Lattice Boltzmann models for rarefied flows

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Collisionless Boltzmann (Vlasov) Equation
and Modelling of Self-Gravitating Systems and Plasmas
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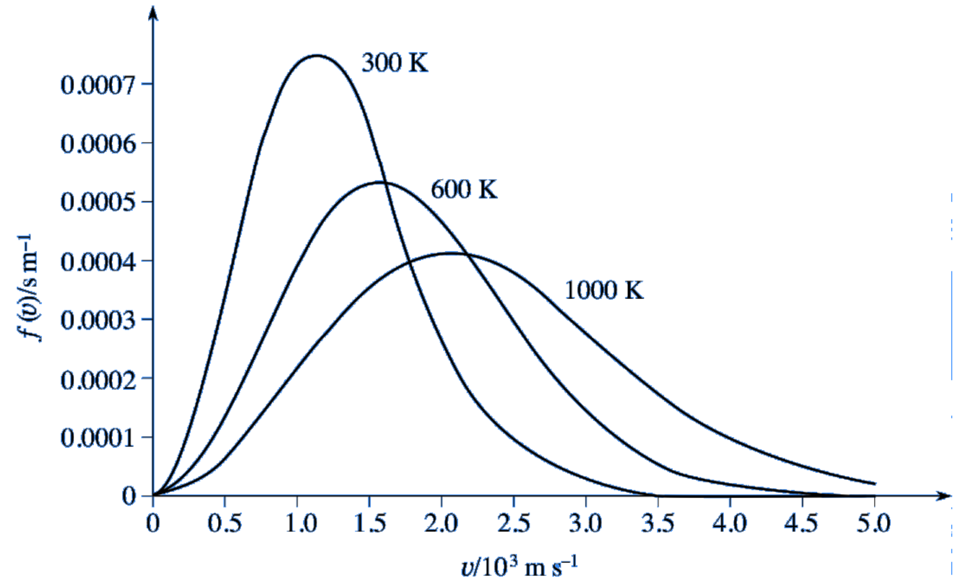
Section 1

Introduction

Mesoscale approach

Macroscopic: n, $\boldsymbol{u}$, T

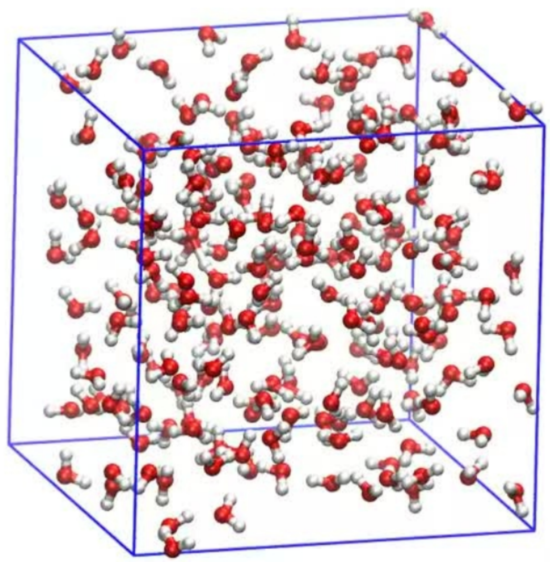
(Navier-Stokes-Fourier)



Mesoscopic: $f(\boldsymbol{x}, \boldsymbol{p}, t)$

(Boltzmann)

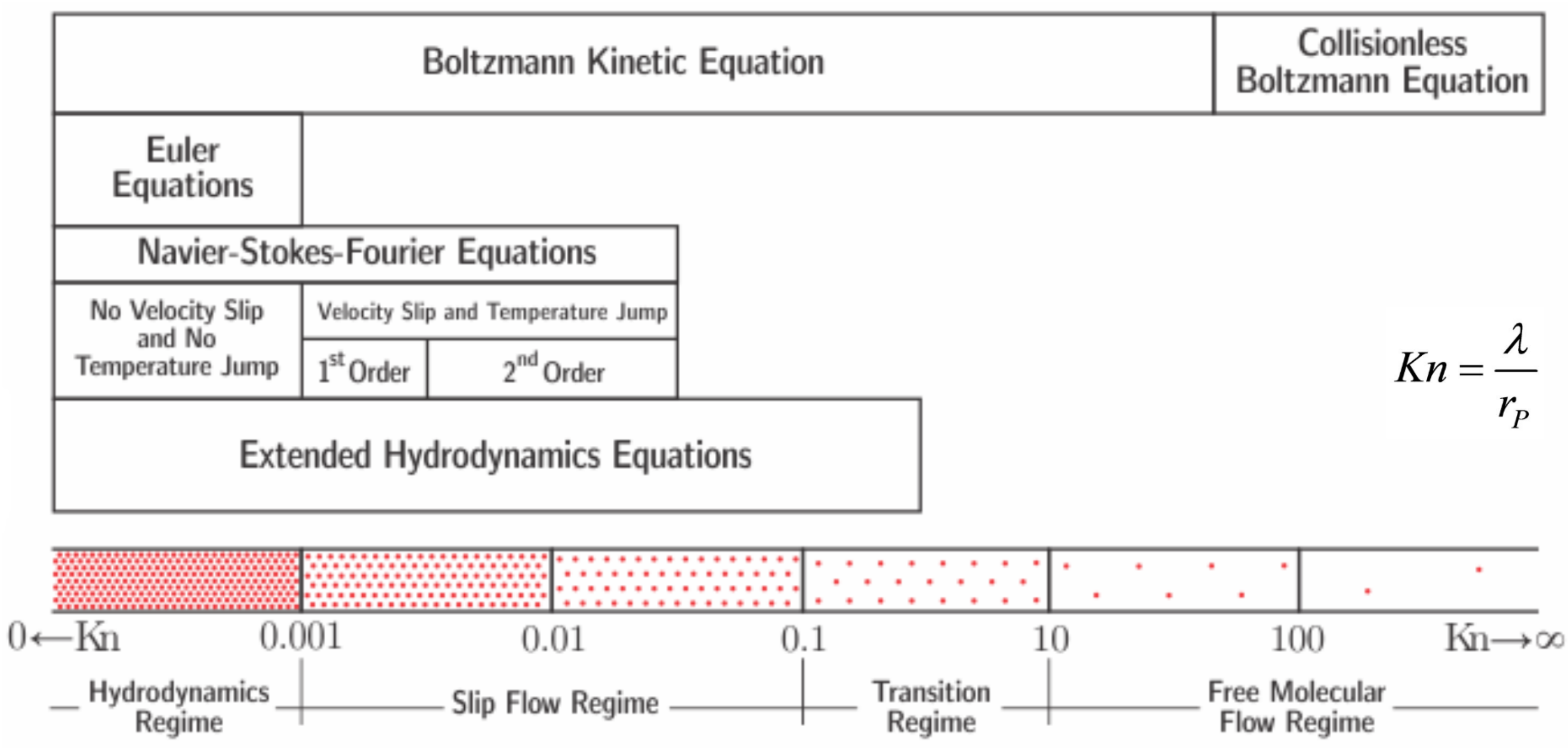
$N_A =$ 6.02×10^{23}



Microscopic: $(\boldsymbol{x}_i, \boldsymbol{p}_i)$

(MD)

Rarefied flows: Knudsen number (Kn)



- $\mathrm{Kn} = \lambda / r_p$ ($\lambda \equiv$ mean free path; $r_p \equiv$ characteristic channel length).
- Hydordynamic regime (NSF): $\mathrm{Kn} \to 0$.
- Ballistic regime (Vlasov): $\mathrm{Kn} \to \infty$.

Lattice Boltzmann (LB): main ingredients

1. Discretisation of the momentum space (Gauss quadratures);
2. Polynomial representation of $f^{(\rm eq)}$ in the BGK collision term;
3. Replacement of $\nabla_{\boldsymbol{p}} f$ using a "suitable" expression;
4. Numerical method for time evolution and spatial advection (RK-3 + WENO-5);
5. Boundary conditions.

The LB method ensures the exact recovery of the conservation eqs. for n, $\rho \boldsymbol{u}$ and E (for thermal models).

Section 2

Planar shocks

Non-relativistic case: equation

- Let us consider the Sod shock tube problem:

$$f(z,t=0) = \begin{cases} f_L^{(\mathrm{eq})}, & z<0 \qquad (n_L, P_L, u_L) = (1,1,0) \\ f_R^{(\mathrm{eq})}, & z>0 \qquad (n_R, P_R, u_R) = (0.125, 0.1, 0) \end{cases}.$$

- The Boltzmann equation reduces to:

$$\partial_t f + \frac{p_z}{m}\partial_z f = -\frac{1}{\tau}(f - f^{(\mathrm{eq})}_{\mathrm{M-B}}), \qquad f^{(\mathrm{eq})}_{\mathrm{M-B}} = \frac{n}{(2\pi m T)^{3/2}} \exp\left[-\frac{(\boldsymbol{p} - m\boldsymbol{u})^2}{2mT}\right],$$

where the BGK relaxation time approximation was used for $J[f]$.[1]

- The p_x and p_y degrees of freedom can be integrated:

$$\begin{pmatrix} g \\ h \end{pmatrix} = \int dp_x dp_y\, f \begin{pmatrix} 1 \\ p_x^2 + p_y^2 \end{pmatrix}, \qquad \partial_t \begin{pmatrix} g \\ h \end{pmatrix} + \frac{p_z}{m}\partial_z \begin{pmatrix} g \\ h \end{pmatrix} = -\frac{1}{\tau}\begin{pmatrix} g - g^{(\mathrm{eq})} \\ h - h^{(\mathrm{eq})} \end{pmatrix},$$

where $h^{(\mathrm{eq})} = 2mTg^{(\mathrm{eq})}$ and

$$g^{(\mathrm{eq})} = \frac{n}{\sqrt{2\pi m T}} \exp\left[-\frac{(p_z - m u_z)^2}{2mT}\right].$$

[1] P. L. Bhatnagar, E. P. Gross, M. Krook, Phys. Rev. **94**, 511–525 (1954).

Non-relativistic case: Discretisation

- p_z are discretised using the Gauss-Hermite quadrature method:[2,3]

$$\int_{-\infty}^{\infty} dp_z \, g \, P_s(p_z) = \sum_{k=1}^{Q} g_k \, P_s(p_{z,k}), \qquad g_k = \frac{w_k}{\omega(p_z)} g(p_k),$$

where $H_Q(p_{z,k}) = 0$, $w_k = Q!/[H_{Q+1}(p_{z,k})]^2$ and $\omega(p_z) = e^{-p_z^2/2}/\sqrt{2\pi}$.

- $g_k^{(\mathrm{eq})}$ is truncated at order $N < Q$ w.r.t. the Hermite polynomials:

$$g_k^{(\mathrm{eq})} = w_k \sum_{\ell=0}^{N} H_\ell(p_k) \sum_{s=0}^{\lfloor \ell/2 \rfloor} \frac{1}{2^s s! (\ell - 2s)!} (mT - 1)^s (mu)^{\ell - 2s}, \qquad h_k^{(\mathrm{eq})} = 2mT g_k^{(\mathrm{eq})}.$$

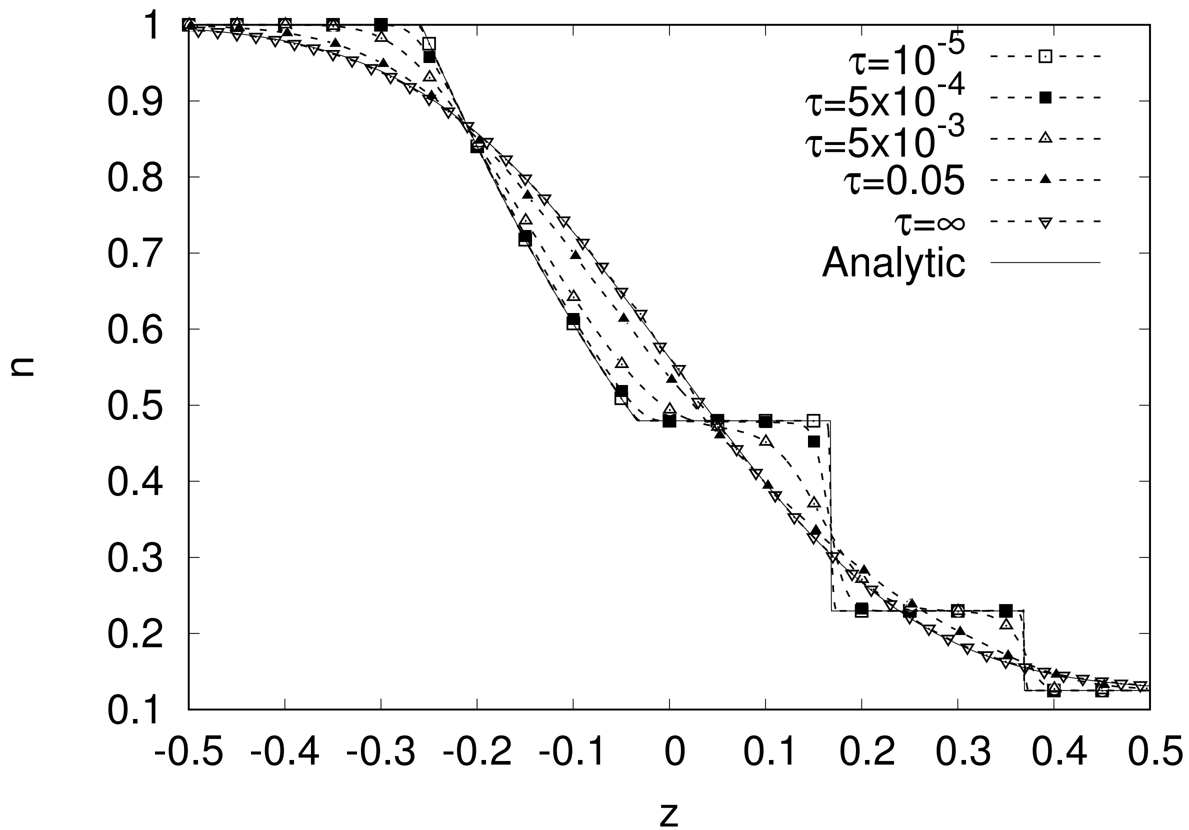
- The space is discretised according to $z_i = -0.5 + \frac{1}{Z}(i - 0.5)$ $(1 \le i \le Z)$.
- The following boundary conditions are imposed:[4]

$$g_{-2,k} = g_{-1,k} = g_{0,k} = g_{k,L}^{(\mathrm{eq})},$$
$$g_{Z+1,k} = g_{Z+2,k} = g_{Z+3,k} = g_{k,R}^{(\mathrm{eq})},$$

and similarly for $h_{i,k}$.

[2] X. W. Shan, X. F. Yuan, and H. D. Chen, J. Fluid. Mech. **550**, 413 (2006).
[3] V. E. Ambruș, V. Sofonea, J. Comput. Phys. **316** (2016) 760.
[4] Y. Gan, A. Xu, G. Zhang, Y. Li, Phys. Rev. E **83** (2011) 056704.

