Approximating gravitational collapse for dust by Vlasov matter

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Motivation 1.0

The New York Times

Computer Defies Einstein's Theory

By JOHN NOBLE WILFORD
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A supercomputer at Cornell University, simulating a tremendous gravitational collapse in the universe, has startled and confounded astrophysicists by producing results that should not be possible according to Einstein's general theory of relativity.

Vlasov matter in general relativity

Suppose a fixed spacetime with metric $g_{\alpha\beta}$ is given. Let m be the mass of the particles to be described, and denote by P the mass shell, which is the subset of the tangent bundle of spacetime given by the equation $g_{\alpha\beta}p^\alpha p^\beta = -m^2$.

The unknown in the Vlasov equation is the phase space density of particles f which is a non-negative function on P. The geodesic flow of $g_{\alpha\beta}$ defines a vector field on the tangent bundle which is tangent to P and so can be restricted to it. Denote the restricted vector field by X. The Vlasov equation then reads $Xf = 0$.

As we will see below, the density f gives rise to an energy-momentum tensor and the Vlasov equation can then be coupled to the Einstein equations and the Einstein-Vlasov (EV) system results.

Dust from Vlasov

The Vlasov equation is linear in f and distributional solutions make sense. One class of distributional solutions is given by

$$f(x^\gamma, p^a) = -u_0 |g|^{-1/2} \rho(x^\gamma) \delta(p^a - u^a),$$

where $\rho \geq 0$ and $u^a(x^\gamma)$ is a mapping from spacetime into the mass shell and u_0 is given by u^a from the mass shell relation.

Solutions of the EV system where the phase space density f has this form are in one-to-one correspondence with dust solutions of the Einstein equations with density ρ and four-velocity u^α.

Dust may thus be considered as a *singular* case of matter described by the Vlasov equation.

Criticism raised by Alan Rendall '92

- Shapiro and Teukolsky first made simulations for the Vlasov-Poisson (VP) system. They then generalized this code to the relativistic case.
- In their VP code they took data close to dust (since the dust solution is known) and found that the kinetic energy and the potential energy diverge as the singularity was approached.
- Shapiro and Teukolsky considered this as support for the reliability of their numerical code. Dust and Vlasov matter, however, behave very differently in some situations.
- Pfaffelmoser and Lions/Perthame showed in the early '90s that global existence holds for the Vlasov-Poisson system. In particular, the kinetic energy and the potential energy do not blow up.

Remark

- Chul-Moon Yoo et al. recently published a new simulation with no conclusive result.
- In a collaboration project with Ellery Ames and Oliver Rinne we have developed an axisymmetric code and one aim is to make an independent study of the simulation by Shapiro and Teukolsky.

Motivation 2.0 (main motivation for this project)

As we will discuss in some detail soon the Einstein-Dust system can be solved in a semi-explicit way. The data can be divided into type (a) and type (b) where:

- data (a) form trapped surfaces, and thus black holes
- data (b) form naked singularities

Our main aim is to study the EV system with data close to either type (a) or type (b).

For data of type (a) the aim is to show that also solutions of the EV system develop trapped surfaces and for data close to type (b) the aim is *also* to show that trapped surfaces form to rule out naked singularities.

The Oppenheimer-Snyder collapse from 1939 has had an immense impact in general relativity.

A huge literature on dust collapse has since then been produced. In these works so called co-moving coordinates are used.

In kinetic theory the movement of the particles is complex and one cannot speak about co-moving coordinates literally. However, in order to compare solutions for dust and for the EV system it is very convenient to use the same type of coordinates.

Hence, in my present project with Gerhard Rein we have used these coordinates also for the EV system.

Metric

The metric reads:

$$ds^2 = -\,dt^2 + e^{2\lambda(t,r)}dr^2 + R^2(t,r)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right).$$

Asymptotic flatness means that the metric quantities λ and R satisfy the boundary conditions

$$\lim_{r\to\infty}\lambda(t,r) = 0, \quad \lim_{r\to\infty}\frac{R(t,r)}{r} = 1.$$

A regular center requires that

$$\lim_{r\to 0}\frac{re^{\lambda(t,r)}}{R(t,r)} = 1.$$

For a metric of this form the non-trivial components of the Einstein equations

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

are found to be

$$\begin{aligned}
2RR_t\lambda_t + 2Re^{-2\lambda}R_r\lambda_r + R_t^2 - 2Re^{-2\lambda}R_{rr} - e^{-2\lambda}R_r^2 + 1 &= 8\pi R^2 T_{00}, \\
R_r\lambda_t - R_{rt} &= 4\pi R T_{01}, \\
-2Re^{2\lambda}R_{tt} - e^{2\lambda}R_t^2 + R_r^2 - e^{2\lambda} &= 8\pi R^2 T_{11}, \\
-R\lambda_t^2 - R_t\lambda_t - R\lambda_{tt} - R_{tt} - e^{-2\lambda}R_r\lambda_r + e^{-2\lambda}R_{rr} &= \frac{8\pi}{R} T_{22};
\end{aligned}$$

the also non-trivial 33 component is a multiple of the 22 component.

