????

Manuel Saorín
Departamento de Matemáticas
Universidad de Murcia, Aptdo. 4021
30100 Espinardo, Murcia
SPAIN
msaorinc@um.es

Jan Stovicek
Institut for Matematiske Fag, NTNU
N-7491 Trondheim
NORWAY
stovicek@math.ntnu.no

Abstract

*

SILTING THEORY REVISITED

Manuel Saorín
(joint with Alexandra Zvonareva)

CONFERENCE ON 'ADVANCES IN REPRESENTATION THEORY OF ALGEBRAS: GEOMETRY AND HOMOLOGY'

Centre International de Rencontres Mathématiques
Luminy, Marseille (FRANCE)
September 6, 2017

In this talk all triangulated categories that appear are K-categories with split idempotents, for some commutative ring K, and all subcategories are full subcategories.

DEFINITION.- If $\mathcal{D}$ is a triangulated category, a pair of subcategories $(\mathcal{U}, \mathcal{V})$ is called a *torsion pair* when $\mathcal{U}$ and $\mathcal{V}$ are closed under direct summands and extensions and $\mathcal{D} = \mathcal{U} \star \mathcal{V}$ (i.e. each object $M \in \mathcal{D}$ appears in a triangle $U \longrightarrow M \longrightarrow V \stackrel{+}{\longrightarrow}$, where $U \in \mathcal{U}$ and $V \in \mathcal{V}$). This torsion pair is called:

1. **A *t-s:tructure*, when $\mathcal{U}[1] \subseteq \mathcal{U}$ (equivalently, when $\mathcal{V}[-1] \subseteq \mathcal{V}$). In this case $\mathcal{H} = \mathcal{U} \cap \mathcal{V}[1]$ is called the *heart* of the t-structure.**

2. **A *co-t-structure* (or *weight structure*) when $\mathcal{U}[-1] \subseteq \mathcal{U}$ (equivalently, when $\mathcal{V}[1] \subseteq \mathcal{V}$). In this case $\mathcal{C} := \mathcal{U}[1] \cap \mathcal{V}$ is called the *co-heart* of the co-t-structure.**

The torsion pair is called *left (resp. right) bounded* when $\mathcal{D} = \bigcup_{n \in \mathbb{Z}} \mathcal{U}[n]$ (resp. $\mathcal{D} = \bigcup_{n \in \mathbb{Z}} \mathcal{V}[n]$). It is called *bounded* when it is left and right bounded.

DEFINITION.- **A set of objects $\mathcal{T} \subset \mathcal{D}$ is called a *silting set* when the following two conditions hold:**

a) **$\mathcal{T}$ is *nonpositive* (i.e. Hom$_{\mathcal{D}}(T, T'[k]) = 0$, for all $T, T' \in \mathcal{T}$ and all integers $k > 0$);**

b) **thick$_{\mathcal{D}}(\mathcal{T}) = \mathcal{D}$.**

Two nonpositive sets $\mathcal{T}$ and $\mathcal{T}'$ are said to be *equivalent* when add$(\mathcal{T})$ = add$(\mathcal{T}')$. A *silting object* of $\mathcal{D}$ is any object T such that $\{T\}$ is a silting set.

DEFINITION.- **A subcategory $\mathcal{X} \subseteq \mathcal{D}$ is called *suspended* (resp. *co-suspended*) when it is closed under extensions, $\mathcal{X}[1] \subseteq \mathcal{X}$ (resp. $\mathcal{X}[-1] \subseteq \mathcal{X}$) and, during this talk, we will assume also that it is closed under taking direct summands. If $\mathcal{S}$ is any set, then susp$_{\mathcal{D}}(\mathcal{S})$ (resp. cosusp$_{\mathcal{D}}(\mathcal{S})$) will denote the smallest suspended (resp. co-suspended) subcategory of $\mathcal{D}$ which contains $\mathcal{S}$.**

THEOREM (König-Yang 2014, Keller-Nicolás).- Let Λ be a finite dimensional algebra over a field. There is a bijection between:

1. **Equivalence classes of silting objects of $\mathcal{D}(\Lambda)^c \cong \mathcal{K}^b($proj$-\Lambda)$.**
2. **Bounded co-t-structures in $\mathcal{K}^b($proj $- \Lambda)$.**
3. **Bounded t-structures in $\mathcal{D}^b(\Lambda -$ mod$)$ whose heart is the category of finite dimensional modules over a finite dimensional algebra.**
4. **Simple-minded collections in $\mathcal{D}^b($mod $- \Lambda)$.**

DEFINITION.- A set of objects $\mathcal{S}$ is said to be *weakly preenveloping* in $\mathcal{D}$ when the following two conditions hold:

1. **For each object $M \in \mathcal{D}$, one has Hom$_{\mathcal{D}}(M, ?[k])_{|\mathcal{S}} = 0$ for $k \gg 0$;**
2. **If $\mathbb{N}_M := \{k \in \mathbb{N}$: Hom$_{\mathcal{D}}(M, ?[k])_{|\mathcal{S}} \neq 0\}$ is nonempty and $m =$ max$(\mathbb{N}_M)$, then M has an add$(\mathcal{S})[m]$-preenvelope ($=$ left add$(\mathcal{S})[m]$-approximation).**

REMARK.- One clearly has the dual concept of *weakly precovering* set in $\mathcal{D}$.

EXAMPLE.- If $\mathcal{D}$ is Hom-finite (over some commutative ring K) and $\mathcal{S}$ is a finite set, then the following two conditions are equivalent:

1. **$\mathcal{S}$ is weakly preenveloping (resp. weakly precovering);**
2. **For each object M of D, one has $\mathbf{Hom}_{\mathcal{D}}(M, ?[k])_{|\mathcal{S}} = 0$ (resp. $\mathbf{Hom}_{\mathcal{D}}(?[-k], M)_{|\mathcal{S}} = 0$) for $k \gg 0$.**

In particular, if Λ is an Artin algebra the following assertions are equivalent for a <u>finite</u> nonpositive set $\mathcal{T}$ in $\mathcal{D}^b(\mathbf{mod} - \Lambda)$:

a) **$\mathcal{T}$ is weakly preenveloping (resp. weakly precovering) in $\mathcal{D}^b(\mathbf{mod} - \Lambda)$;**

b) **(Up to quasi-isomorphism) $\mathcal{T} \subseteq \mathcal{K}^b(\mathbf{inj} - \Lambda)$ (resp. $\mathcal{T} \subseteq \mathcal{K}^b(\mathbf{proj} - \Lambda)$).**

THEOREM (S.-Zvonareva).- Let $\mathcal{D}$ be a skeletally small triangulated category. The assignment $\mathcal{T} \rightsquigarrow ({}^{\perp_{\geq 0}}\mathcal{T}, \mathbf{susp}_{\mathcal{D}}(\mathcal{T}))$ (resp. $\mathcal{T} \rightsquigarrow (\mathbf{cosusp}_{\mathcal{D}}(\mathcal{T})[-1], \mathcal{T}^{\perp_{>0}})$) defines a bijection between the set of equivalence classes of weakly preenveloping (resp. weakly precovering) nonpositive sets and the set of left (resp right) bounded co-t-structures in $\mathcal{D}$. Its inverse takes $\rho = (\mathcal{U}, \mathcal{V})$ to (the equivalence class of) a set of representatives of the objects in the co-heart $\mathcal{C}_\rho = \mathcal{U}[1] \cap \mathcal{V}$.

Moreover, this bijection restricts to a bijection between the set of (equivalence classes of) silting sets and the set of bounded co-t-structures in $\mathcal{D}$.

QUESTION.- Let Λ be an Artin algebra. Does there exist (weakly precovering) <u>infinite</u> nonpositive sets in $\mathcal{K}^b(\textbf{proj} - \Lambda)$?