Non-relativistic: n

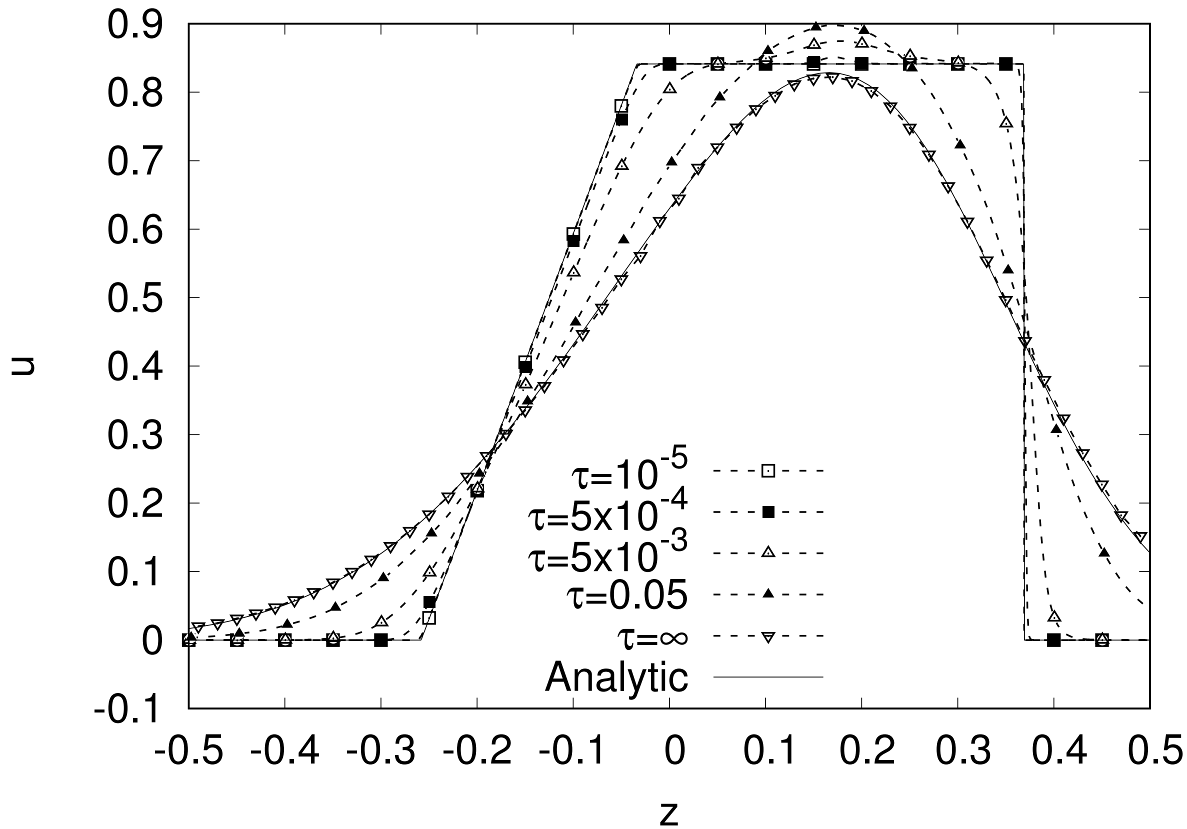


Solution in the collisionless regime:[5]

$$n(z,t) = \frac{n_L}{2}\operatorname{erfc}\left(\frac{z}{t}\sqrt{\frac{m}{2T_L}}\right) + \frac{n_R}{2}\operatorname{erfc}\left(-\frac{z}{t}\sqrt{\frac{m}{2T_R}}\right).$$

[5] Z. Guo, R. Wang, K. Xu, Phys. Rev. E **91** (2015) 033313.

Non-relativistic: u

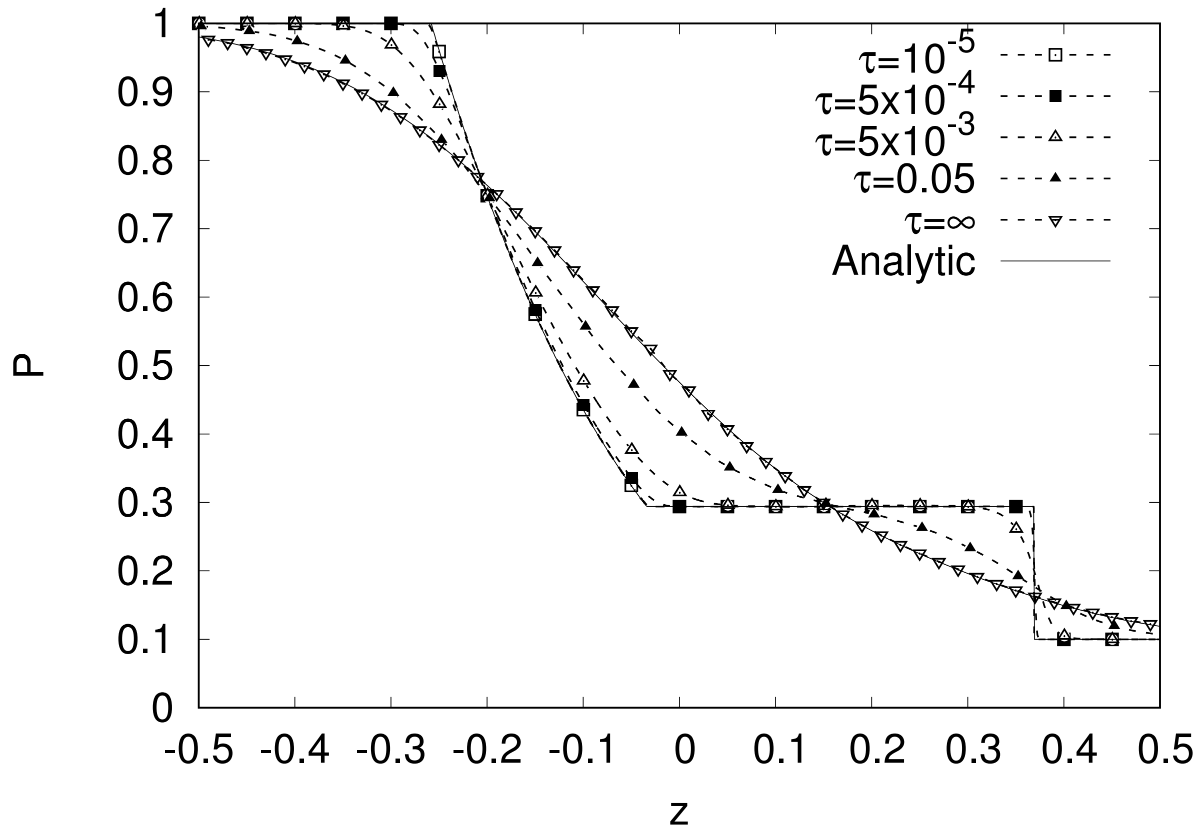


Solution in the collisionless regime:

$$\rho u = n_L \sqrt{\frac{m T_L}{2\pi}} \exp\left(-\frac{m z^2}{2t^2 T_L}\right) - n_R \sqrt{\frac{m T_R}{2\pi}} \exp\left(-\frac{m z^2}{2t^2 T_R}\right).$$

[5] Z. Guo, R. Wang, K. Xu, Phys. Rev. E **91** (2015) 033313.

Non-relativistic: P



Solution in the collisionless regime:

$$\frac{3}{2}nT + \frac{1}{2}\rho u^2 = \frac{n_L T_L}{4}\left[3\text{erfc}\left(\frac{z}{t}\sqrt{\frac{m}{2T_L}}\right) + \frac{z}{t}\sqrt{\frac{2m}{\pi T_L}}e^{-mz^2/2t^2T_L}\right] + \frac{n_R T_R}{4}\left[3\text{erfc}\left(-\frac{z}{t}\sqrt{\frac{m}{2T_R}}\right) - \frac{z}{t}\sqrt{\frac{2m}{\pi T_R}}e^{-mz^2/2t^2T_R}\right].$$

[5] Z. Guo, R. Wang, K. Xu, Phys. Rev. E **91** (2015) 033313.

Ultrarelativistic case: equation

- The advection part is $1D$:

$$\partial_t f + \xi \partial_z f = -\frac{\gamma_L(1-\beta_L\xi)}{\tau}[f - f^{(\rm eq)}_{\rm M-J}], \qquad f^{(\rm eq)}_{\rm M-J} = \frac{n}{8\pi T^3}\exp\left[-\frac{p\gamma_L}{T}(1-\beta_L\xi)\right],$$

where the Anderson-Witting SRT approximation was used for $J[f]$.[6]

- The momentum space is discretised using spherical coordinates:

$$p_x = p\sin\theta\cos\varphi, \qquad p_y = p\sin\theta\sin\varphi, \qquad p_z = p\cos\theta,$$

while $\xi = \cos\theta$.

- Integrating over the φ degree of freedom, p and ξ are discretised following the Gauss-Laguerre and Gauss-Legendre prescriptions:[7]

$$M_{s,r} = \int \frac{d^3p}{p^0} f\, p^{s+1}\xi^r = 2\pi \int_0^\infty dp\, e^{-p} p^2 \int_{-1}^1 d\xi (e^p f) p^s \xi^r = \sum_{k=1}^{Q_L}\sum_{j=1}^{Q_\xi} f_{jk} p_k^s \xi_j^r,$$

where $f_{jk} = w_k^L w_j^P e^{p_k} f(p_k, \xi_j)$, while $L_{Q_L}(p_k) = 0$ and $P_{Q_\xi}(\xi_j) = 0$.

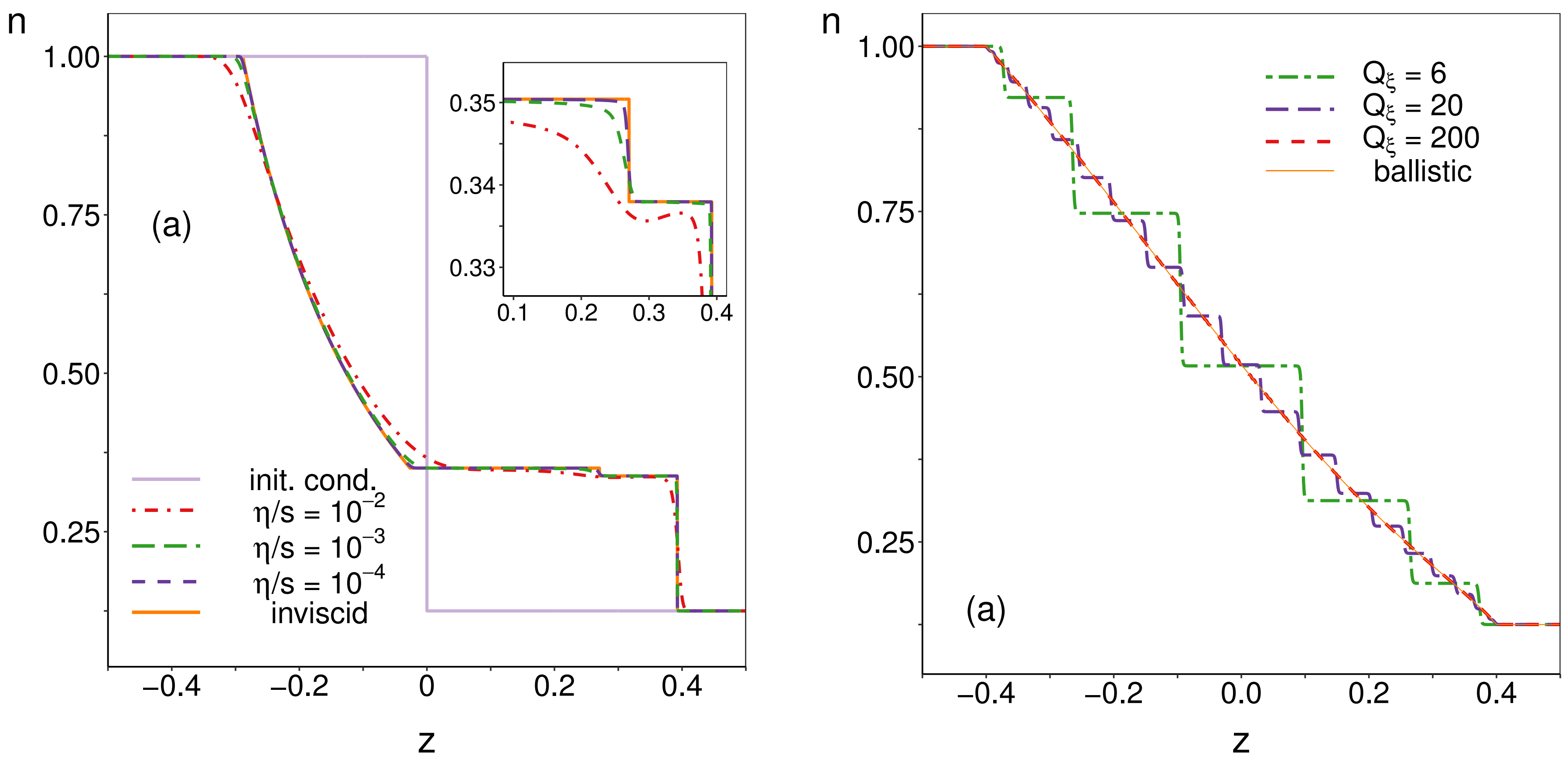
- The quadrature weights w_k^L and w_j^P are given by:

$$w_k^L = \frac{(Q_L+1)(Q_L+2)p_k}{[(Q_L+1)L^{(2)}_{Q_L+1}(p_k)]^2}, \qquad w_j^P = \frac{2(1-\xi_j^2)}{[(Q_\xi+1)P_{Q_\xi+1}(\xi_j)]^2}.$$

- **For the planar case: $Q_L = 2$ and** $Q_\varphi = 1$.

[6] J. L. Anderson, H. R. Witting, Physica **74**, 466–488 (1974); 489–495 (1974).
[7] R. Blaga, V. E. Ambruș, arXiv:1612.01287 [physics.flu-dyn].

