Coxeter energy of graphs and algebras

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Introduction

The Coxeter spectrum $\mathbf{spec}(\mathrm{Cox}_\Lambda) \subseteq \mathbb{C}$ of a finite-dimensional algebra Λ reflects several properties of Λ, $\mathrm{mod}\,\Lambda$ and $\mathcal{D}^b(\Lambda)$.

$\mathbf{spec}(\mathrm{Cox}_\Lambda)$ reveals/reflects the interplay between representation theory of algebras and:

- Lie theory,
- group theory,
- (spectral) graph theory,
- algebraic geometry,
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- [Gutman-Zhou, 2006]: *Laplacian energy* of Δ:

$$\mathcal{LE}(\Delta) = \sum_{\nu \in \mathbf{spec}(L_\Delta)} |\nu - \overline{\nu}|,$$

  - $L_\Delta = \mathrm{diag}(d_1, \ldots, d_n) - \mathrm{Ad}_\Delta \in \mathbb{M}_n(\mathbb{Z})$ – Laplacian matrix of Δ, $d_i = \deg(i)$, $i \in \Delta_0$,
  - $\overline{\nu} := \frac{\sum_{\nu \in \mathbf{spec}(L_\Delta)} \nu}{n} = \frac{\mathrm{tr}(L_\Delta)}{n}$.

Coxeter energy

The Coxeter matrix $\mathrm{Cox}_{\mathcal{G}} \in \mathbb{M}_n(\mathbb{Z})$ and the Coxeter spectrum **spec**$(\mathrm{Cox}_{\mathcal{G}}) \subseteq \mathbb{C}$ is defined for each of the following objects $\mathcal{G}$:

- $\mathcal{G} = \Lambda$ – fin.-dim. k-algebra with $\mathrm{gl.dim}\,\Lambda < \infty$
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- *Coxeter energy* of $\mathcal{G}$: $\mathcal{CE}(\mathcal{G}) = \sum_{\lambda \in \mathbf{spec}(\mathrm{Cox}_{\mathcal{G}})} |\lambda|.$
- *Normalized Coxeter energy* of $\mathcal{G}$: $\overline{\mathcal{CE}}(\mathcal{G}) =$
  $$\sum_{\lambda \in \mathbf{spec}(\mathrm{Cox}_{\mathcal{G}})} (|\lambda| + \lambda) = \sum_{\lambda \in \mathbf{spec}(\mathrm{Cox}_{\mathcal{G}})} |\lambda| + \mathrm{tr}(\mathrm{Cox}_{\mathcal{G}}).$$

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Ex. $Q = 1 \to 2 \to 3$, $\mathrm{Cox}_Q = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$, $\mathbf{spec} = \{-1, i, -i\}$;

$$\mathcal{CE}(Q) = 3, \quad \overline{\mathcal{CE}}(Q) = 3 - 1 = 2.$$

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Note: $\mathcal{CE}$ and $\overline{\mathcal{CE}}$ for are invariant under derived equivalence (resp. bilinear Gram congruence).

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2. $\mathcal{CE}(Q) \geq n$,
3. $\overline{\mathcal{CE}}(Q) \geq 0$,
4. $\overline{\mathcal{CE}}(Q) = 0$ *if and only if* $Q_1 = \emptyset$,
5. *if $Q = Q' \,\dot{\cup}\, Q''$ then*
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[Apply Happel's results on Hochschild cohomology of algebras.]

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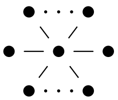
[Properties analogous to Gutman's $\mathcal{E}$.]

Trees

T – a tree with $|T_0| = n \geq 1$.

- [Gutman, 1977]: $\mathcal{E}(S_n) \leq \mathcal{E}(T) \leq \mathcal{E}(\mathbb{A}_n)$ (non-trivial).

S_n :



maximal star :=
star of type $[1, 1, \ldots, 1]$

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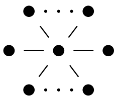

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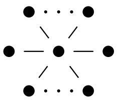

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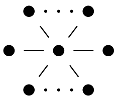

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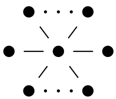

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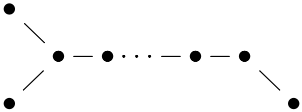
  Ex.