The Vlasov equation

The Vlasov equation reads

$$\partial_t f + e^{-\lambda}\frac{w}{p^0}\,\partial_r f + \left(-w\lambda_t + \frac{e^{-\lambda}R_r L}{R^3 p^0}\right)\partial_w f = 0,$$

where

$$p^0 = \sqrt{1 + w^2 + \frac{L}{R^2}},\ w := e^{-\lambda}p_1,\ L := (p_2)^2 + \frac{1}{\sin^2\theta}(p_3)^2.$$

In order to close the system we have to define the energy momentum tensor in terms of f and the metric. In general,

$$T_{\alpha\beta} = |g|^{-1/2}\int p_\alpha p_\beta f \frac{dp_1 dp_2 dp_3}{p^0},$$

where $|g|$ denotes the modulus of the determinant of the metric.

Energy momentum tensor

In the above coordinates and using the restriction to the mass shell,

$$
\begin{aligned}
T_{00}(t,r) &= \frac{\pi}{R^2}\int_{-\infty}^{\infty}\int_0^{\infty} p_0 f(t,r,w,L)\,dL\,dw =: \rho(t,r),\\
T_{01}(t,r) &= -\frac{\pi}{R^2}e^{\lambda}\int_{-\infty}^{\infty}\int_0^{\infty} wf(t,r,w,L)\,dL\,dw =: -e^{\lambda}j,\\
T_{11}(t,r) &= \frac{\pi}{R^2}e^{2\lambda}\int_{-\infty}^{\infty}\int_0^{\infty} \frac{w^2}{p^0}f(t,r,w,L)\,dL\,dw =: e^{2\lambda}p,\\
T_{22}(t,r) &= \frac{\pi}{2R^2}\int_{-\infty}^{\infty}\int_0^{\infty} \frac{L}{p^0}f(t,r,w,L)\,dL\,dw =: q.
\end{aligned}
$$

Rmk: For dust $j = p = q = 0$.

The dust case

The Einstein-dust system consists of the Euler equations

$$u^\alpha \nabla_\alpha u_\beta := u^\alpha \left(\partial_\alpha u_\beta - \Gamma^\gamma_{\alpha\beta} u_\gamma \right) = 0, \tag{5}$$

$$u^\alpha \nabla_\alpha \rho + \rho \nabla_\alpha u^\alpha = 0 \tag{6}$$

coupled to the Einstein equations with energy momentum tensor

$$T_{\alpha\beta} = \rho u_\alpha u_\beta. \tag{7}$$

Here ρ is the energy density and u^α is the four velocity.

We use comoving coordinates and require that the four velocity field initially is given by

$$u^{\alpha}_{|t=0} = (1, 0, 0, 0), \; u_{\alpha|t=0} = (-1, 0, 0, 0).$$

Since $\Gamma^0_{0\beta} = 0$ for a metric of the form (1),

$$u^{\alpha} = (1, 0, 0, 0), \; u_{\alpha} = (-1, 0, 0, 0)$$

solves (5). The energy momentum tensor takes the form

$$T_{00} = \rho, \quad T_{\alpha\beta} = 0 \text{ for } (\alpha, \beta) \neq (0, 0),$$

and the continuity equation (6) becomes

$$\partial_t \rho + \left(\partial_t \lambda + 2 \frac{\partial_t R}{R} \right) \rho = 0.$$

Solution of the Einstein-dust system

As initial condition we prescribe $\rho(0,r) = \mathring{\rho}(r)$, $R(0,r) = \mathring{R}(r)$ and $\partial_t R(0,r) = \mathring{v}(r)$. Given these we define

$$F(r) := 8\pi \int_0^r \mathring{R}^2(s)\mathring{R}'(s)\mathring{\rho}(s)\,ds =: 2\mathring{m}(r),$$

and

$$W^{-2}(r) := (\mathring{v}(r))^2 + 1 - \frac{F(r)}{\mathring{R}(r)}.$$

Let $R = R(t,r)$ solve *the master equation*

$$\partial_t R(t,r) = -\sqrt{\frac{F(r)}{R(t,r)} + W^{-2}(r) - 1},$$

then λ and ρ are given by

$$e^{\lambda(t,r)} = W(r)\,\partial_r R(t,r),$$

$$\rho(t,r) = \frac{\mathring{R}^2(r)\mathring{R}'(r)}{R^2(t,r)\partial_r R(t,r)}\mathring{\rho}(r).$$

Solving the master equation

The master equation reads

$$\partial_t R(t,r) = -\sqrt{\frac{F(r)}{R(t,r)} + f(r)}, \tag{8}$$

where $f(r) = (W(r))^{-2} - 1$.