Ultrarelativistic: n

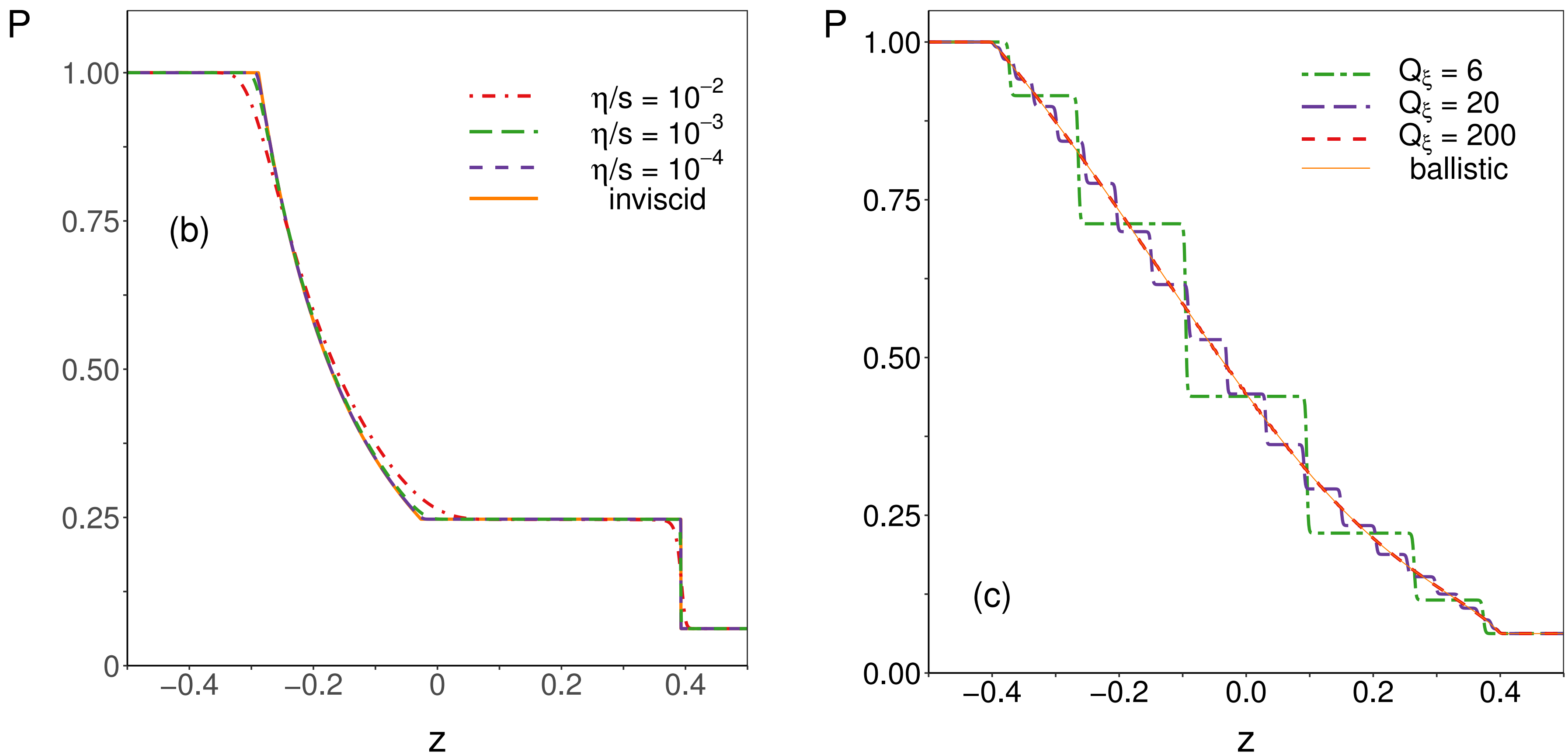


- $\eta/s \equiv$ ratio of shear viscosity to entropy density.
- Ballistic limit:

$$n_{\rm Eck} = \sqrt{\left(\frac{n_{\rm L}+n_{\rm R}}{2} - \frac{n_{\rm L}-n_{\rm R}}{2}\,\frac{z}{t}\right)^2 - \left(\frac{n_{\rm L}-n_{\rm R}}{4}\right)^2\left(1-\frac{z^2}{t^2}\right)^2}$$

[7] R. Blaga, V. E. Ambruș, arXiv:1612.01287 [physics.flu-dyn].

Ultrarelativistic: P



Ballistic limit:

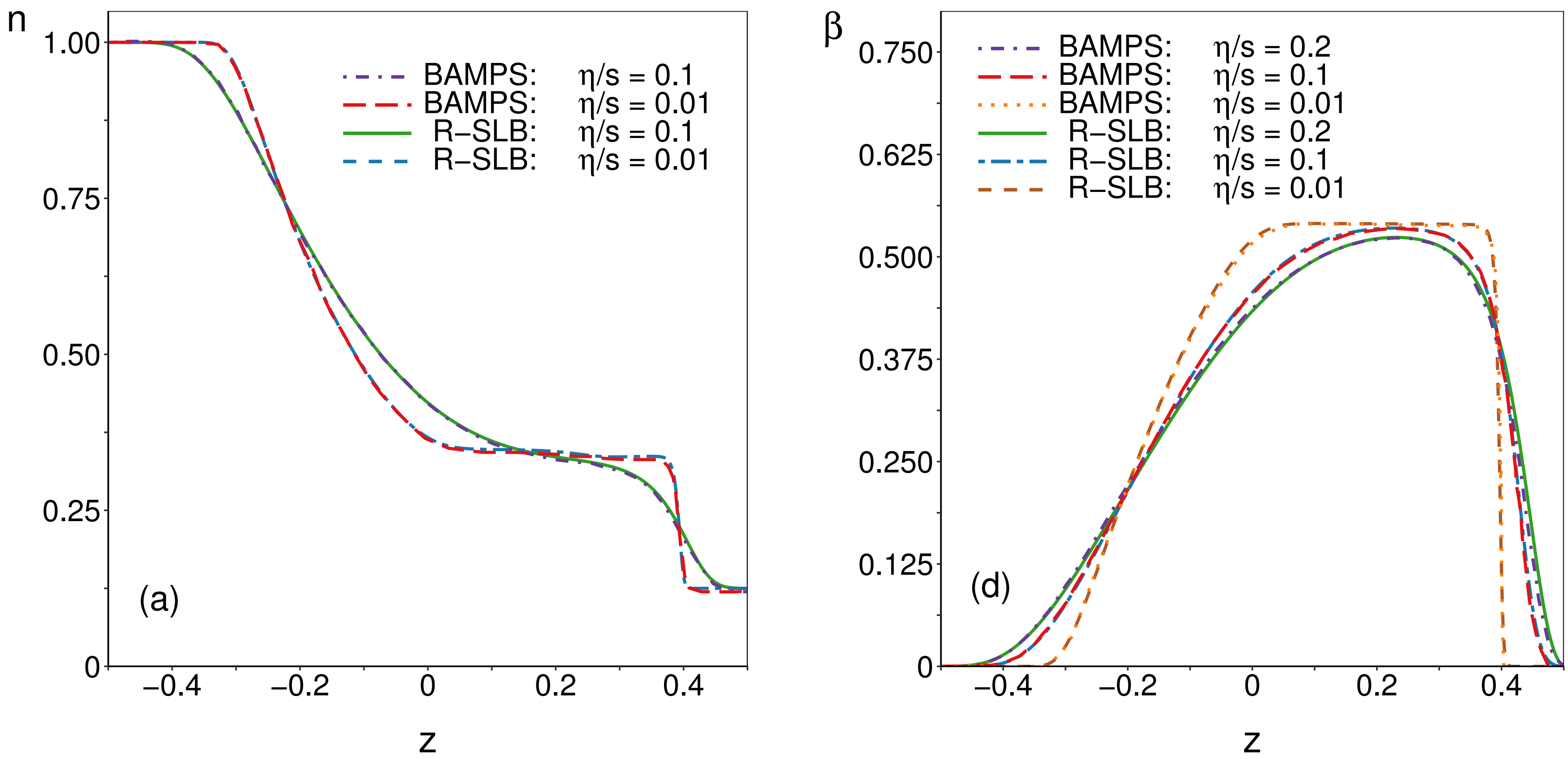
$$T^{00} = \frac{3}{2}(P_{\rm L} + P_{\rm R}) - \frac{3}{2}(P_{\rm L} - P_{\rm R})\frac{z}{t},$$

$$T^{0z} = \frac{3}{4}(P_{\rm L} - P_{\rm R})\left(1 - \frac{z^2}{t^2}\right),$$

$$T^{zz} = \frac{1}{2}(P_{\rm L} + P_{\rm R}) - \frac{1}{2}(P_{\rm L} - P_{\rm R})\left(\frac{z}{t}\right)^3,$$

[7] R. Blaga, V. E. Ambruş, arXiv:1612.01287 [physics.flu-dyn].

R-SLB vs BAMPS: n and β

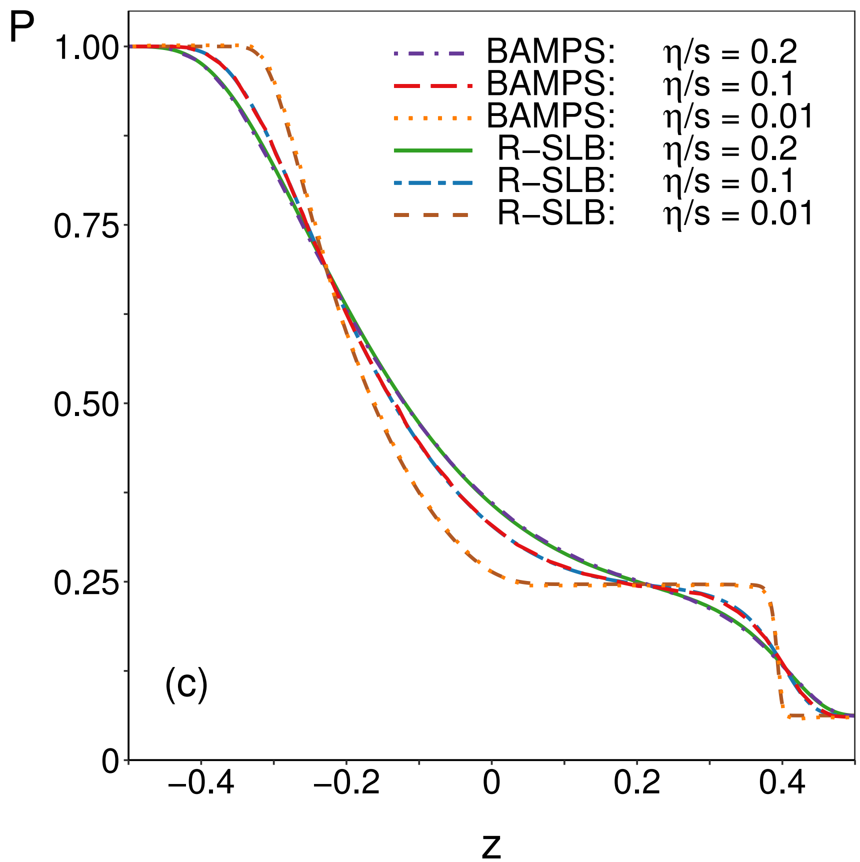


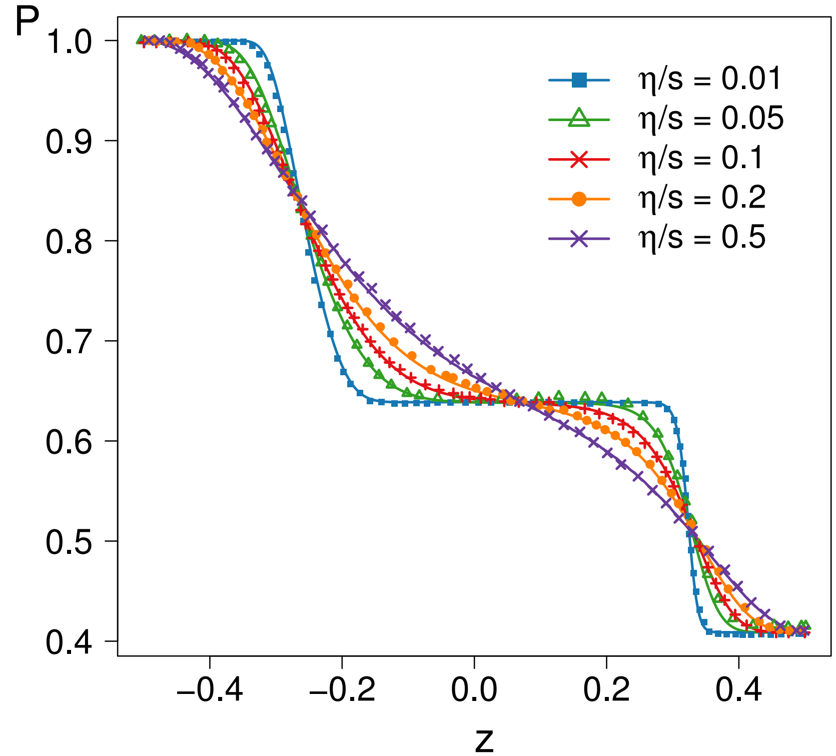
BAMPS $\equiv$ Boltzmann approach to multiparton scattering.[8]

[8] I. Bouras, E. Molnár, H. Niemi, Z. Xu, A. El, O. Fochler, C. Greiner, D. H. Rischke, Phys. Rev. Lett. **103** (2009) 032301.

[7] R. Blaga, V. E. Ambruș, arXiv:1612.01287 [physics.flu-dyn].

R-SLB vs BAMPS: P





BAMPS ≡ Boltzmann approach to multiparton scattering.

[8] I. Bouras, E. Molnár, H. Niemi, Z. Xu, A. El, O. Fochler, C. Greiner, D. H. Rischke, Phys. Rev. Lett. **103** (2009) 032301.

[6] R. Blaga, V. E. Ambruș, arXiv:1612.01287 [physics.flu-dyn].

Section 3

Spherical shocks

Non-relativistic case: Equation

- Let us consider the Sod shock tube problem with spherical symmetry:

$$f(r,t=0) = \begin{cases} f_L^{(\mathrm{eq})}, & r < r_d \qquad (n_L, P_L, u_L) = (1, 1, 0) \\ f_R^{(\mathrm{eq})}, & r > r_d \qquad (n_R, P_R, u_R) = (0.125, 0.1, 0) \end{cases}.$$

- Spherical coordinates: r, ϑ, ϕ and $p_{\hat{r}}, p_{\hat{\vartheta}}, p_{\hat{\phi}}$:

$$\partial_t f + \frac{p\xi}{mr^2}\partial_r(fr^2) + \frac{p}{mr}\partial_\xi[(1-\xi^2)f] = -\frac{1}{\tau}(f - f_{\mathrm{M-B}}^{(\mathrm{eq})}),$$

where $p^{\hat{r}} = p\cos\theta$, $p^{\hat{\vartheta}} = p\sin\theta\cos\varphi$ and $p^{\hat{\phi}} = p\sin\theta\sin\varphi$, while $\xi = \cos\theta$.

- f can be expanded w.r.t. ξ:

$$f = \sum_{s=0}^{\infty} \frac{2s+1}{2}\mathcal{F}_s P_s(\xi).$$

- Thus, $\partial_\xi[(1-\xi^2)f]$ can be written as:

$$\partial_\xi[(1-\xi^2)f] = \int_{-1}^{1} d\xi' \, \mathcal{K}(\xi,\xi') f(\xi'),$$

where

$$\mathcal{K}(\xi,\xi') = \sum_{s=1}^{\infty} \frac{s(s+1)}{2} P_s(\xi)[P_{s+1}(\xi') - P_{s-1}(\xi')].$$

Non-relativistic case: Discretisation

- p, ξ and φ are discretised using the Gauss-Laguerre, Gauss-Legendre and Mysovskikh quadratures:[9]

$$\int d^3p f\, p_x^\ell p_y^s p_z^r = \int_0^\infty dp\, p^2 \int_0^\pi d\theta\, \sin\theta \int_0^{2\pi} d\varphi\, f p^{\ell+s+r} (\sin\theta)^{\ell+s} (\cos\theta)^r (\cos\varphi)^\ell (\sin\varphi)^s$$
$$= \sum_{k=1}^{Q_p} \sum_{j=1}^{Q_\xi} \sum_{i=1}^{Q_\varphi} f_{ijk}\, p_k^{\ell+s+r} (\sin\theta_j)^{\ell+s} (\cos\theta_j)^r (\cos\varphi_i)^\ell (\sin\varphi_i)^s.$$

- The discrete magnitudes p_k are obtained by solving $L_{Q_L}^{(1/2)}(p_k^2) = 0$.
- The elevations ξ_j are obtained by solving $P_{Q_\xi}(\xi_j) = 0$.
- $\varphi_i = 2\pi(i-1)/Q_\varphi$.
- The discrete populations f_{ijk} are:

$$f_{ijk} = \frac{\pi}{Q_\varphi} w_j^\xi e^{p_k^2} w_k^L\, f(p_k, \xi_j, \varphi_i).$$

- The derivative $\partial_\xi[(1-\xi)^2 f] \rightarrow \{\partial_\xi[(1-\xi)^2 f]\}_{ijk} = \sum_{j'=1}^{Q_\xi} \mathcal{K}^\xi_{j,j'} f_{i,j',k}$, where

$$\mathcal{K}^\xi_{j,j'} = w_j^\xi \sum_{s=1}^{Q_\xi} \frac{s(s+1)}{2} P_s(\xi_j)[P_{s+1}(\xi_{j'}) - P_{s-1}(\xi_{j'})]..$$

[9] V. E. Ambruș, V. Sofonea, Phys. Rev. E **86** (2012) 016708.