COROLLARY.- Let Λ be an Artin algebra and consider the sets $\mathcal{X}_i$ (i=1,2,3) whose elements are, respectively:

1. **The equivalence classes of nonpositive objects in $\mathcal{K}^b(\textbf{inj}-\Lambda)$ (resp. $\mathcal{K}^b(\textbf{proj} - \Lambda)$);**
2. **The left (resp. right) bounded co-t-structures in $\mathcal{K}^b(\textbf{inj} - \Lambda)$ (resp. $\mathcal{K}^b(\textbf{proj} - \Lambda)$);**
3. **The left (resp. right) bounded co-t-structures in $\mathcal{D}^b(\textbf{mod} - \Lambda)$.**

There is a bijection $\mathcal{X}_2 \stackrel{\cong}{\longleftrightarrow} \mathcal{X}_3$ and an injection $\mathcal{X}_1 \rightarrowtail \mathcal{X}_2$. If the answer to the previous question is 'No' for all Artin algebras, this latter map is also bijective. In any case, the latter map induces a bijection between the equivalences classes of silting objects and the bounded co-t-structures in $\mathcal{K}^b(\textbf{inj} - \Lambda)$ (resp. $\mathcal{K}^b(\textbf{proj} - \Lambda)$.

DEFINITION.- **Let $\mathcal{T}$ be a set of objects in $\mathcal{D}$. We say that $\mathcal{T}$ is *partial silting* when the following two conditions hold:**

1. $(\mathcal{U}_{\mathcal{T}}, \mathcal{V}_{\mathcal{T}}) := ({}^{\perp}(\mathcal{T}^{\perp_{\leq 0}}), \mathcal{T}^{\perp_{\leq 0}})$ **is a t-structure in $\mathcal{D}$;**
2. $\textbf{Hom}_{\mathcal{D}}(T, ?[1])$ **vanishes on $\mathcal{U}_{\mathcal{T}}$, for each $T \in \mathcal{T}$.**

If, in addition, $\mathcal{T}$ generates $\mathcal{D}$ (i.e. $\mathcal{T}^{\perp_{i\in\mathbb{Z}}} = 0$), we will say that $\mathcal{T}$ is *partial silting generating* set in $\mathcal{D}$.

REMARK.- **Any partial silting set is nonpositive since $\bigcup_{k\geq 0} \mathcal{T}[k] \subset \mathcal{U}_{\mathcal{T}}$. Furthermore, when $\mathcal{D}$ has (set-indexed) coproducts it is even *strongly nonpositive*, i.e. $\textbf{Hom}_{\mathcal{D}}(T, ?[k])$ vanishes in $\textbf{Add}(\mathcal{T})$, for each $T \in \mathcal{T}$ and each integer $k > 0$.**

EXAMPLES.-

1. **If $\mathcal{D}$ has coproducts, then any nonpositive set of compact objects is a partial silting set. Such a set is generating if, and only if, $\mathcal{T}$ is a silting set in $\mathcal{D}^c$.**

2. **If A is any algebra, then a *semi-tilting complex* ([Wei-2013]) is a bounded complex of projectives $P^\bullet$ such that $\mathbf{Hom}_{\mathcal{D}(A)}(P^\bullet, P^{\bullet(I)}[k]) = 0$, for all sets I and integers $k > 0$, and such that $A \in \mathbf{thick}_{\mathcal{D}(A)}(\mathbf{Add}(P^\bullet))$. Such a complex is a partial silting generating object of $\mathcal{D}(A)$. In particular, any big (n-)tilting A-module is a partial silting generating object of $\mathcal{D}(A)$.**

3. **Suppose that $\mathcal{D}$ is a thick subcategory of a triangulated category $\mathcal{E}$ and that $\mathcal{T} \subset \mathbf{Ob}(\mathcal{D})$ is a set which is partial silting in $\mathcal{E}$. If the associated t-structure in $\mathcal{E}$ restricts to $\mathcal{D}$, then T is partial silting in $\mathcal{D}$.**