Using separation of variables the following function turns up:

$$G(y) := \begin{cases} \frac{\arcsin\sqrt{y}}{y^{3/2}} - \frac{\sqrt{1-y}}{y} & , \quad 0 < y \le 1, \\ \frac{2}{3} & , \quad y = 0, \\ -\frac{\operatorname{arcsinh}\sqrt{-y}}{(-y)^{3/2}} - \frac{\sqrt{1-y}}{y} & , \quad -\infty < y < 0. \end{cases}$$

If we define

$$t_0(r) := \frac{\mathring{R}(r)^{3/2} G\left(-\frac{\mathring{R}(r) f(r)}{F(r)}\right)}{\sqrt{F(r)}}$$

then for $0 \leq t \leq t_0(r),\ r > 0$ the implicit relation

$$t_0(r) - t = \frac{R^{3/2} G\left(-\frac{R f(r)}{F(r)}\right)}{\sqrt{F(r)}}$$

defines the desired solution to (8).

Since $\lim_{t \to t_0(r)} R(t, r) = 0,\ t_0(r)$ is the coordinate time at which the solution blows up at r which is the time by which the dust particle which started out initially at r has reached the center.

The general condition for a trapped surface in the given coordinates is that along all radial null geodesics

$$\frac{d}{d\tau}R(t,r) = \frac{dt}{d\tau}\left(\partial_t R + \partial_r R \frac{dr}{dt}\right) < 0,$$

which for dust results in the condition $R(t,r) < F(r) = 2\mathring{m}(r)$.

If we take initial data with no trapped surface, there is a unique time $t_H(r) \in (0, t_0(r))$ such that $R(t_H(r), r) = F(r)$, given by the relation

$$t_0(r) - t_H(r) = F(r)G(-f(r)).$$

$t_H(r)$ characterizes the time when an apparent horizon forms at r.

Hence we have derived expressions for both the blow up time $t_0(r)$ and the time when a trapped surface forms $t_H(r)$.

The Oppenheimer-Snyder solution

A special case is the Oppenheimer-Snyder solution for which $W(r) \equiv 1$ and

$$\mathring{\rho} = c\, 1_{[0,1]}.$$

Then

$$F(r) = 2\mathring{m}(r) = \frac{8\pi}{3} c r^3,$$

and

$$R(t,r) = \left(1 - \sqrt{6\pi c}\, t\right)^{2/3} r,$$

which is the expected Friedmann form;

$$t_0(r) = \frac{1}{\sqrt{6\pi c}}, \quad t_H(r) = \frac{1}{\sqrt{6\pi c}} - \frac{16\pi c}{9} r^3.$$

Remark on stability and instability

A general feature of the dust solutions is that they form trapped surfaces and blow up in finite time. In particular, for the Oppenheimer-Snyder solution both $t_0(r)$ and $t_H(r)$ are explicitly given.

Moreover, note that the amplitude c of the constant density can be taken arbitrary small. Still a trapped surface and blow up occur in finite, but longer, time. *Hence, the Einstein-Dust system might be said to be unstable.*

How does this relate to the stability results for the EV system? Recall the results:

- Rein and Rendall '92 in spherical symmetry.
- Lindblad and Taylor '17 in the general case
- Fajman, Joudioux and Smulevici '17 in the general case

Roughly, to approximate dust we choose

$$\mathring{f}(x, v) = h_\epsilon(v)\mathring{\rho}(x),$$

where h_ϵ is approximating a Dirac delta function, and $\mathring{\rho}(x) = c1_{[0,1]}$.

Hence we have two parameters, ϵ and c. If we fix c and let $\epsilon \to 0$ then $\mathring{f} \to \infty$, whereas if we fix ϵ and let $c \to 0$ then $\mathring{f} \to 0$.

In the former case the above stability results do not apply whereas in the latter they do.

This simple observation shows a fundamental difference between dust and Vlasov matter!

Remark on naked singularities

In the Oppenheimer-Snyder collapse the density is homogeneous within the matter; $\rho(t,r)=\rho(t)$. It is well-known that inhomogeneous data can be prescribed which lead to naked singularities for dust.

Numerically this was first studied by Eardley and Smarr in 1979 and then a rigorous proof was given by Christodoulou 1984:

- Time functions in numerical relativity: Marginally bound dust collapse, *Phys. Rev.* D 19, 2239 (1979).
- Violation of Cosmic Censorship in the Gravitational Collapse of a Dust Cloud, *Commun. Math. Phys.* 93, 171-195 (1984).

Recall that one of the aims of the present project is to show that the naked singularities will not be present if dust is replaced by Vlasov matter. (*This is still an open issue.*)

Main result

A loose formulation of our main result reads as follows.

Theorem

Let an Oppenheimer-Snyder solution be given with blow-up time t_0. Then for any $T < t_0$ there is initial data for the Einstein-Vlasov system such that the corresponding solution exists on $[0, T]$ and the solution is arbitrary close to the Oppenheimer-Snyder solution. In particular a trapped surface forms in the evolution and a black hole spacetime results.

Thank You!