Cohomology of partial smash products

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$A \times_\alpha G$ the partial smash product

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Plan

Relate it with $\mathrm{H}^*(A, M)$ and some "partial group cohomology" of G with coefficients somewhere.

Definition of partial actions

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$\{\alpha_g : D_{g^{-1}} \to D_g\}_{g \in G}$ a collection of algebra isomorphisms

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(1) $D_e = A$, and $\alpha_e = \mathrm{id}_A$;

(2) $\alpha_h^{-1}(D_h \cap D_{g^{-1}}) \subset D_{(gh)^{-1}}$;

(3) If $x \in \alpha_h^{-1}(D_h \cap D_{g^{-1}})$, then $\alpha_g\alpha_h(x) = \alpha_{gh}(x)$.

Remarks

1. Although α_{gh} is only an extension of $\alpha_g\alpha_h$, we always have

$$\alpha_g\alpha_h\alpha_{h^{-1}} = \alpha_{gh}\alpha_{h^{-1}};$$
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2.

$$A = \sum_{g\in G} A_g \text{ is a } G\text{-graded algebra} \Rightarrow A_gA_h \subset A_{gh}$$

If $A_gA_{g^{-1}}A_g = A_g, \ \forall g \in G$, then

$$A_gA_hA_{h^{-1}} = A_{gh}A_{h^{-1}};$$
$$A_{g^{-1}}A_gA_h = A_{g^{-1}}A_{gh}.$$

Motivation/Examples

Restriction

G acts on A, B a unital ideal of A

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C^*-algebras: when dealing with algebras generated by partial isometries on a Hilbert space.

- The Cuntz-Krieger algebras [Exel, Laca, Quigg, 2002].
- The Hecke algebras for protonormal subgroups [Exel, 2008].
- The Leavitt path algebras [Gonçalves, Öinert and Royer, 2014].

Definition of partial smash product

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$$A \times_\alpha G = \sum_{g \in G} D_g \# g$$

$$(a_g \# g)(b_h \# h) = \alpha_g(\alpha_g^{-1}(a_g) b_h) \# gh$$

Definition of partial G-module

V a K-vector space,

$$\pi : G \to \mathrm{End}_{\mathrm{K}}(V)$$

such that:

(A) $\pi(e) = \mathrm{id}_V$;

(B) $\pi(s)\pi(t)\pi(t^{-1}) = \pi(st)\pi(t^{-1})$;

(C) $\pi(s^{-1})\pi(s)\pi(t) = \pi(s^{-1})\pi(st)$.

PARTIAL GROUP ALGEBRA $K_{\mathrm{par}}\, G$

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$$\mathsf{K}_{\mathsf{par}}\, G = \mathrm{K}S(G), \qquad S(G) =< [g] : g \in G >$$

with relations:

(1) $[e] = 1$;

(2) $[s^{-1}][s][t] = [s^{-1}][st]$;

(3) $[s][t][t^{-1}] = [st][t^{-1}]$; for all $s, t \in G$.

Theorem [M. Dokuchaev, R. Exel, P. Piccione, 2000]

The category Par G-mod is equivalent to the category $\mathsf{K}_{\mathsf{par}}\, G$-mod.

Partial invariants

Partial invariants

$$V^{G_{par}} = \{v \in V : [g]v = [g][g^{-1}]v \quad \text{for all } g \in G\}$$

$\mathsf{K}_{\mathsf{par}}\, G$ is G-graded

$$\mathsf{K}_{\mathsf{par}}\, G = \sum_{g \in G} B_g$$

$$B_g =< [h_1][h_2]...[h_n] : g = h_1 h_2 ... h_n >$$

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In particular

$$B := B_e =< e_g = [g][g^{-1}] : g \in G >$$

is a commutative algebra generated by central idempotents.

Proposition [AAR, 2017]

$$(-)^{G_{par}} \simeq \mathsf{Hom}_{\mathsf{K}_{\mathsf{par}}\,G}(B, -)$$

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Definition of partial group cohomology

$$\mathrm{H}^n_{par}(G, V) = \mathrm{Ext}^n_{\mathsf{K}_{\mathrm{par}}\, G}(B, V)$$

the right derived functor of $(-)^{G_{par}} \simeq \mathrm{Hom}_{\mathsf{K}_{\mathrm{par}}\, G}(B, -)$.

Lemma

Every B-module is flat.

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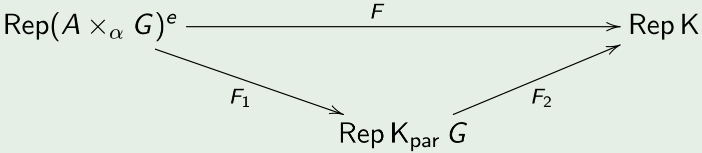
Proof:
Any finitely generated ideal I of B is principal and generated by an idempotent.

THEOREM [AAR, 2017]

For any $A \times_\alpha G$-bimodule M there is a third quadrant cohomology spectral sequence starting with E_2 and converging to $H^*(A \times_\alpha G, M)$:

$$E_2^{p,q} = H_{par}^q(G, H^p(A, M)) \Rightarrow H^{p+q}(A \times_\alpha G, M).$$



$$F(M) = \mathrm{Hom}_{(A\times_\alpha G)^e}(A\times_\alpha G, M)$$

$$F_1(M) = \mathrm{Hom}_{A^e}(A, M)$$

$$F_2(X) = \mathrm{Hom}_{\mathsf{K}_{\mathrm{par}}\, G}(B, X)$$

Proof:

$$\begin{array}{ccc} \mathrm{Rep}(A \times_\alpha G)^e & \xrightarrow{\quad F \quad} & \mathrm{Rep}\,\mathsf{K} \\ & {\scriptstyle F_1} \searrow \quad \nearrow {\scriptstyle F_2} & \\ & \mathrm{Rep}\,\mathsf{K}_{\mathrm{par}}\,G & \end{array}$$

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$$\mathrm{Hom}_{\mathsf{K}_{\mathrm{par}}\, G}(-, \mathrm{Hom}_{A^e}(A, M)) \simeq \mathrm{Hom}_{(A \times_\alpha G)^e}(- \otimes_B (A \times_\alpha G), M)$$

Spectral sequence

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