\Omega e^{p\beta(t)\phi}\, dx$$

$$\leq \|\langle f_+(\cdot,t)\rangle\|_{L^p}\|e^{p\beta(t)\phi}\|_{L^1}^{\frac{p-1}{p}}$$

$$\Rightarrow 1 \leq p \leq \infty \quad \|e^{p\beta(t)\phi}\|_{L^p} \leq \|\langle f_+(\cdot,t)\rangle\|_{L^p}$$

$$+\text{ellipticicity}\|E = \nabla\phi\|_{L^\infty} \leq C\|\langle f_+(\cdot,t)\rangle\|_{L^1}^{\frac{1}{d}}\|\langle f_+(\cdot,t)\rangle\|_{L^\infty}^{\frac{d-1}{d}}$$

# III The Liouville Dynamic for Ions. A standard a-priori estimate for Vlasov equation.

The solution $f_+(x,v,t)$ of the ions equation is the pushforward of $f_+(x,v,0)$ by the flow given by the equations

$$\dot{X}(s) = V(s) \quad \dot{V}(s) = -\nabla_x \phi(X(s))$$

and the specular reflexion when ever $X(s)$ meets $\partial\Omega$. Hence one has

$$\begin{aligned} ||V(t)| - |V(0)|| &\leq \int_0^t \sup_{x\in\Omega} |\nabla\phi(x,s)| ds \\ &\leq C \int_0^t \|\langle f_+(\cdot,s)\rangle\|_{L^1}^{\frac{1}{d}} \|\langle f_+(\cdot,s)\rangle\|_{L^\infty}^{\frac{d-1}{d}} ds \end{aligned} \tag{14}$$

Then with $|v| \geq K_0 \Rightarrow f_+(x,v,0) = 0$ one has

$$f_+(x,v,0) = 0 \quad \text{for} \quad |v| \geq K_0 \Rightarrow f_+(x,v,t) = 0 \Rightarrow$$

$$\|\langle f_+(\cdot,t)\rangle\|_{L^\infty} \leq C_0(K_0 + \int_0^t \|\nabla\phi(\cdot,s)\|_{L^\infty} \, ds)^d$$

# Gronwall type a priori estimate:

Hence for $X(t) = (\|\langle f_+(\cdot,t)\rangle\|_{L^\infty})^{\frac{1}{d}}$ (for $d=2$) a linear and for ($d>2$) a non linear Gronwall estimate:

$$\begin{aligned}
&\|\langle f_+(\cdot,t)\rangle\|_{L^\infty}^{\frac{1}{d}} \leq \\
&\leq C_0(K_0 + \int_0^t (\|\langle f_+(\cdot,s)\rangle\|_{L^\infty}^{\frac{1}{d}})^{d-1}\,ds \\
&\Rightarrow\ X \leq C_0 + C_0\int_0^t X^{d-1}(s)ds
\end{aligned} \tag{15}$$

Moreover for $d=3$ the estimate can be made global following the proof of Schaeffer for the classical Vlasov equation or of Pallard in a periodic box. Pallard, Christophe A refined existence criterion for the relativistic Vlasov-Maxwell system. Commun. Math. Sci. 13 (2015), no. 2, 347–354.

# A Priori Estimate from Averaging Lemma

$$\partial_t f_+ + v \cdot \nabla_x f_+ = -\nabla_v (E f_+).$$

$E \in L^\infty$ and $f \in L^1 \cap L^\infty$.

$$\|f_+\|^2_{L^2(0,T;L^2(\Omega\times\mathcal{R}^3))} \leq \|f_+\|_{L^\infty} \|f_+\|_{L^1(0,T;L^1(\Omega\times\mathcal{R}^3))} \leq \|f_+\|_{L^\infty} \|\rho_+\|_{L^1(0,T)}$$

and

$$\|E f_+\|_{L^2(0,T;L^2(\Omega\times\mathcal{R}^3))} \leq \|E\|_{L^\infty} \|f_+\|_{L^2(0,T;L^2(\Omega\times\mathcal{R}^3))}.$$

$$\Rightarrow \int_{\mathcal{R}^3} f_+(t,x,v)\varphi(v)\, dv \in H^{1/4}((0,T)\times\Omega)$$

together with the uniform bound

$$\left\| \int_{\mathcal{R}^3} f_+(\cdot,\cdot,v)\varphi(v)\, dv \right\|_{H^{1/4}((0,T)\times\Omega)} \leq C_\varphi \|E\|_{L^\infty} \|f_+\|_{L^2(0,T;L^2(\Omega\times\mathcal{R}^3))}$$

for any test function $\varphi(v)$ in $C^\infty_c(\mathcal{R}^3)$.