	$[2,2,7]$		$[1,2,8]$ $= \mathbb{E}_{12}$
$\mathcal{E}$	14.525	$>$	14.473
$\mathcal{CE}$	12.223	$>$	12.054

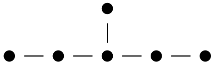
(spherical, finite) Dynkin graphs

$\mathbb{A}_n$

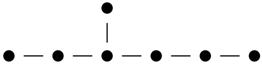
$\mathbb{D}_n$



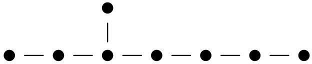
$\mathbb{E}_6$



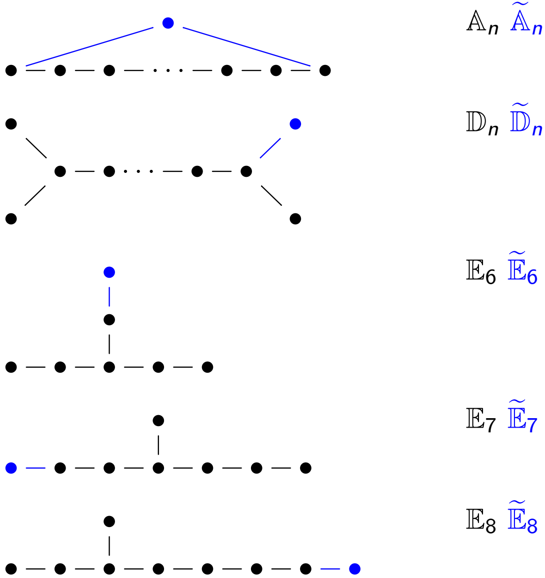
$\mathbb{E}_7$



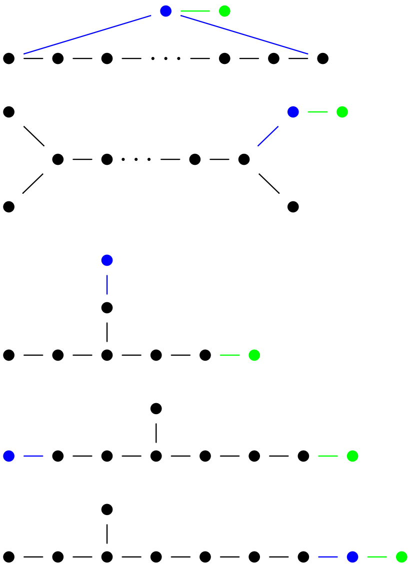
$\mathbb{E}_8$



Euclidean / extended Dynkin / affine Dynkin graphs



over-extended Dynkin / hyperbolic Dynkin graphs



$\mathbb{A}_n$ $\widetilde{\mathbb{A}}_n$ $\widetilde{\widetilde{\mathbb{A}}}_n$

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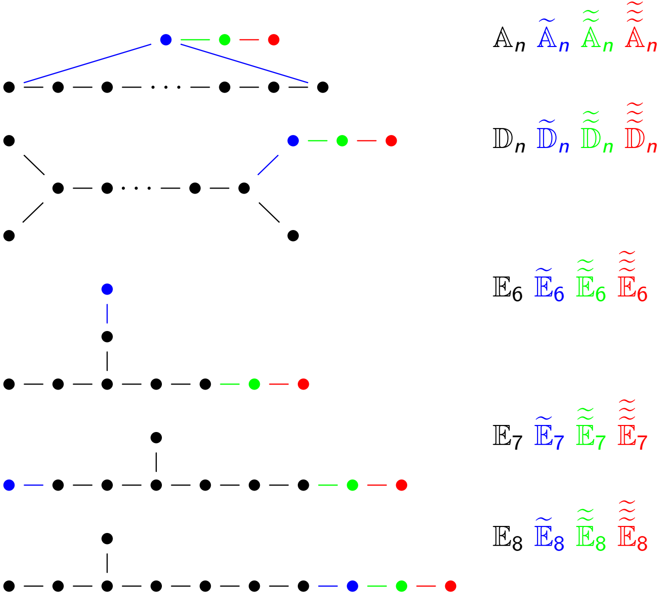
very-extended Dynkin graphs