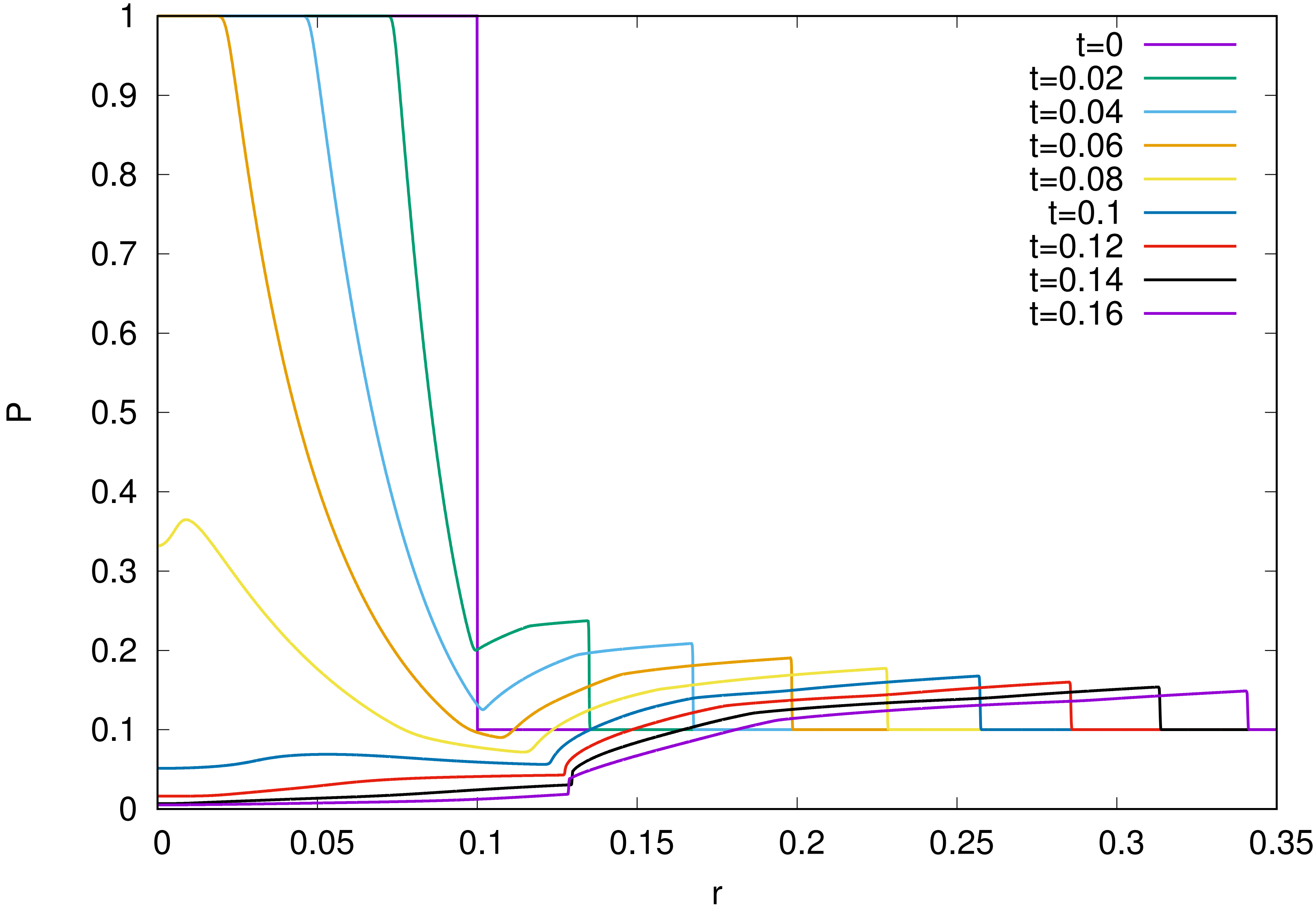
Ultrarelativistic case: equation

- The relativistic Boltzmann equation reads:

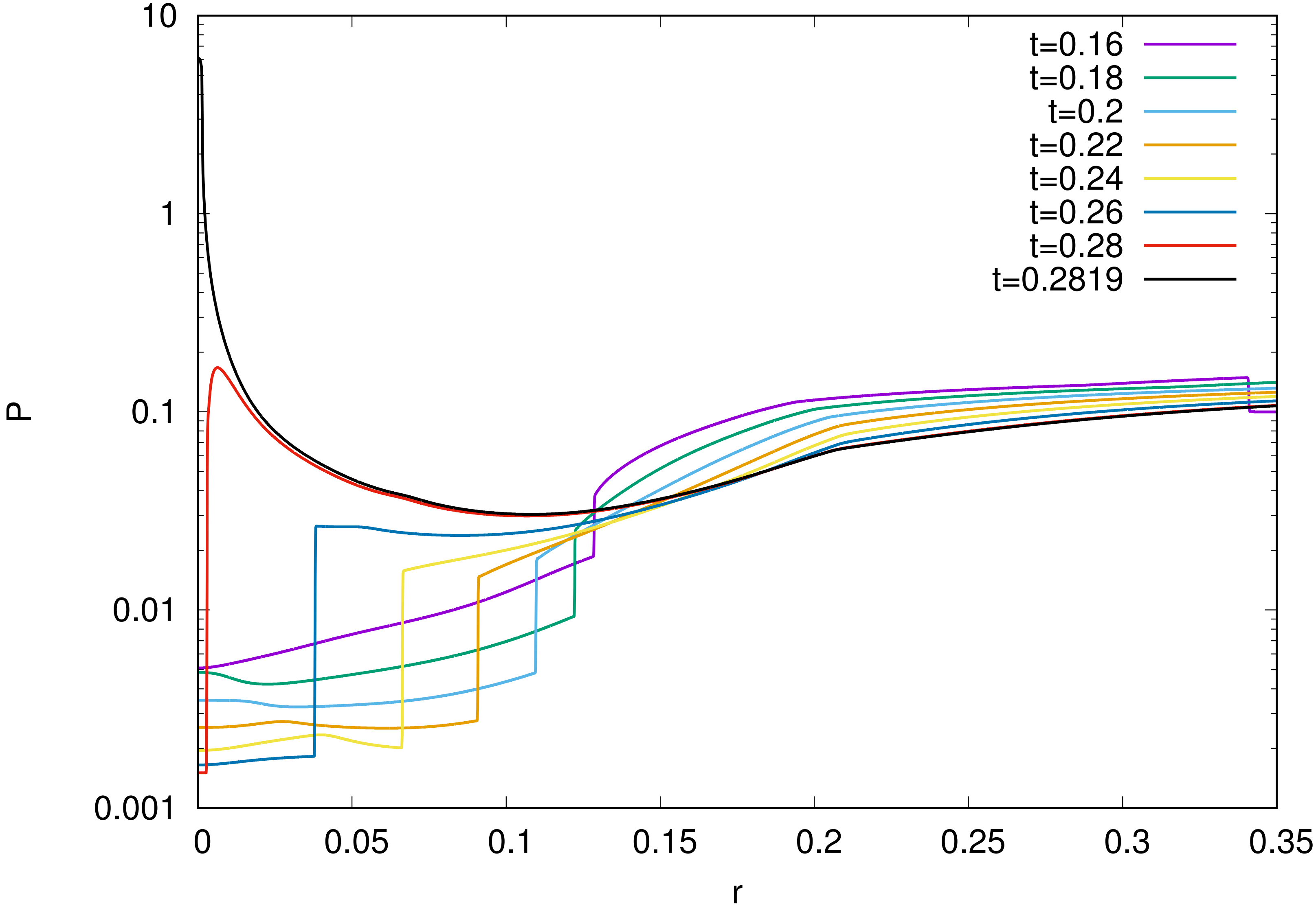
$$\partial_t f + \frac{\xi}{r^2}\partial_r(fr^2) + \partial_\xi[(1-\xi^2)f] = -\frac{\gamma_L}{\tau}(1-\beta_L\xi)[f - f^{(\mathrm{eq})}_{\mathrm{M-J}}].$$

- The derivative w.r.t. ξ is computed in exactly the same way as in the non-relativistic case.
- The momentum space is discretised in the same way as for the planar shock (including $Q_\varphi = 1$).

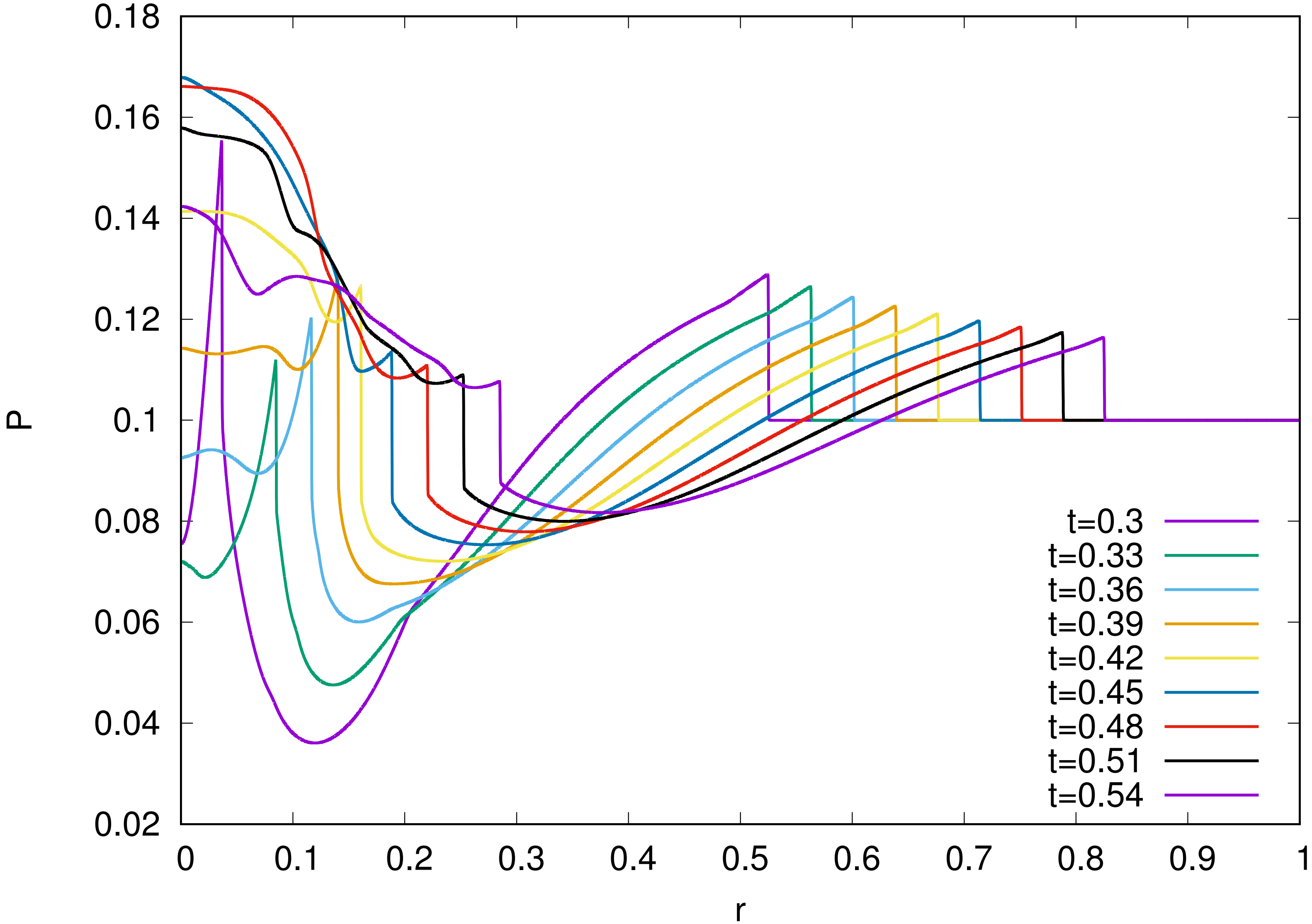
Non-relativistic: Inviscid limit: Primary shock



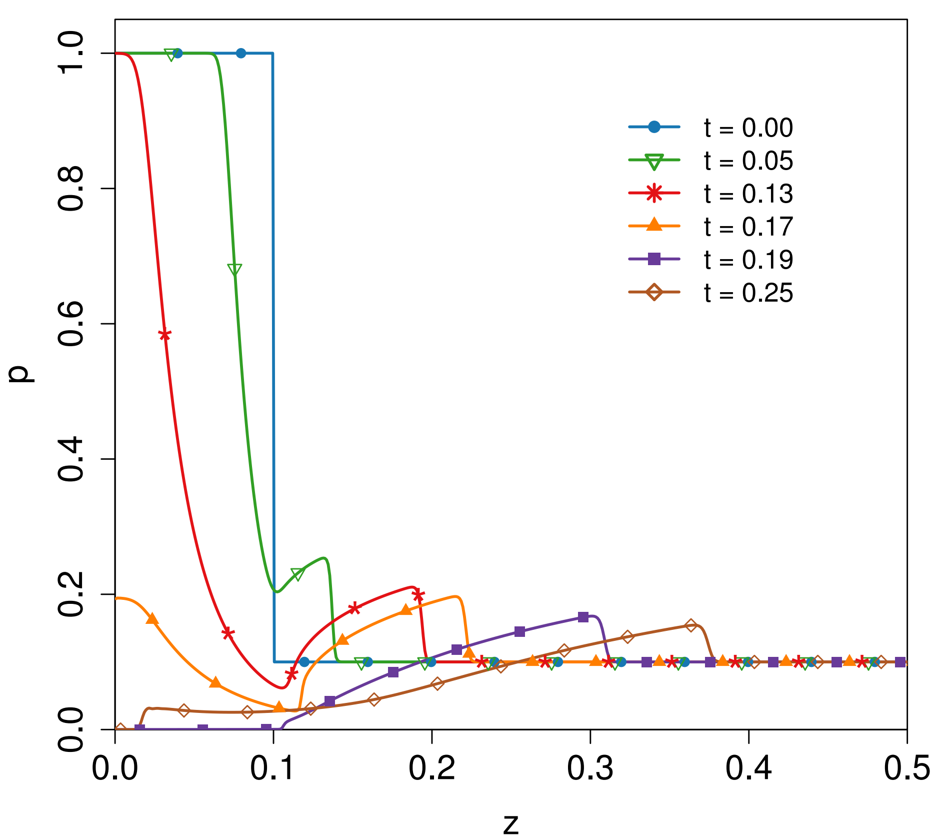
Non-relativistic: Inviscid limit: Reverse shock