4. **If A is a finite dimensional algebra (or even a homologically finite dimensional homologically nonpositive dg algebra) over a field K, then any silting object of $\mathcal{D}^c(A)$ is a partial silting generating object of $\mathcal{D}^b(\mathbf{mod} - A)$.**

PROPOSITION.- Let $\mathcal{T}$ be a <u>precovering</u> partial silting set in $\mathcal{D}$ and let $(\mathcal{U}_{\mathcal{T}}, \mathcal{V}_{\mathcal{T}}) := ({}^{\perp}(\mathcal{T}^{\perp_{\leq 0}}), \mathcal{T}^{\perp_{\leq 0}})$ be the associated t-structure. The heart $\mathcal{H}_{\mathcal{T}} = \mathcal{U}_{\mathcal{T}} \cap \mathcal{V}_{\mathcal{T}}[1]$ is equivalent to the category mod $-\,\mathcal{T}$ of finitely presented $\mathcal{T}$-modules. When $\mathcal{T}$ is finite and $T := \coprod_{T' \in \mathcal{T}} T'$, then $B = \text{End}_{\mathcal{D}}(T)$ is a right coherent algebra and $\mathcal{H}_{\mathcal{T}} \cong$ mod $-\, B$.

PROPOSITION.- Let $\tau = (\mathcal{U}, \mathcal{V})$ be a left nondegenerate t-structure in $\mathcal{D}$ whose heart $\mathcal{H} = \mathcal{U} \cap \mathcal{V}[1]$ is equivalent to the category mod $-\,B$ of finitely presented (right) modules over a right coherent algebra B. Fix an equivalence F : mod $-\; B \stackrel{\cong}{\longrightarrow} \mathcal{H}$ and put $P = F(B)$. If the functor $\text{Hom}_{\mathcal{H}}(P, \tilde{H}(?)) : \mathcal{D} \stackrel{\cong}{\longrightarrow}$ Mod $-\, K$ is representable, where $\tilde{H} : \mathcal{D} \longrightarrow \mathcal{H}$ is the cohomological functor given by the t-structure, then there exists a precovering partial silting object T of $\mathcal{D}$ such that $\tau = ({}^{\perp}(T^{\perp_{\leq 0}}), T^{\perp_{\leq 0}})$.

THEOREM (Bondal-Van den Bergh, Rouquier).- Let K be a commutative noetherian ring and let $\mathcal{D}$ be an Hom-finite triangulated K-category. If $\mathcal{D}$ has a strong generator, then each locally finitely presented cohomological functor $H : \mathcal{D} \longrightarrow \text{Mod} - K$ is representable. If, in addition, $\mathcal{D}$ is Ext-finite, then each locally finite cohomological functor $F : \mathcal{D} \longrightarrow \text{Mod} - K$ is locally finitely presented, whence representable.

THEOREM.- Let $\mathcal{D}$ be an Ext-finite triangulated K-category, where K is a commutative noetherian ring, and assume that $\mathcal{D}$ has a strong generator. If τ is a bounded t-structure in $\mathcal{D}$ whose heart is equivalent to mod$-B$, for some coherent K-algebra B, then $\tau({}^{\perp}(T^{\perp_{\leq 0}}), T^{\perp_{\leq 0}})$, for some precovering generating partial silting set T in $\mathcal{D}$.

EXAMPLES.-

1. $\mathcal{D} = \mathcal{D}^b(\mathbf{mod} - A)$, **where** A **is a Noether algebra of finite global dimension.**
2. $\mathcal{D} = \mathcal{D}^b(\mathbb{X})$, **where** $\mathbb{X}$ **is a regular projective scheme of finite type over a field.**
3. $\mathcal{D} = \mathcal{D}^b(\mathbf{coh}(\mathbb{X}))/\mathbf{per}(\mathbb{X})$, **where** $\mathbb{X}$ **is a Gorenstein separated scheme of finite type over a perfect field.**