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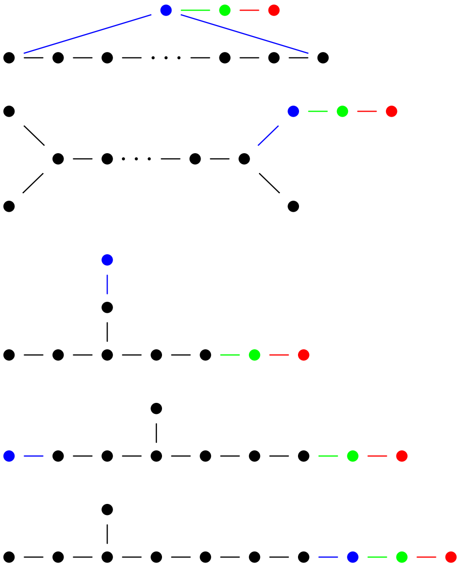

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Additionally we set: $\mathbb{E}_n := [1, 2, n-4]$, $n \geq 6$.

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Theorem

T – tree, $n = |T_0| \ge 6$, $\{\lambda_1, \ldots, \lambda_s\} = \mathbf{spec}(\mathrm{Cox}_T) \cap \mathbb{R}_{>1}$ (as a multiset) $[\lambda_i =$ "spikes", $T =$ "s-spike tree"].

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2. *If $\mathcal{CE}(T) > \mathcal{CE}(\mathbb{A}_n)$ then $\mathcal{CE}(T) \geq \mathcal{CE}(\mathbb{X}_n)$, where $\mathbb{X}_n = \widetilde{\widetilde{\mathbb{D}}}_4, \widetilde{\widetilde{\mathbb{D}}}_5, \widetilde{\widetilde{\mathbb{E}}}_6, \widetilde{\widetilde{\mathbb{E}}}_7, \widetilde{\widetilde{\mathbb{E}}}_8 = \mathbb{E}_{10}$ or $\mathbb{E}_n$, $n \geq 11$, resp. ("$=$" holds $\Leftrightarrow$ $T = \mathbb{X}_n$).*

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3. *If T is a 1-spike tree* (=: Salem tree), *then* $\mathcal{CE}(T) \leq \mathcal{CE}(S_n) = 2n - 5$ *("=" holds* $\Leftrightarrow$ $T = S_n$*).*

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2. *If* $\mathcal{CE}(T) > \mathcal{CE}(\mathbb{A}_n)$ *then* $\mathcal{CE}(T) \ge \mathcal{CE}(\mathbb{X}_n)$, *where* $\mathbb{X}_n = \widetilde{\widetilde{\mathbb{D}}}_4, \widetilde{\widetilde{\mathbb{D}}}_5, \widetilde{\widetilde{\mathbb{E}}}_6, \widetilde{\widetilde{\mathbb{E}}}_7, \widetilde{\widetilde{\mathbb{E}}}_8 = \mathbb{E}_{10}$ *or* $\mathbb{E}_n$, $n \ge 11$, *resp.* *("=" holds* $\Leftrightarrow$ $T = \mathbb{X}_n$).
3. *If T is a 1-spike tree* (=: Salem tree), *then* $\mathcal{CE}(T) \le \mathcal{CE}(S_n) = 2n - 5$ *("=" holds* $\Leftrightarrow$ $T = S_n$).
4. *If* $\deg(T) \le 3$ *then* $\mathcal{CE}(T) < \mathcal{CE}(S_n)$.

Trees: $\mathcal{CE}(\mathbb{A}_n) \le \mathcal{CE}(T) \le \mathcal{CE}(S_n)$?