# Construction of a solution by iteration I

- Start from $(\beta_n, \phi_n)$ solution of the elliptic problem

$$-\Delta\phi_n + e^{\beta_n\phi_n} = \langle f_n(x,v,t)\rangle = \rho_n(t)$$
$$\int_\Omega e^{\beta_n\phi_n}\, dx = m_0$$
$$\frac{m_0 d}{\beta_n} + \frac{1}{2}\int_\Omega |\nabla\phi_n|^2\, dx = \mathcal{E}_0 - \int_\Omega \langle \frac{|v|^2}{2} f_n(x,v,t)\rangle dx = \mathcal{E}_n(t)$$

- Then construct $f_{n+1}$ by solving the linearized Vlasov equation

$$\partial_t f_{n+1} + v\cdot\nabla_x f_{n+1} - \nabla_x\phi_n\cdot\nabla_v f_{n+1} = 0$$

with the same initial data $f_{n+1}(x,v,0) = f_{0,+}(x,v)$ and now with density and energy $(\rho_{n+1}(t), \mathcal{E}_{n+1}(t))$.

# Construction of a solution by iteration I

Modulo extraction of a subsequence $(f_n(x, v, t), \rho_n(x, t), \mathcal{E}(t), \beta(t)$ and $\phi_n(x, t)$ weakly converge and satisfy all the above a priori estimates!!!

With the averaging lemma and the fact that support of $f_n(x, v, t)$ remains bounded in velocity space the functions $\rho_n(x, t)$ and $\mathrm{E}_n(t)$ converge almost every where .

With the uniqueness of the solution of the elliptic problem

$$(\rho(x, t), \mathcal{E}(t)) \mapsto (\beta(t), \phi(x, t))$$

same is true for the sequence $(\beta_n(t), \phi_n(x, t))$!!

We same type of estimates one has the following

### Theorem (Uniqueness)

*Let $T > 0$. There exists at most one weak solution $(f_+, \phi, \beta)$ to the ion problem with v-compactly supported initial data $f_{0,+}$ so that*

$$\sup_{t\in[0,T]} \sup_{x\in\Omega} \|\nabla_v f_1\|_{L^2(\mathbb{R}^d)} + \int_0^T \|\nabla\phi(s,\cdot)\|_{L^\infty(\Omega)}\, ds < \infty. \qquad (16)$$

### Remark

*The estimate $\int_0^T \|\nabla\phi(s,\cdot)\|_{L^\infty(\Omega)}\, ds < \infty$. has been established above under the only hypothesis that initial data be of compact support in $v$ space.*
*The hypothesis $\sup_{t\in[0,T]} \sup_{x\in\Omega} \|\nabla_v f_1\|_{L^2(\mathbb{R}^d)} < \infty$ would be trivial in the absence of boundary effect but up to now not so clear in the presence of specular reflection!*

# Arnold type stability results

- *With the a-priory estimates one can get global in time non linear stability results. Such results seem useful because the estimate persist are remain valid for weak limits of solutions.*
- *In many cases the Boltzmann Maxwell relation is used with a prescribed time independent ?. Hence it is convenient to validate this choice.*

For the stability of the electrons with a given constant ion density and specular reflexion boundary condition

$$\begin{aligned} &\partial_t f + v \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_v f = \mathcal{C}_-(f) \\ &-\Delta \phi + \langle f(t) \rangle = I(x) \end{aligned} \tag{17}$$

near a stationary Boltzmann-Maxwell solution , with the relative entropy

$$\mathcal{H}(f|\overline{f}) := \iint_{\Omega \times \mathbb{R}^3} \left[ f \log \left( \frac{f}{\overline{f}} \right) - f + \overline{f} \right](x, v)\, dxdv,$$

one has:

**Theorem**

*Let* $(\overline{f}, \overline{\phi})$ *be any stationary solution :*

$$\overline{f}(x,v) = (\frac{\beta}{2\pi})^{d/2} e^{-\beta(\frac{|v|^2}{2} - \overline{\phi}(x))}, \quad -\Delta\overline{\phi} + e^{\beta\overline{\phi}} = \langle \overline{f} \rangle \tag{18}$$

*and* $(f, \phi)$ *be any weak solution of the system (17) then:*

$$\begin{aligned} &\frac{d}{dt}(\mathcal{H}(f|\overline{f}) + \frac{\beta}{2}\int_\Omega |\nabla\phi - \nabla\overline{\phi}|^2 \, dx) = D(f) \\ &\textit{with} \quad D(f) := -\int_\Omega \langle \mathcal{C}_+(f) \log f \rangle dx \end{aligned} \tag{19}$$

Proof.

$$\frac{d}{dt}\mathcal{H}(f|\overline{f}) + D(f) = -\int_\Omega \langle (1+\log \overline{f})\partial_t f(x,v,t)\rangle \, dx$$

$$= \int_\Omega \langle (1 + \frac{d}{2}\log(\frac{\beta}{2\pi}) - \beta(\frac{|v|^2}{2} - \overline{\phi}))\partial_t f(x,v,t)\rangle \, dx$$

$$= -\beta \int_\Omega \partial_t \langle \frac{|v|^2}{2} f(x,v,t)\rangle - \overline{\phi}\partial_t \langle f(x,v,t)\rangle dx$$

$$= \beta \int_{\Omega 3} (\nabla_x \phi - \nabla_x \overline{\phi}) \cdot \langle vf\rangle \, dx$$

$$= \beta \int_\Omega (\phi - \overline{\phi})\partial_t \langle f\rangle \, dx = -\frac{\beta}{2}\frac{d}{dt}\int_\Omega |\nabla\phi - \nabla\overline{\phi}|^2 \, dx$$

$$\Rightarrow \frac{d}{dt}(\mathcal{H}(f|\overline{f}) + \int_\Omega |\nabla\phi - \nabla\overline{\phi}|^2 \, dx) + D(f) \leq 0$$

□

Starting from

$$(f_-^\epsilon, f_+^\epsilon) \Rightarrow ((\frac{\beta(t)}{2\pi})^{\frac{d}{2}} e^{-\beta(t)(\frac{|v|^2}{2}-\phi(x,t))}, f_+(x,v,t))$$

$$-\lambda^2(\epsilon)\Delta\phi + e^{\beta(t)\phi(x,t)} = \langle f_+(t)\rangle \quad \partial_{\vec{n}}\phi = 0 \text{ on } \partial\Omega\,,$$

$$\partial_t f_+ + v\cdot\nabla_x f_+ - \nabla_x\phi\cdot\nabla_v f_+ = 0\,.$$

$$\int_\Omega \frac{|v|^2}{2}\langle f_+(t)\rangle dx + \frac{m_0 d}{\beta(t)} + \frac{1}{2}\int_\Omega |\nabla_x\phi(x,t)|^2 dx = \mathcal{E}_0\,.$$

One may consider the formal quasi neutral limit $\lambda^2(\epsilon) \to 0$. That would give:

$$e^{\beta(t)\phi(x,t)} = \langle f_+(t)\rangle \,,$$

$$\partial_t f_+ + v\cdot\nabla_x f_+ - \nabla_x\left(\frac{1}{\beta(t)}\log\langle f_+\rangle\right)\cdot\nabla_v f_+ = 0\,.$$

$$\int_\Omega \frac{|v|^2}{2}\langle f_+(t)\rangle dx + \frac{m_0 d}{\beta(t)} + \frac{1}{2\beta^2(t)}\int_\Omega |\nabla_x \log\langle f_+\rangle|^2 dx = \mathcal{E}_0\,.$$

For this limit the problem may be ill posed. Well posed with a simple bump, in $1d$ a Penrose criteria and uniformly with respect to $\epsilon$ by a generalization of this Penrose criteria.!!!!

Basic reference:

Han-Kwan, Daniel; Rousset, Frédéric Quasineutral limit for Vlasov-Poisson with Penrose stable data. Ann. Sci. Ec. Norm. Supr. (4) 49 (2016), no. 6, 1445D/21495.

Less sophisticated

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