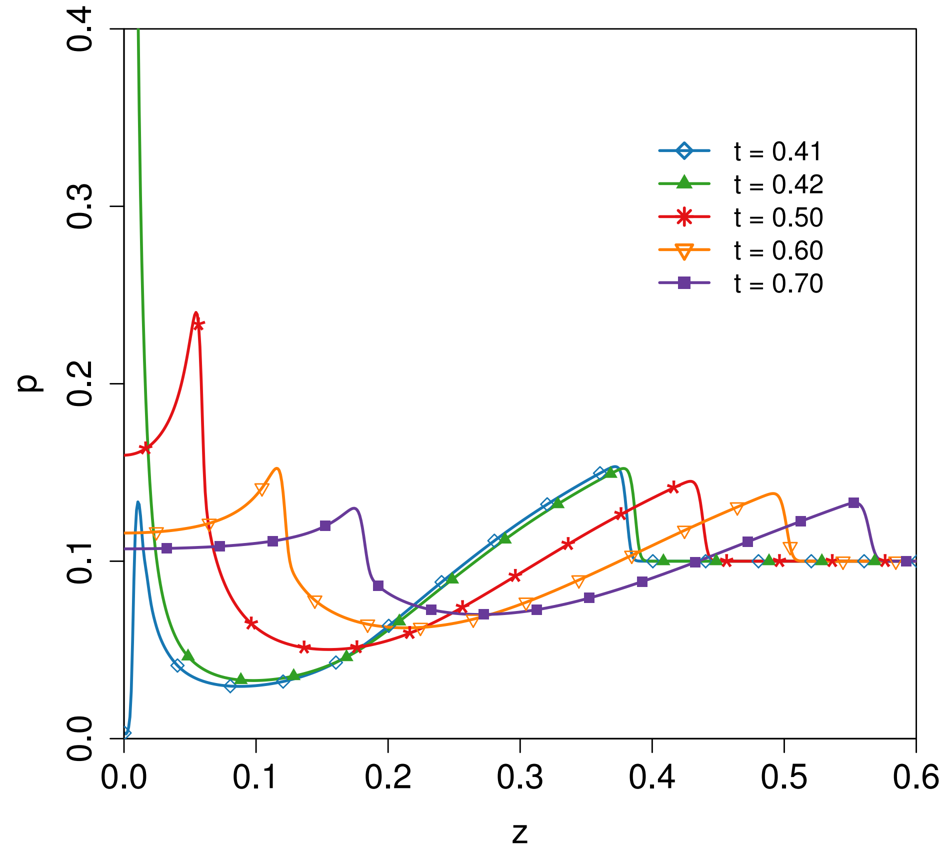


Non-relativistic: Inviscid limit: Secondary shock



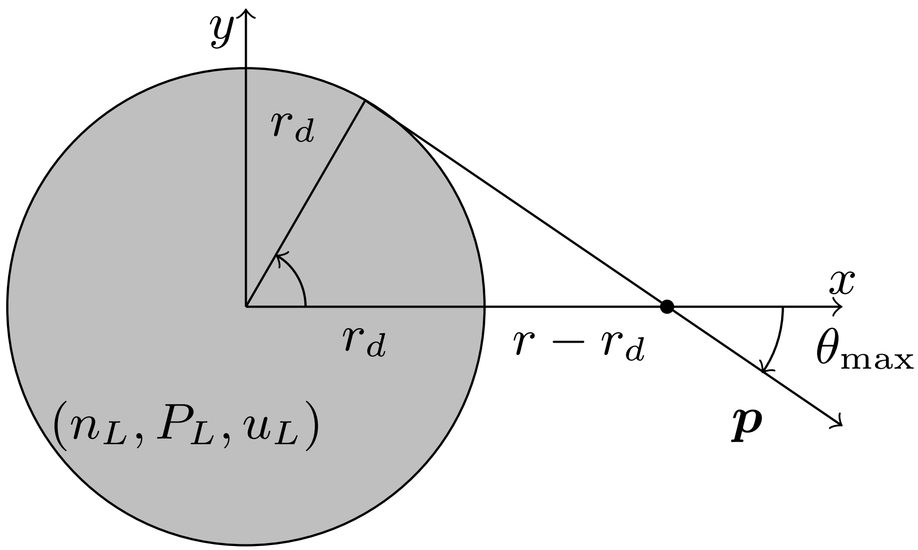
Ultra-relativistic: Inviscid limit[10]





[10] T. P. Downes, P. Duffy, S. S. Komissarov, Mon. Not. R. Astron. Soc. **332** (2002) 144–154.

Non-relativistic: Collisionless limit: solution



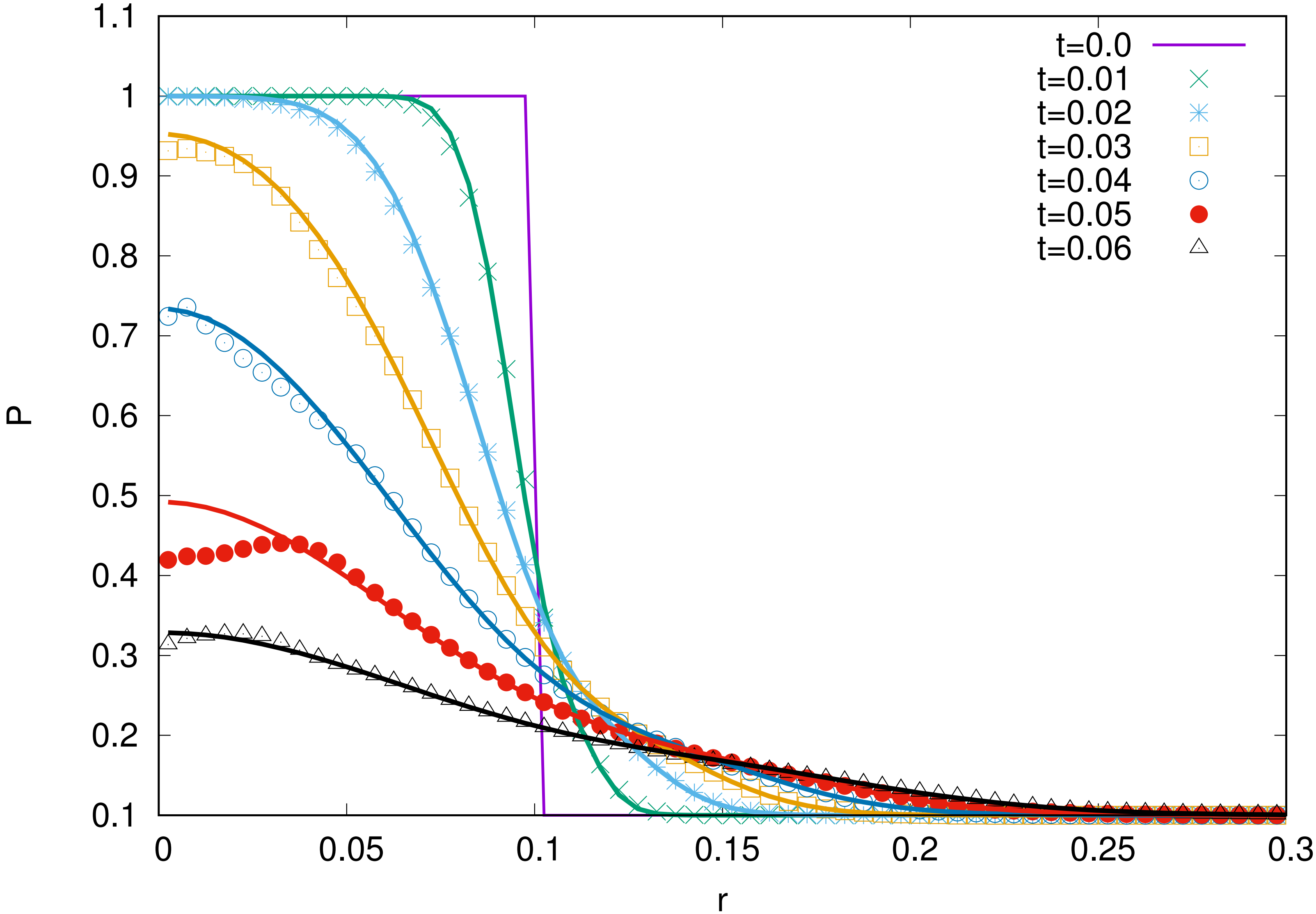
$$f(r,p,\theta,t) = \begin{cases} f_L^{(\rm eq)}, & \theta < \theta_{\max} \text{ and } p_- < p < p_+ \\ f_R^{(\rm eq)}, & \text{otherwise.} \end{cases}, \qquad \theta_{\max} = \begin{cases} \arcsin \frac{r_d}{r} & r > r_d \\ \pi & r < r_d \end{cases},$$

while $p_\pm = \frac{m}{t}(r\cos\theta \pm \sqrt{r_d^2 - r^2 \sin^2\theta})$ ($p_- = 0$ when $r < r_d$).
Solution ($\zeta_{\pm,L/R} \equiv p_\pm / \sqrt{2mT_{L/R}}$):

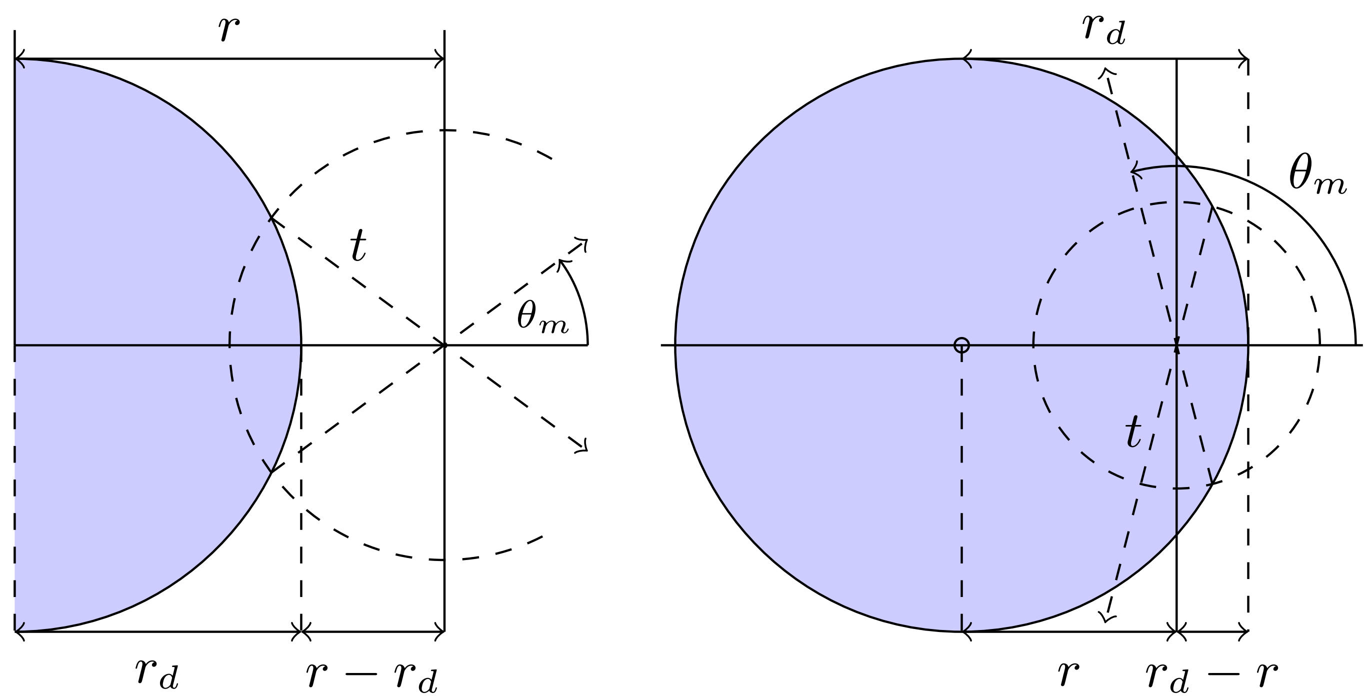
$$n = n_R + \int_0^{\theta_{\max}} d\theta\, \sin\theta \left\{ \frac{n_L}{\sqrt{\pi}} \left[\left(\zeta_{-,L} e^{-\zeta_{-,L}^2} - \frac{\sqrt{\pi}}{2} \text{erf}\zeta_{-,L} \right) - (+ \leftrightarrow -) \right] - [L \leftrightarrow R] \right\},$$

etc.

Non-relativistic: Ballistic limit



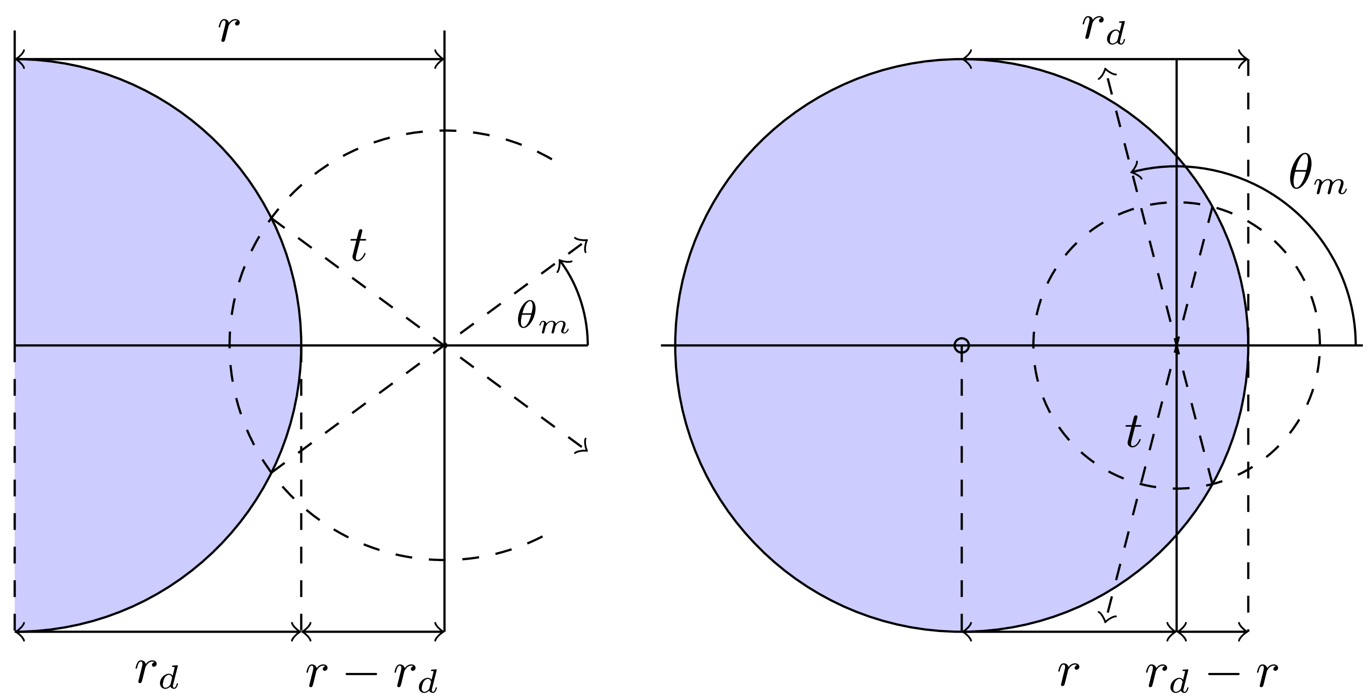
Ultra-relativistic: Collisionless limit[11]



- $f(r - r_d > t) = f_R$ (causality: no L particles reached r);
- $f(r_d - r > t) = f_L$ (causality: no R particles reached r);
- $f(r + r_d < t) = f_R$ (all L particles have flown away);

[11] C. Greiner, D.-H. Rischke, Phys. Rev. C **54** (1996) 1360–1365.