Theorem

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Representation type

Applying Theorem $+$ $\mathcal{CE}(\widetilde{\mathbb{A}}_{n-1}) = n$ (+ some work):

Corollary

Let Λ be a (basic) fin.-dim. hereditary algebra over a field $k = \overline{k}$ having $n \geq 1$ pairwise non-isomorphic simple modules. Then

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Proposition

Let Λ be a piecewise hereditary algebra with $n = \mathrm{rk}(K_0(\Lambda))$. Then

- $\overline{\mathcal{CE}}(\Lambda) < n-1 \;\Leftrightarrow\; \Lambda \cong_{\mathrm{der}} \mathbb{X}(p_1, \ldots, p_s)$ *& $s > 3$;*
- $\overline{\mathcal{CE}}(\Lambda) = n-1 \;\Leftrightarrow\; (\Lambda \cong_{\mathrm{der}} \mathbb{X}(p_1, \ldots, p_s)$ *& $t = 3$) or ($\Lambda \cong_{\mathrm{der}} kQ$ & Q is Dynkin or Euclidean tree);*
- $\overline{\mathcal{CE}}(\Lambda) > n-1 \Leftrightarrow \Lambda \cong_{\mathrm{der}} kQ$ *& $Q \notin$ (Dyn. or Euclid. tree).*

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Theorem [Lenzing-de la Peña]

Let T be a tree with $s \geq 0$ spikes and $r \geq 0$ ramifications. Then

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Remark. 6 is maximal:

$\forall_{s \geq 1}\ \mathrm{Join}_s(S_6)$ has s spikes and $n = 6s + 1$ vertices.

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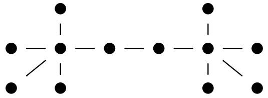
Ex. T with 2 spikes $\lambda_1 \approx 2.2369$ and $\lambda_2 \approx 3.4269$ and 1 non-cyc. irred. factor $f(t) = t^6 - 5\,t^5 + 4\,t^4 + 4\,t^3 + 4\,t^2 - 5\,t + 1$ of the Coxeter polynomial $\text{cox}_T(t) = f(t) \cdot (t+1)^6$:



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(Counter)ex. spec$(\mathrm{Cox}_{S_6}) = (\rho, 1/\rho, -1, -1, -1, -1)$, $\rho \in \mathbb{R}_{>1}$.

Coulson integral formula and other Coxeter energies

Δ = (simple, undirected) graph with $n \geq 1$ vertices.

$\mathrm{Ad}_\Delta \in \mathbb{M}_n(\mathbb{Z})$ – symmetric adjacency matrix of Δ.

Energy of Δ:

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Proposition

$\mathcal{G}$ = (bi)graph (or quiver or k-algebra) with $n \geq 1$ vertices. Then

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \left(n - it \frac{f'(it)}{f(it)} \right) \mathrm{d}t = \sum_{\lambda \in \mathbf{spec}(\mathrm{Cox}_\mathcal{G})} |\mathrm{Re}(\lambda)| =: \mathcal{CE}_{\mathrm{re}}(\mathcal{G}),$$

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Other Coxeter energies

$\mathcal{G}$ = (bi)graph <u>or</u> acyclic quiver <u>or</u> *k*-algebra with $\mathrm{gl.dim}\,\mathcal{G}<\infty$.

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- $\mathcal{CE}_{\mathrm{re}}(S_n) = \mathcal{CE}(S_n) = 2n-5$, $\mathcal{CE}_{\mathrm{im}}(S_n) = 0$ $(\forall_{n\geq 5})$;
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$$\begin{array}{ccccccccc} & & \mathcal{CE}_{\mathrm{re}}(\mathbb{A}_n) & \leq & \mathcal{CE}_{\mathrm{re}}(T) & \leq & \mathcal{CE}_{\mathrm{re}}(S_n) & = & 2n-5 \\ 0 & = & \mathcal{CE}_{\mathrm{im}}(S_n) & \leq & \mathcal{CE}_{\mathrm{im}}(T) & \leq & \mathcal{CE}_{\mathrm{im}}(\mathbb{A}_n) & & \end{array}$$

for any tree T with $n = |T_0|$?