Ultra-relativistic: Collisionless limit: Solution



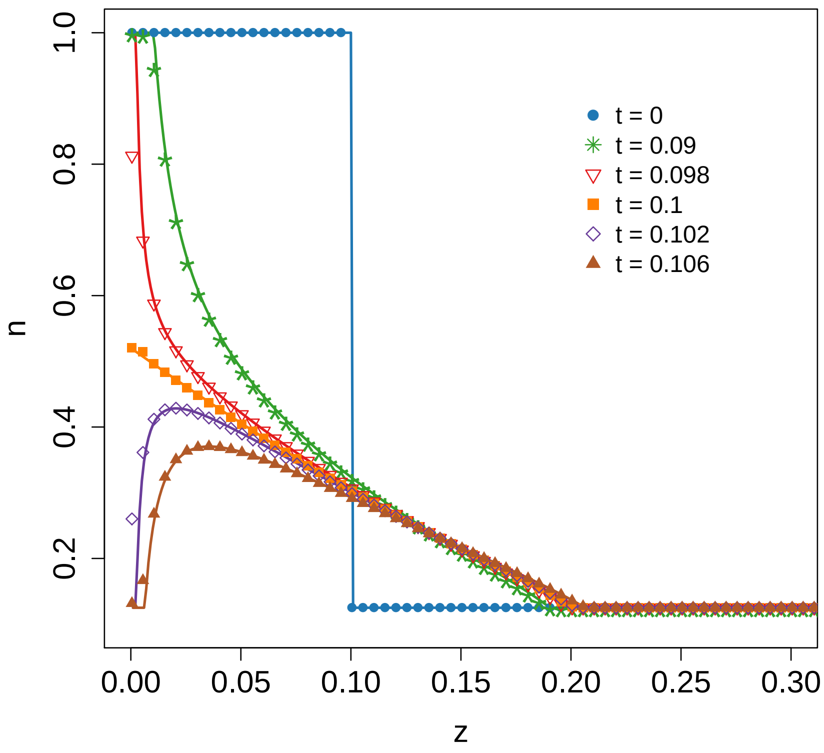
When $r > r_d$ and $r - r_d < t < r + r_d$:

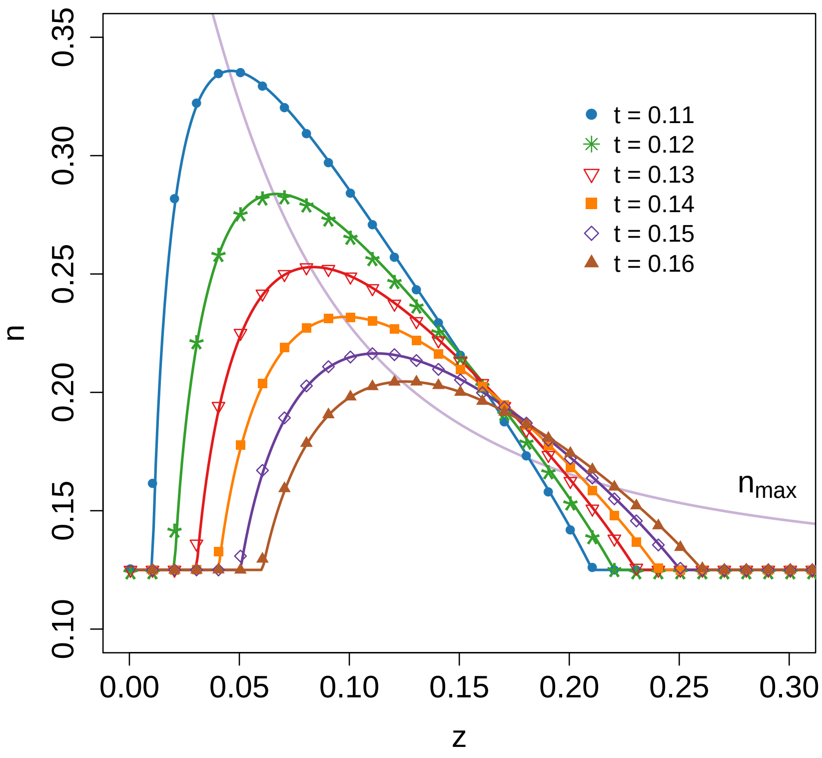
$$f(r, t; \theta, \varphi) = \begin{cases} f_1, & 0 < \theta < \theta_m, \\ f_2, & \text{otherwise}, \end{cases} \qquad \theta_m = \begin{cases} \arccos \frac{r^2 + t^2 - r_d^2}{2rt}, & r - r_d < t < r + r_d, \\ 0, & \text{otherwise}. \end{cases}$$

When $r < r_d$ and $r_d - r < t$:

$$f(r, t; \theta, \varphi) = \begin{cases} f_2, & \theta_m < \theta < \pi, \\ f_1, & \text{otherwise}. \end{cases}, \qquad \theta_m = \begin{cases} \pi, & r_d - r > t, \\ \arccos \frac{r^2 + t^2 - r_d^2}{2rt}, & r_d - r < t < r_d + r, \\ 0, & t > r_d + r. \end{cases}$$

Ultra-relativistic: Ballistic limit





Section 4

Cylindrical shocks

Non-relativistic case: Equation

- Let us consider the Sod shock tube problem with cylindrical symmetry:

$$f(R, t = 0) = \begin{cases} f_L^{(\mathrm{eq})}, & R < R_d \qquad (n_L, P_L, u_L) = (1, 1, 0) \\ f_R^{(\mathrm{eq})}, & R > R_d \qquad (n_R, P_R, u_R) = (0.125, 0.1, 0) \end{cases}.$$

- Cylindrical coordinates: R, ϕ, z and $p_{\hat{R}}, p_{\hat{\phi}}, p_{\hat{z}}$:

$$\partial_t f + \frac{p_\perp \cos\varphi}{mR}\partial_R(fR) - \frac{p_\perp}{mR}\partial_\varphi(f \sin\varphi) = -\frac{1}{\tau}(f - f_{\mathrm{M-B}}^{(\mathrm{eq})}),$$

  where $p^{\hat{R}} = p_\perp \cos\varphi$ and $p^{\hat{\phi}} = p_\perp \sin\varphi$
- f can be expanded w.r.t. φ:

$$f = \frac{1}{2\pi}A_0 + \frac{1}{\pi}\sum_{j=1}^{\infty}\left[A_j \cos(j\varphi) + B_j \sin(j\varphi)\right].$$

- Thus, $\partial_\varphi(f \sin\varphi)$ can be written as:

$$\partial_\varphi(f\sin\varphi) = \int_0^{2\pi} d\varphi' \, \mathcal{K}(\varphi, \varphi') f(\varphi'),$$

  where

$$2\pi\mathcal{K}(\varphi,\varphi') = \cos\varphi + \sum_{m=1}^{\infty}\left\{(m+1)\cos[m\varphi' - (m+1)\varphi] - (m-1)\cos[m\varphi' - (m-1)\varphi]\right\}.$$

- The $p_{\hat{z}}$ degree of freedom can be integrated out:

$$g = \int_{-\infty}^{\infty} dp_z \, f, \qquad h = \int_{-\infty}^{\infty} dp_z \, f \, p_z^2.$$

Non-relativistic case: Discretisation

- $p_\perp$ and φ are discretised using the Gauss-Laguerre and Mysovskikh[12] quadrature methods:

$$\int_{-\infty}^{\infty} dp_x \int_{-\infty}^{\infty} dp_y\, g\, p_x^s p_y^r = \int_0^{\infty} dp_\perp\, p_\perp \int_0^{2\pi} d\varphi\, g p_\perp^{s+r} (\cos\varphi)^s (\sin\varphi)^r$$

$$= \sum_{k=1}^{Q_\perp} \sum_{j=1}^{Q_\varphi} g_{jk}\, p_{\perp,k}^{s+r} (\cos\varphi_j)^s (\sin\varphi_j)^r.$$

- The discrete magnitudes $p_{\perp,k}$ are obtained by solving $L_{Q_L}(p_{\perp,k}^2) = 0$, while $\varphi_j = 2\pi(j-1)/Q_\varphi$. The discrete populations g_{jk} are:

$$g_{jk} = \frac{\pi}{Q_\varphi} e^{p_k^2} w_k^L\, g(p_{\perp,k}, \varphi_j), \qquad w_k^L = \frac{p_k^2}{[(Q_\perp + 1) L_{Q_\perp}(p_k^2)]^2}.$$

- The derivative $\partial_\varphi(f \sin\varphi) \to [\partial_\varphi(f\sin\varphi)]_{jk} = \sum_{j'=1}^{Q_\varphi} \mathcal{K}^\varphi_{j,j'} f_{j',k}$, where

$$\mathcal{K}^\varphi_{j,j'} = \frac{1}{Q_\varphi} \left\{ \sum_{n=1}^{\lfloor Q_\phi/2 \rfloor} n \cos[n\varphi_j - (n-1)\varphi_{j'}] - \sum_{n=1}^{\lfloor Q_\varphi/2\rfloor - 1} n \cos[n\varphi_j - (n+1)\varphi_{j'}] \right\}.$$

- $g_{jk}^{(\mathrm{eq})} = \frac{2\pi n}{Q_\varphi} F_k E_{jk}$ is implemented using:

$$F_k = \frac{w_k}{2\pi} \sum_{\ell=0}^{N_\perp} (1 - 2mT)^\ell L_\ell(p_k^2), \qquad E_{jk} = w_j \sum_{s=0}^{\lfloor N_\varphi \rfloor} \frac{1}{s!} \left(-\frac{m\boldsymbol{u}^2}{2T}\right) \sum_{r=0}^{N_\varphi - 2s} \frac{1}{r!} \left(\frac{\boldsymbol{p}\cdot\boldsymbol{u}}{T}\right)^r.$$

[12] I. P. Mysovskikh, Soviet Math. Dokl. **36** (1988) 229–322.
[9] V. E. Ambruș, V. Sofonea, Phys. Rev. E **86** (2012) 016708.

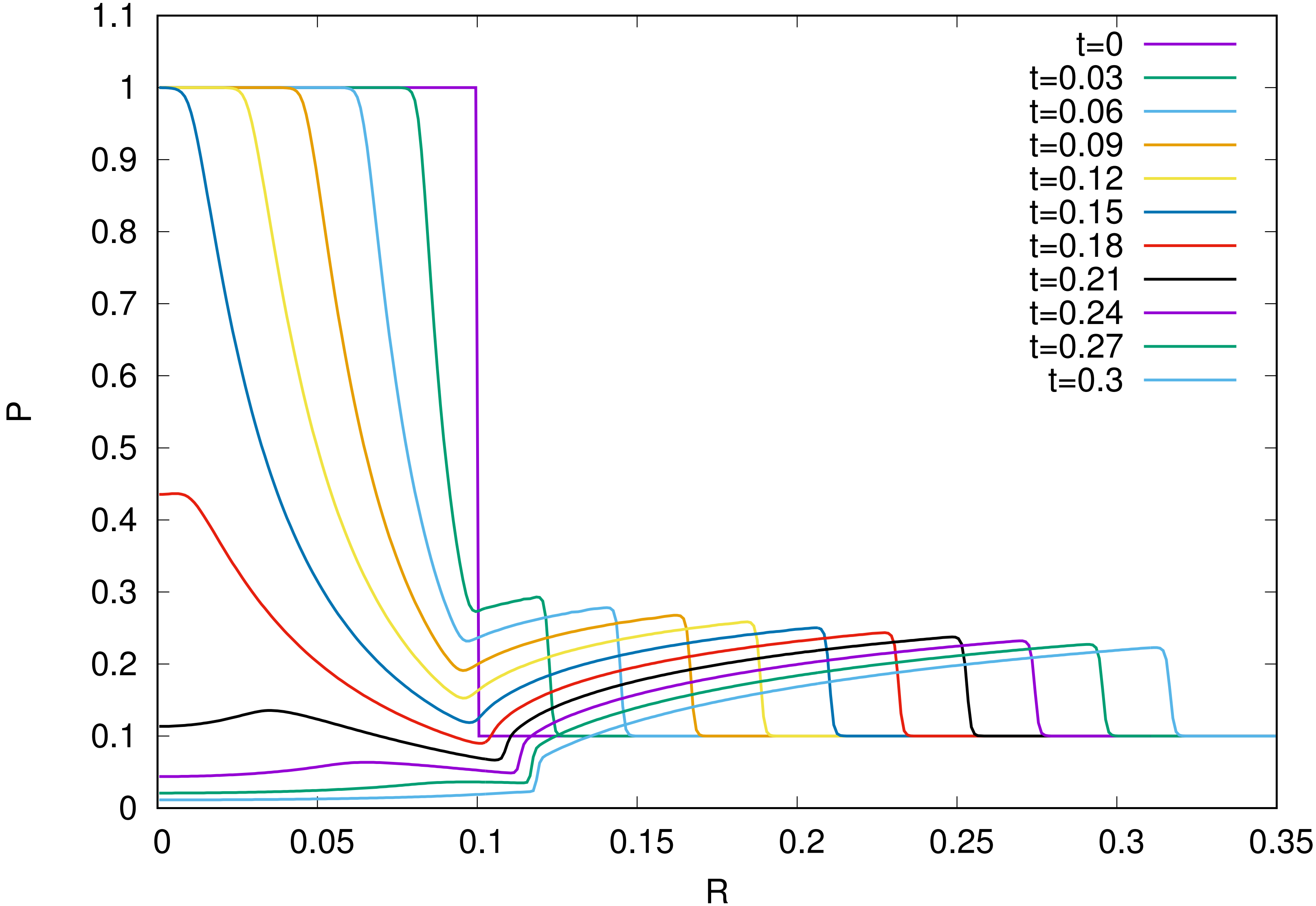
Ultrarelativistic case: equation

- Since $f^{(\mathrm{eq})}_{\mathrm{M-J}}$ depends on $p = \sqrt{p_x^2 + p_y^2 + p_z^2}$ (instead of p^2), it is not convenient to switch to $p_\perp$.
- The relativistic Boltzmann equation reads:

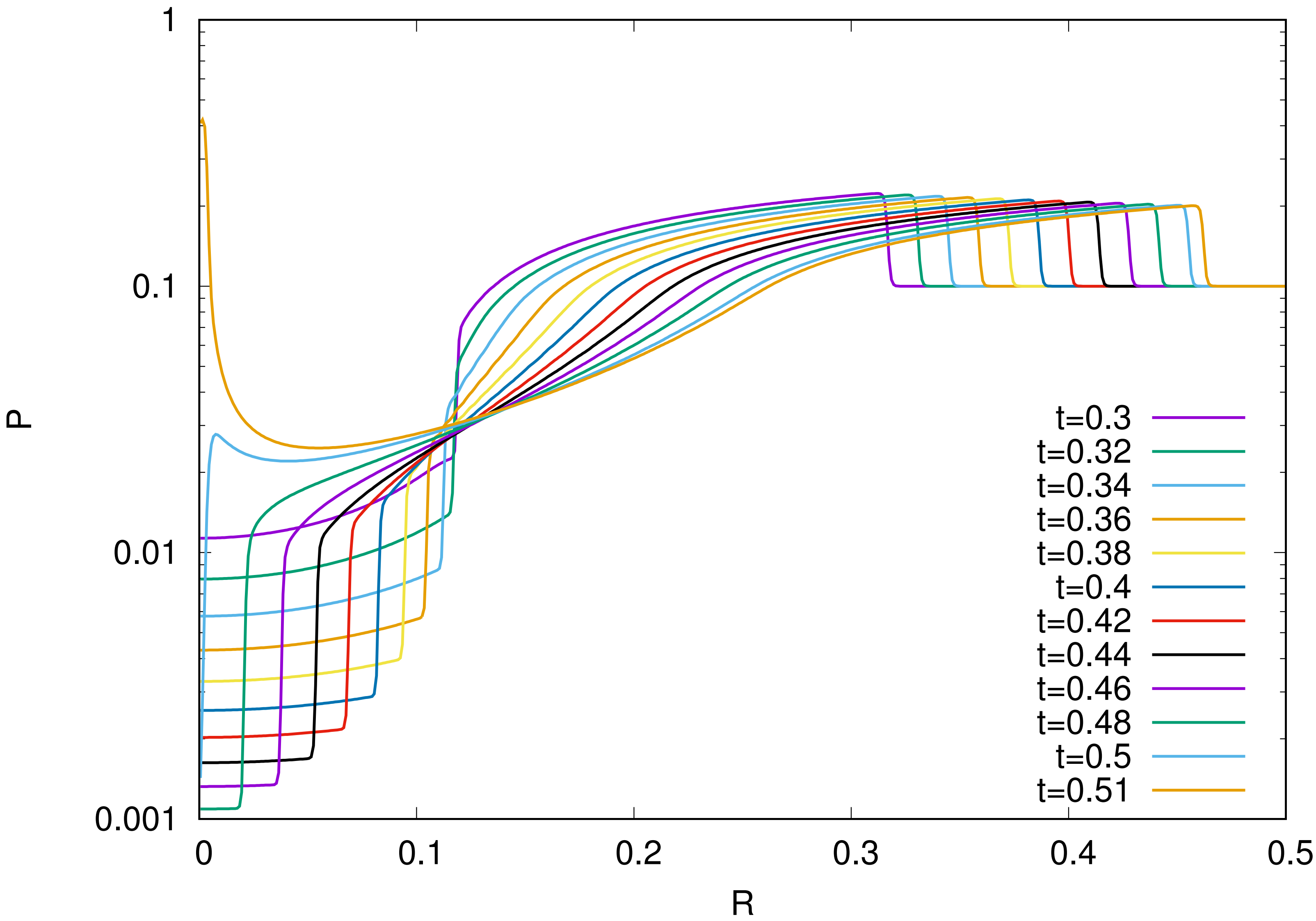
$$\partial_t f + \frac{\sin\theta\cos\varphi}{mR}\partial_R(fR) - \frac{\sin\theta}{mR}\partial_\varphi(f\sin\varphi) \\ = -\frac{\gamma_L}{\tau}(1 - \sin\theta\cos\varphi\beta_L)[f - f^{(\mathrm{eq})}_{\mathrm{M-J}}].$$

- The derivative w.r.t. φ is computed in exactly the same way as in the non-relativistic case.
- The momentum space is discretised in the same way as for the planar shock (now $Q_\varphi > 1$!).

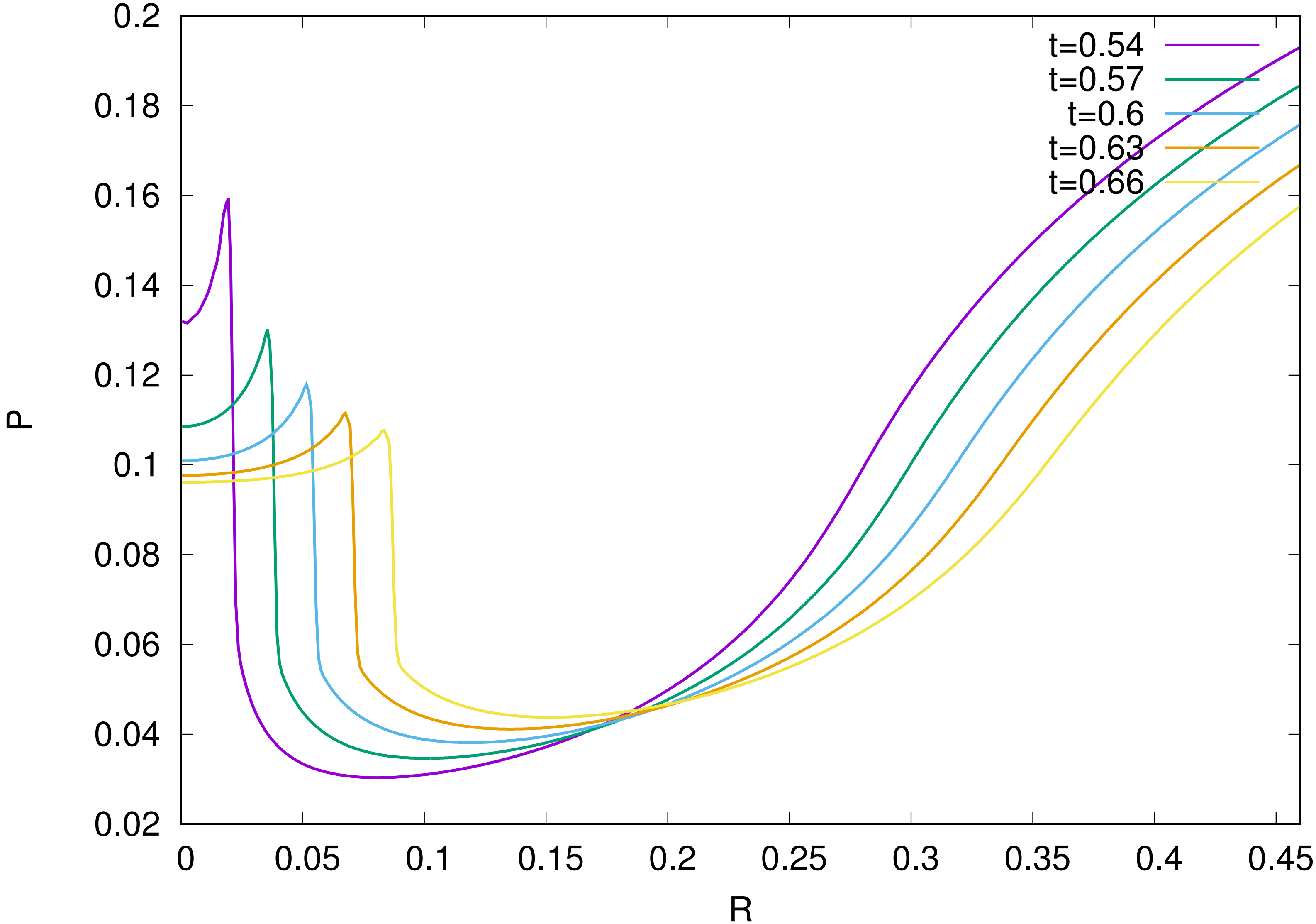
Ultra-relativistic: Inviscid limit: Primary shock



Ultra-relativistic: Inviscid limit: Reverse shock



Ultra-relativistic: Inviscid limit: Secondary shock



Non-relativistic: Collisionless limit: solution

$$g(R, p_\perp, \varphi, t) = \begin{cases} g_L^{(\mathrm{eq})}, & |\varphi| < \varphi_{\max} \text{ and } p_- < p_\perp < p_+ \\ g_R^{(\mathrm{eq})}, & \text{otherwise.} \end{cases}$$

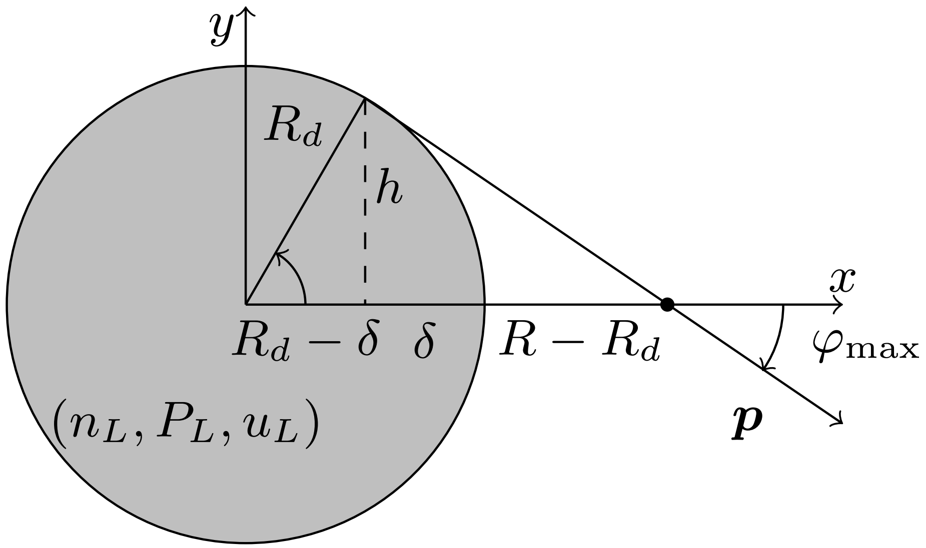
If $R > R_d$:

- $\varphi_{\max} = \arcsin(R_d/R)$;
- $p_\pm = \frac{m}{t}(R \pm \sqrt{R_1^2 - R^2 \sin^2 \varphi})$;

If $R < R_d$:

- $\varphi_{\max} = \pi$;
- $p_- = 0$.

Solution ($\zeta_{\pm, L/R} \equiv p_\pm/\sqrt{2mT_{L/R}}$):

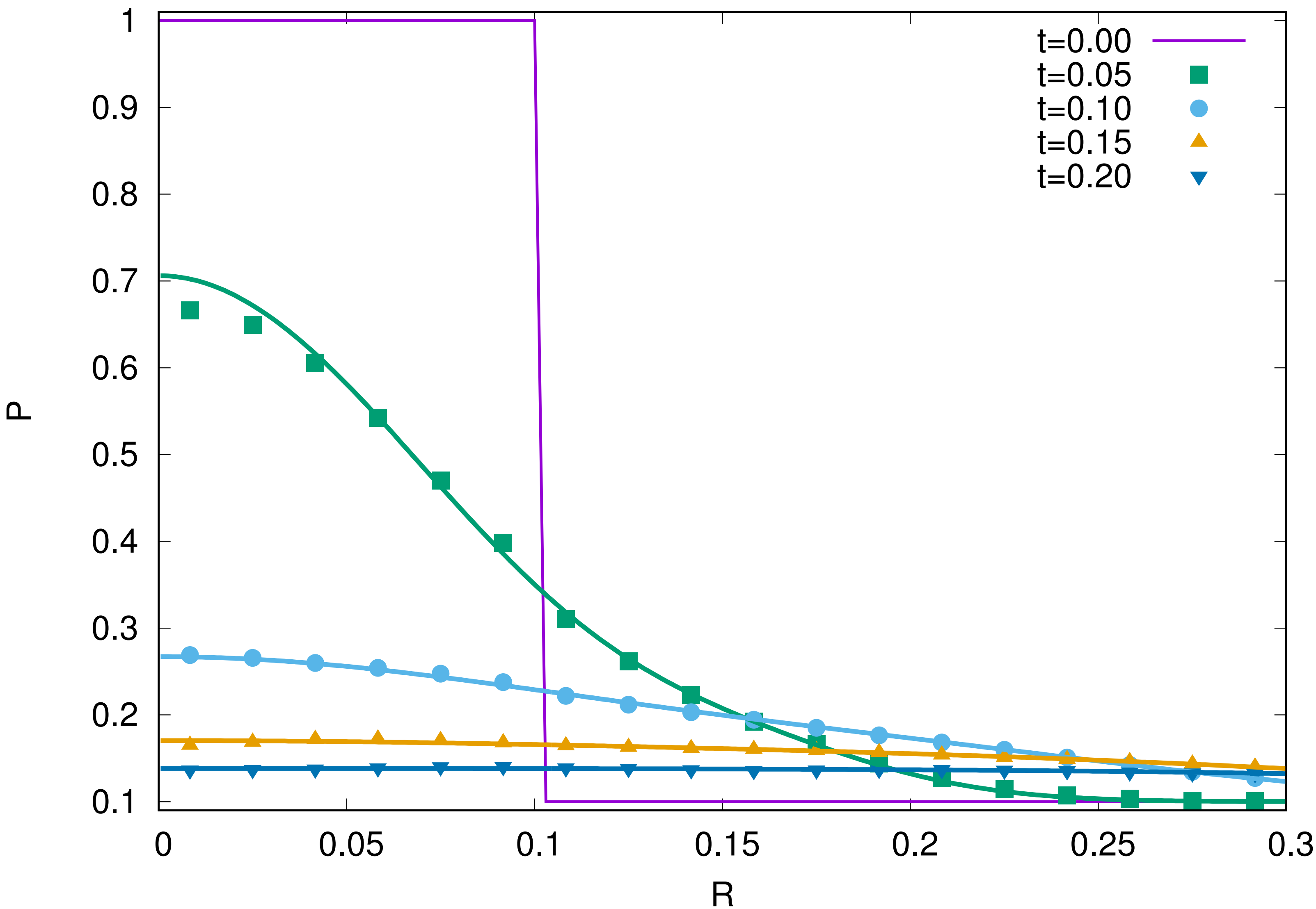


$$n = n_R + \int_0^{\varphi_{\max}} d\varphi \left[\frac{n_L}{\pi}\left(e^{-\zeta_{-,L}^2} - e^{-\zeta_{+,L}^2}\right) - (L \leftrightarrow R)\right],$$

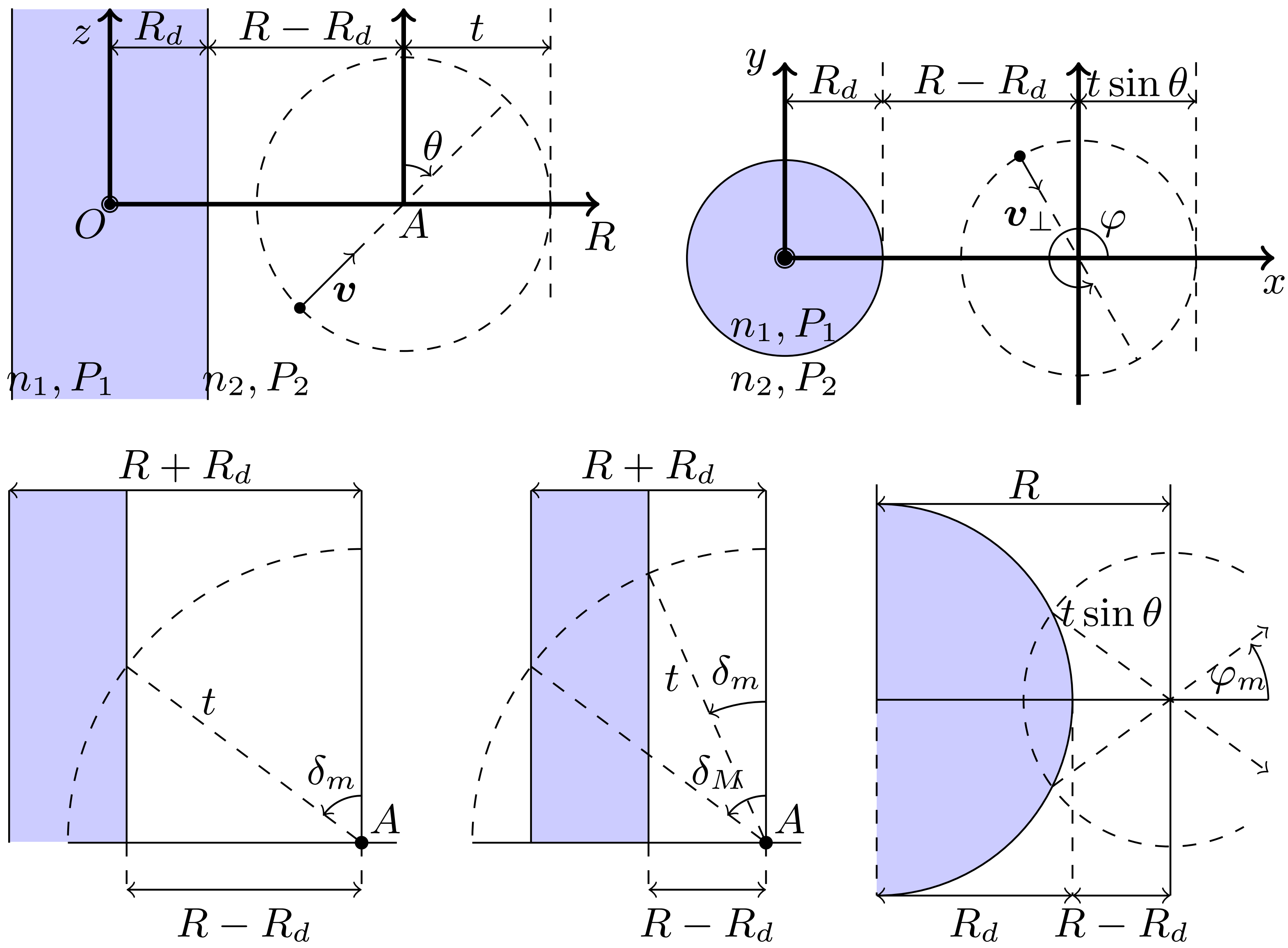
$$\rho u = \int_0^{\varphi_{\max}} d\varphi \, \cos\varphi \left\{\frac{n_L}{\pi}\left[p_- e^{-\zeta_{-,L}^2} - p_+ e^{-\zeta_{+,L}^2} + \sqrt{\frac{\pi m T_L}{2}}(\mathrm{erfc}\zeta_{-,L} - \mathrm{erfc}\zeta_{+,L})\right] - (L \leftrightarrow R)\right\}$$

$$\frac{3}{2}nT + \frac{1}{2}\rho u^2 = \frac{3}{2}n_R T_R + \int_0^{\varphi_{\max}} d\varphi \left\{\frac{n_L T_L}{\pi}\left[\left(\frac{3}{2} + \zeta_{-,L}^2\right)e^{-\zeta_{-,L}^2} - \left(\frac{3}{2} + \zeta_{+,L}^2\right)e^{-\zeta_{+,L}^2}\right] - (L \leftrightarrow R)\right\}$$