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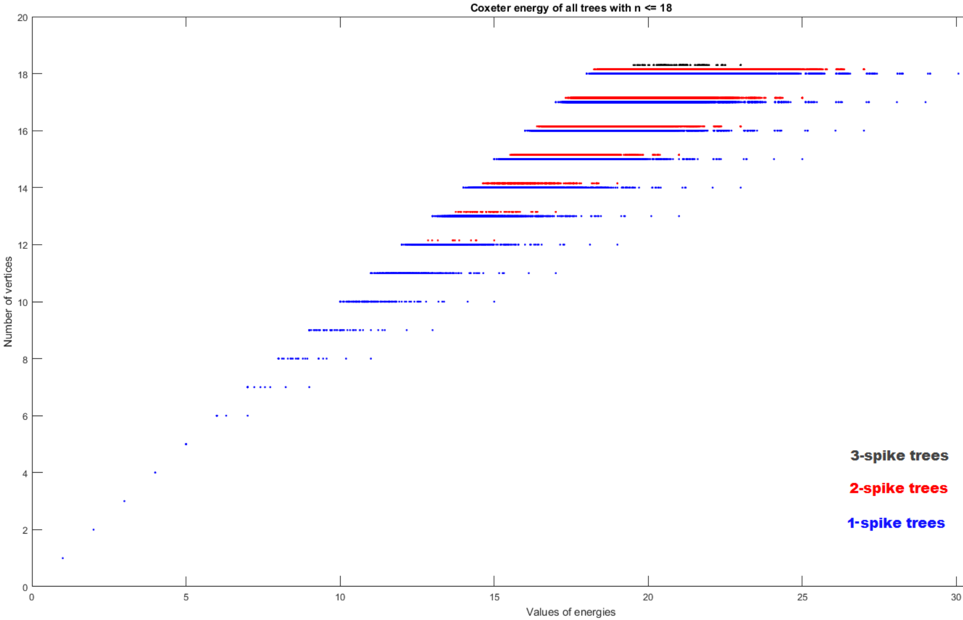
$$\begin{array}{ccccccccc} & & & & [2,3,3] & & & & \\ & & \mathcal{CE}_{\mathrm{re}}(\mathbb{A}_n) & \not\leq & \mathcal{CE}_{\mathrm{re}}(T) & \leq & \mathcal{CE}_{\mathrm{re}}(S_n) & = & 2n-5 \\ 0 & = & \mathcal{CE}_{\mathrm{im}}(S_n) & \leq & \mathcal{CE}_{\mathrm{im}}(T) & \not\leq & \mathcal{CE}_{\mathrm{im}}(\mathbb{A}_n) & & \\ & & & & [2,2,2,2,2,2,2] & & & & \end{array}$$

for any tree T with $n = |T_0|$? Unfortunately, not true.

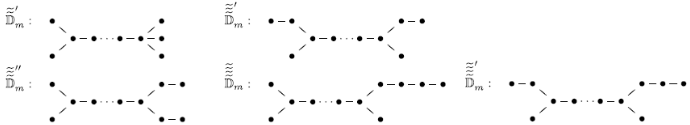
Recall: $\mathcal{CE}(\mathbb{A}_n) \leq \mathcal{CE}(T) \leq \cdots < \mathcal{CE}(\acute{S}_n) < \mathcal{CE}(S_n)$ [rhs: Salem trees, $\deg(T) \leq 3$].



$\mathcal{CE}(\mathbb{A}_n) \leq \mathcal{CE}(T) \leq \mathcal{CE}(S_n) \rightsquigarrow$ "hierarchy of wildness" ?

First few lowest Coxeter energies of wild trees.

6	7	8	9	10	11	12	13	14	15	16	17	18
$\tilde{\tilde{\tilde{\mathbb{D}}}}_4$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_5$	$\tilde{\tilde{\tilde{\mathbb{E}}}}_6$	$\tilde{\tilde{\tilde{\mathbb{E}}}}_7$	$\tilde{\tilde{\tilde{\mathbb{E}}}}_8$	$\mathbb{E}_{11}$	$\mathbb{E}_{12}$	$\mathbb{E}_{13}$	$\mathbb{E}_{14}$	$\mathbb{E}_{15}$	$\mathbb{E}_{16}$	$\mathbb{E}_{17}$	$\mathbb{E}_{18}$
$\tilde{\tilde{\tilde{\mathbb{D}}}}_4'$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_4$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_6$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_7$	$1, 3, 5$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_9$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_{10}$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_{11}$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_{12}$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_{13}$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_{14}$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_{15}$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_{16}$
	$\tilde{\tilde{\tilde{\mathbb{D}}}}_4'$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_5$	$2, 2, 4$	$1, 4, 4$	$1, 3, 6$	$1, 3, 7$	$1, 3, 8$	$1, 3, 9$	$1, 3, 10$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_{13}'$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_{14}'$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_{15}'$
	$\tilde{\tilde{\tilde{\mathbb{D}}}}_5'$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_5'$	$2, 3, 3$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_8$	$1, 4, 5$	$1, 4, 6$	$1, 4, 7$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_{11}'$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_{12}'$	$1, 3, 11$	$1, 3, 12$	$1, 3, 13$
	$\acute{S}_7$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_5''$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_6$	$2, 2, 5$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_8$	$1, 5, 5$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_{10}$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_{11}$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_{12}$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_{13}$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_{14}$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_{15}$
	S_7	$\tilde{\tilde{\tilde{\mathbb{D}}}}_4$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_6'$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_7$	$2, 2, 6$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_9$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_{10}'$	$1, 4, 8$	$1, 4, 9$	$1, 4, 10$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_{13}'$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_{14}'$
		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$



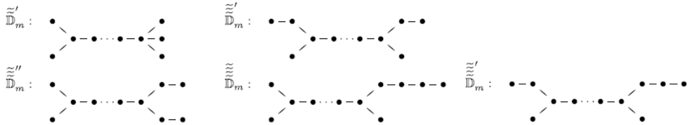
$\tilde{\tilde{\tilde{\mathbb{D}}}}_4 = \acute{S}_6$, $\tilde{\tilde{\tilde{\mathbb{D}}}}_4' = S_6$, $\tilde{\tilde{\tilde{\mathbb{D}}}}_4' = \tilde{\tilde{\tilde{\mathbb{D}}}}_4'' = [1, 1, 2, 2]$,

$[2, 2, 4] = \tilde{\tilde{\tilde{\mathbb{E}}}}_6$, $[1, 3, 5] = \tilde{\tilde{\tilde{\mathbb{E}}}}_7$, $[1, 2, 7] = \tilde{\tilde{\tilde{\mathbb{E}}}}_8 = \mathbb{E}_{11}$.

$\mathcal{CE}(\mathbb{A}_n) \leq \mathcal{CE}(T) \leq \mathcal{CE}(S_n) \leftrightsquigarrow$ "hierarchy of wildness" ?

First few lowest Coxeter energies of wild trees.