Non-relativistic: Ballistic limit



Ultra-relativistic: Collisionless limit: solution ($R > R_d$)



Ultra-relativistic: Collisionless limit: solution $(R > R_d)$

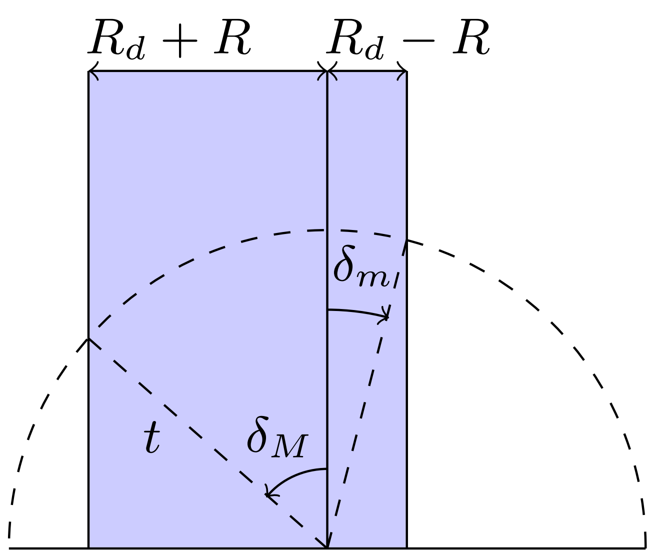
$$f(R > R_d, t; \theta, \varphi) = \begin{cases} f_1, & \theta \in (\delta_m, \delta_M) \cup (\pi - \delta_M, \pi - \delta_m) \text{ and} \\ & \varphi \in (0, \varphi_m) \cup (2\pi - \varphi_m, 2\pi), \\ f_2, & \text{otherwise,} \end{cases}$$

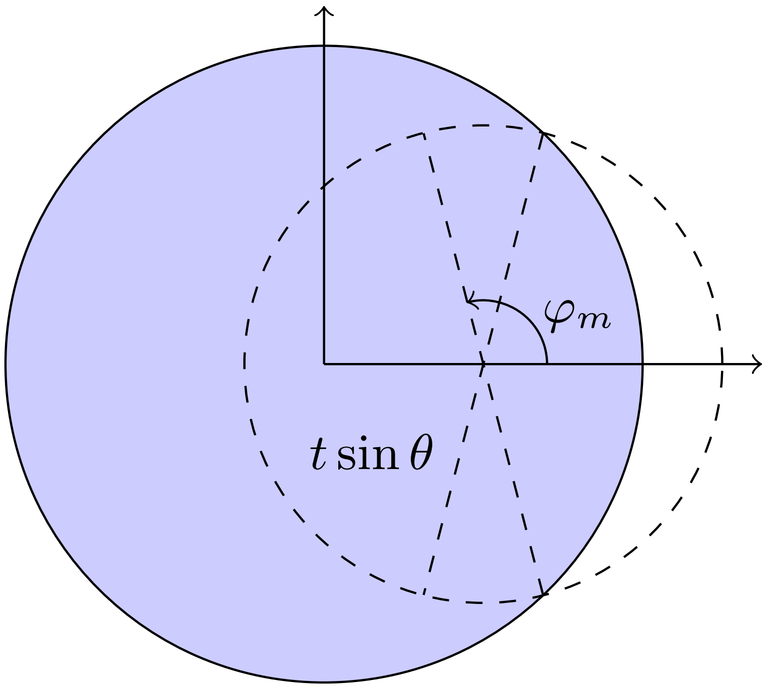
where

$$\delta_m = \begin{cases} \arcsin \frac{|R - R_d|}{t} & |R - R_d| < t \\ \pi/2 & \text{otherwise.} \end{cases} \qquad \delta_M = \begin{cases} \arcsin \frac{R + R_d}{t} & R + R_d < t \\ \pi/2 & \text{otherwise} \end{cases}$$

$$\varphi_m = \begin{cases} \arccos \frac{R^2 + t^2 \sin^2\theta - R_d^2}{2Rt \sin\theta}, & |R - R_d| < t \sin\theta < R + R_d, \\ 0, & \text{otherwise.} \end{cases}$$

Ultra-relativistic: Collisionless limit: solution ($R < R_d$)



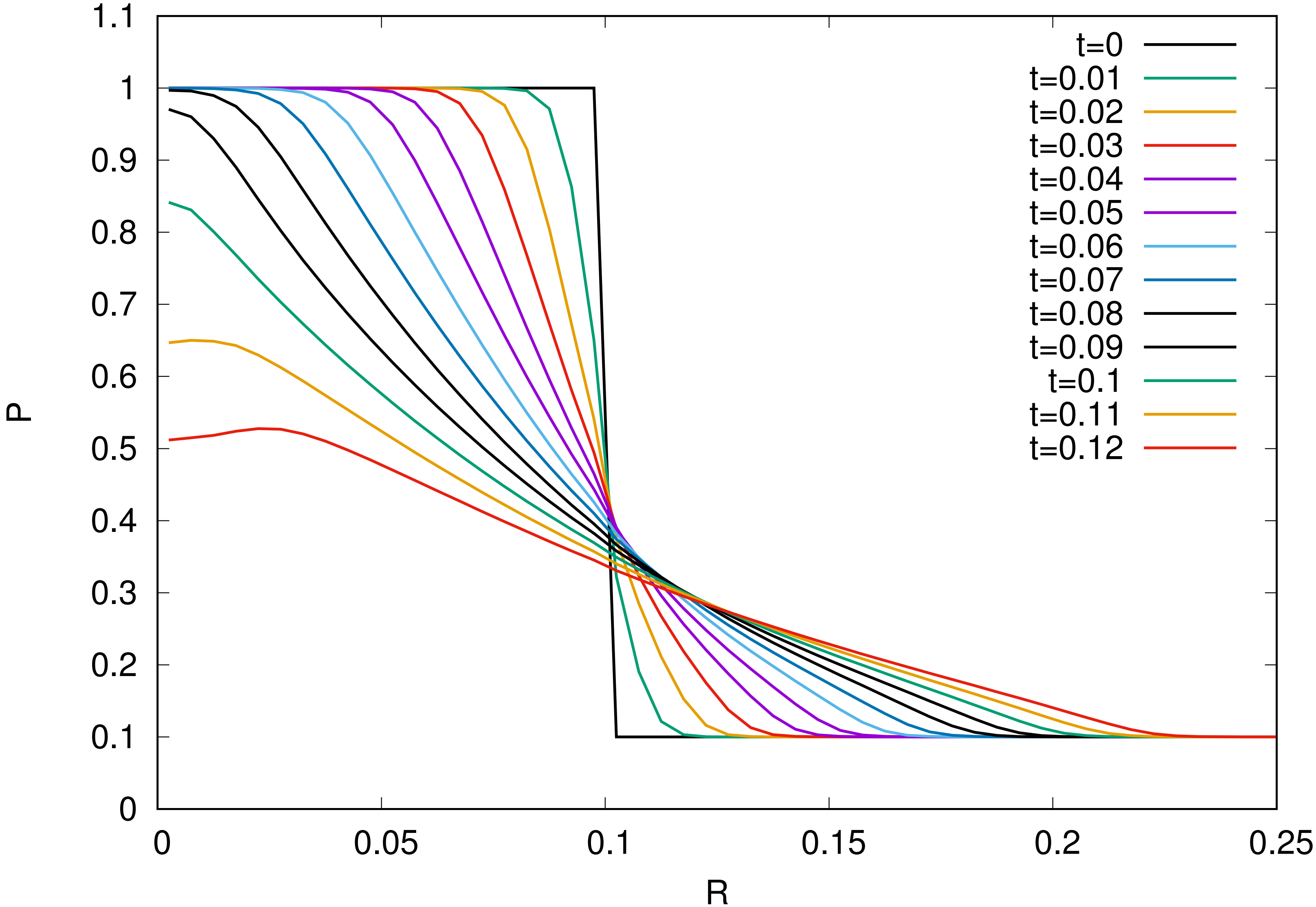


$$f(R,t;\theta,\varphi) = \begin{cases} f_2 & \theta \in (\delta_m, \pi - \delta_m) \text{ and } \varphi \in (\varphi_m, 2\pi - \varphi_m), \\ f_1 & \text{otherwise}, \end{cases}$$

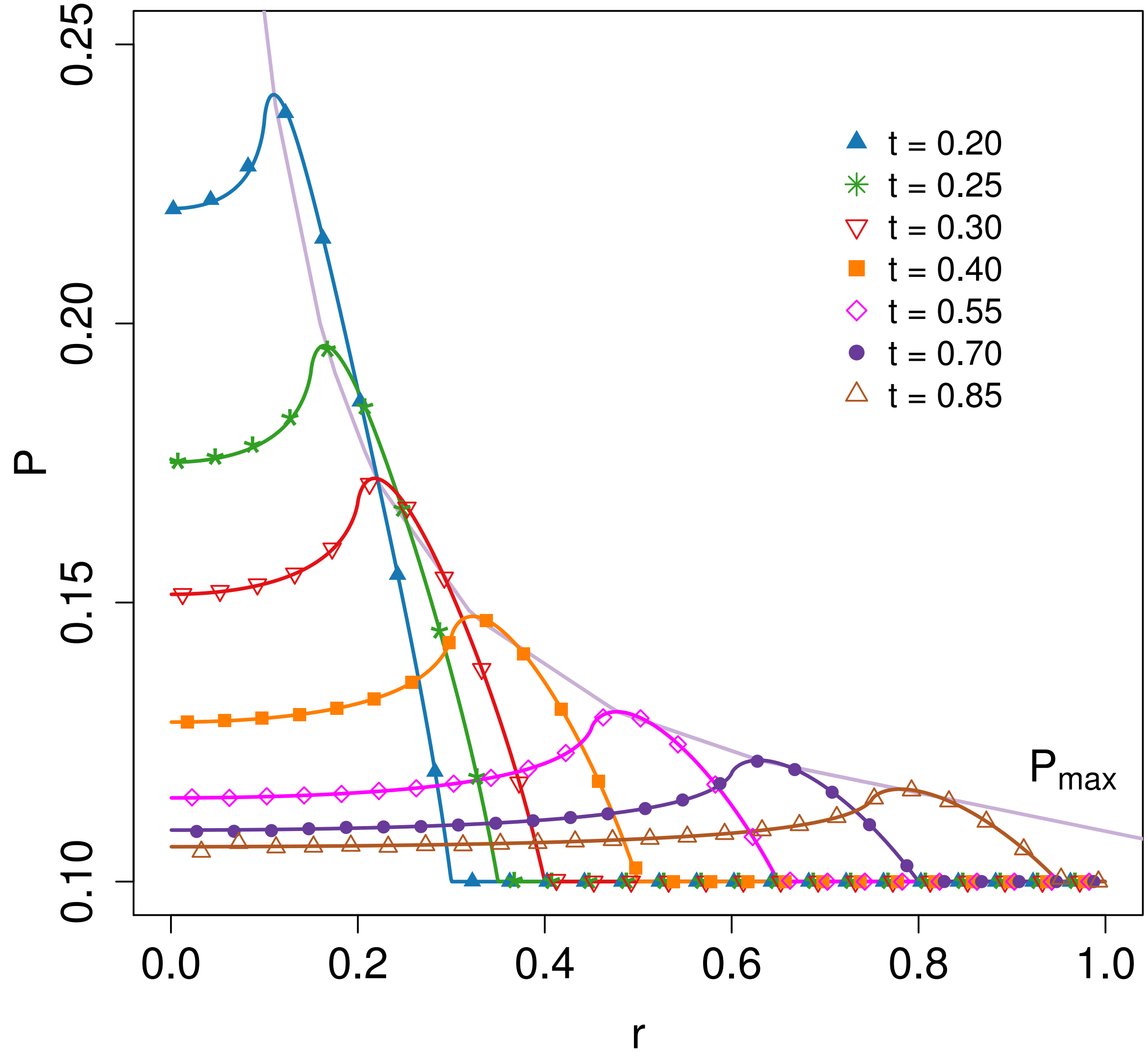
$$\delta_m = \begin{cases} \arcsin \frac{|R - R_d|}{t} & |R - R_d| < t \\ \pi/2 & \text{otherwise}. \end{cases}$$

$$\varphi_m = \begin{cases} \pi, & R_d - R > t\sin\theta, \\ \arccos \frac{R^2 + t^2 \sin^2\theta - R_d^2}{2Rt\sin\theta}, & R_d - R < t\sin\theta < R_d + R, \\ 0, & R_d + R < t\sin\theta. \end{cases}$$

Ultra-relativistic: Ballistic limit



Ultra-relativistic: Ballistic limit



Conclusion

- LB can be applied to study flows from the inviscid to the ballistic regime.
- The discretisation of the momentum space using Gauss quadratures ensures the exact recovery of the moments of f.
- Flows in the inviscid regime require small velocity sets ($Q \sim 6$).
- For the ballistic regime, $Q \sim 100$.
- Flows in arbitrary coordinate systems can be studied using the vielbein formalism.
- Aligning the momentum space directions along the coordinate unit vectors allows the flow symmetries to be preserved.
- The examples shown are $1D$. However, the scheme is directly extendible to more complex geometries.
- In the inviscid regime, the rich phenomenology of cylindrical and spherical shocks was successfully captured.
- In the ballistic regime, the LB method was successfully validated against the analytic result.