6	7	8	9	10	11	12	13	14	15	16	17	18
$\tilde{\tilde{\tilde{\mathbb{D}}}}_4$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_5$	$\tilde{\tilde{\tilde{\mathbb{E}}}}_6$	$\tilde{\tilde{\tilde{\mathbb{E}}}}_7$	$\tilde{\tilde{\tilde{\mathbb{E}}}}_8$	$\mathbb{E}_{11}$	$\mathbb{E}_{12}$	$\mathbb{E}_{13}$	$\mathbb{E}_{14}$	$\mathbb{E}_{15}$	$\mathbb{E}_{16}$	$\mathbb{E}_{17}$	$\mathbb{E}_{18}$
$\tilde{\tilde{\tilde{\mathbb{D}}}}'_4$	$\tilde{\tilde{\tilde{\tilde{\mathbb{D}}}}}_4$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_6$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_7$	1, 3, 5	$\tilde{\tilde{\tilde{\mathbb{D}}}}_9$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_{10}$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_{11}$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_{12}$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_{13}$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_{14}$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_{15}$	$\tilde{\tilde{\tilde{\mathbb{D}}}}_{16}$
	$\tilde{\tilde{\tilde{\tilde{\mathbb{D}}}}}'_4$	$\tilde{\tilde{\tilde{\tilde{\mathbb{D}}}}}_5$	2, 2, 4	1, 4, 4	1, 3, 6	1, 3, 7	1, 3, 8	1, 3, 9	1, 3, 10	$\tilde{\tilde{\tilde{\tilde{\mathbb{D}}}}}'_{13}$	$\tilde{\tilde{\tilde{\tilde{\mathbb{D}}}}}'_{14}$	$\tilde{\tilde{\tilde{\tilde{\mathbb{D}}}}}'_{15}$
	$\tilde{\tilde{\tilde{\mathbb{D}}}}'_5$	$\tilde{\tilde{\tilde{\tilde{\mathbb{D}}}}}'_5$	2, 3, 3	$\tilde{\tilde{\tilde{\mathbb{D}}}}_8$	1, 4, 5	1, 4, 6	1, 4, 7	$\tilde{\tilde{\tilde{\tilde{\mathbb{D}}}}}'_{11}$	$\tilde{\tilde{\tilde{\tilde{\mathbb{D}}}}}'_{12}$	1, 3, 11	1, 3, 12	1, 3, 1
	$\acute{S}_7$	$\tilde{\tilde{\tilde{\tilde{\mathbb{D}}}}}''_5$	$\tilde{\tilde{\tilde{\tilde{\mathbb{D}}}}}_6$	2, 2, 5	$\tilde{\tilde{\tilde{\tilde{\mathbb{D}}}}}_8$	1, 5, 5	$\tilde{\tilde{\tilde{\tilde{\mathbb{D}}}}}_{10}$	$\tilde{\tilde{\tilde{\tilde{\mathbb{D}}}}}_{11}$	$\tilde{\tilde{\tilde{\tilde{\mathbb{D}}}}}_{12}$	$\tilde{\tilde{\tilde{\tilde{\mathbb{D}}}}}_{13}$	$\tilde{\tilde{\tilde{\tilde{\mathbb{D}}}}}_{14}$	$\tilde{\tilde{\tilde{\tilde{\mathbb{D}}}}}_{15}$
	S_7	$\tilde{\tilde{\tilde{\tilde{\mathbb{D}}}}}_4$	$\tilde{\tilde{\tilde{\tilde{\mathbb{D}}}}}'_6$	$\tilde{\tilde{\tilde{\tilde{\mathbb{D}}}}}_7$	2, 2, 6	$\tilde{\tilde{\tilde{\tilde{\mathbb{D}}}}}_9$	$\tilde{\tilde{\tilde{\tilde{\mathbb{D}}}}}'_{10}$	1, 4, 8	1, 4, 9	1, 4, 10	$\tilde{\tilde{\tilde{\tilde{\mathbb{D}}}}}'_{13}$	$\tilde{\tilde{\tilde{\tilde{\mathbb{D}}}}}'_{14}$
		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$



$\tilde{\tilde{\tilde{\mathbb{D}}}}_4 = \acute{S}_6$, $\tilde{\tilde{\tilde{\mathbb{D}}}}'_4 = S_6$, $\tilde{\tilde{\tilde{\tilde{\mathbb{D}}}}}'_4 = \tilde{\tilde{\tilde{\tilde{\mathbb{D}}}}}''_4 = [1, 1, 2, 2]$,

$[2, 2, 4] = \tilde{\tilde{\tilde{\tilde{\mathbb{E}}}}}_6$, $[1, 3, 5] = \tilde{\tilde{\tilde{\tilde{\mathbb{E}}}}}_7$, $[1, 2, 7] = \tilde{\tilde{\tilde{\tilde{\mathbb{E}}}}}_8 = \mathbb{E}_{11}$.

cf.:
Zhang, Xi,
Brüstle-
de la Peña-
Skowroński,...