Generic extensions and Hall polynomials for invariant subspaces of nilpotent linear operators

Justyna Kosakowska,
Nicolaus Copernicus University, Toruń, Poland

A report on a joint project with Mariusz Kaniecki and Stanisław Kasjan

ARTA

September 2017

Plan

1. Hall polynomials for invariant subspaces of nilpotent linear operators (joint work with Stanisław Kasjan)

Plan

1. Hall polynomials for invariant subspaces of nilpotent linear operators (joint work with Stanisław Kasjan)
2. Generic extensions of invariant subspaces of nilpotent linear operators (joint work with Mariusz Kaniecki)

Plan

1. Hall polynomials for invariant subspaces of nilpotent linear operators (joint work with Stanisław Kasjan)
2. Generic extensions of invariant subspaces of nilpotent linear operators (joint work with Mariusz Kaniecki)
3. Connections

Classical case

k – an arbitrary field

Classical case

k – an arbitrary field

$k[x]$ – the algebra of polynomials in one variable

Classical case

k – an arbitrary field

$k[x]$ – the algebra of polynomials in one variable

$\alpha = (\alpha_1, \ldots, \alpha_n)$ – a partition, $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$

Classical case

k – an arbitrary field

$k[x]$ – the algebra of polynomials in one variable

$\alpha = (\alpha_1, \ldots, \alpha_n)$ – a partition, $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$

$N_\alpha = N_\alpha(k) = \bigoplus_{i=1}^{n} k[x]/x^i$ – the nilpotent linear operator of type α

Classical case

k – an arbitrary field

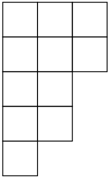
$k[x]$ – the algebra of polynomials in one variable

$\alpha = (\alpha_1, \ldots, \alpha_n)$ – a partition, $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$

$N_\alpha = N_\alpha(k) = \bigoplus_{i=1}^{n} k[x]/x^i$ – the nilpotent linear operator of type α

$$\{\text{nilp. lin. operators}\}/_{\simeq} \xleftrightarrow{1-1} \{\mathit{partitions}\}$$

$$k[x]/x^5 \oplus k[x]/x^4 \oplus k[x]/x^2 \longleftrightarrow$$



Classical case

$\mathcal{N} = \mathcal{N}(k)$ – the category of nilpotent linear operators:

- objects: N_α
- morphisms: homomorphisms of $k[x]$-modules

Classical case

$\mathcal{N} = \mathcal{N}(k)$ – the category of nilpotent linear operators:

- objects: N_α
- morphisms: homomorphisms of $k[x]$-modules

α, β, γ – partitions

Hall numbers:

$$F^{\beta}_{\alpha,\gamma} = F^{\beta}_{\alpha,\gamma}(k) = \#\{U \subseteq N_\beta \; ; \;\; U \simeq N_\alpha \text{ and } N_\beta/U \simeq N_\gamma\}$$

Classical case

Theorem (Hall)

Let α, β, γ be partitions. There exists a polynomial $\varphi^{\beta}_{\alpha,\gamma} \in \mathbb{Z}[T]$, such that for any finite field k:

$$\varphi^{\beta}_{\alpha,\gamma}(\#k) = F^{\beta}_{\alpha,\gamma}(k)$$

Classical case

Theorem (Hall)

Let α, β, γ be partitions. There exists a polynomial $\varphi^{\beta}_{\alpha,\gamma} \in \mathbb{Z}[T]$, such that for any finite field k:

$$\varphi^{\beta}_{\alpha,\gamma}(\#k) = F^{\beta}_{\alpha,\gamma}(k)$$

C. M. Ringel - the existence of Hall polynomials for representation directed algebras.

Classical case

Theorem (Hall)

Let α, β, γ be partitions. There exists a polynomial $\varphi^{\beta}_{\alpha,\gamma} \in \mathbb{Z}[T]$, such that for any finite field k:

$$\varphi^{\beta}_{\alpha,\gamma}(\#k) = F^{\beta}_{\alpha,\gamma}(k)$$

C. M. Ringel - the existence of Hall polynomials for representation directed algebras.

Conjecture (Ringel)

There exist Hall polynomials for all representation finite algebras.

Invariant subspaces of nilpotent linear operators

$\mathcal{S} = \mathcal{S}(k)$ - the category of invariant subspaces of nilpotent linear operators:

Invariant subspaces of nilpotent linear operators

$\mathcal{S} = \mathcal{S}(k)$ - the category of invariant subspaces of nilpotent linear operators:

- objects: triples (N_α, N_β, f), where $f : N_\alpha \to N_\beta$ - injective $k[x]$ - homomorphism
- morphisms: pairs $(h_1, h_2) : (N_\alpha, N_\beta, f) \to (N_\gamma, N_\delta, g)$ such that $h_1 : N_\alpha \to N_\gamma$, $h_2 : N_\beta \to N_\delta$ and the following diagram is commutative

$$\begin{array}{ccc} N_\alpha & \xrightarrow{f} & N_\beta \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} \\ N_\gamma & \xrightarrow{g} & N_\delta \end{array}$$

Invariant subspaces of nilpotent linear operators

$\mathcal{S} = \mathcal{S}(k)$ - the category of invariant subspaces of nilpotent linear operators:

- objects: triples (N_α, N_β, f), where $f : N_\alpha \to N_\beta$ - injective $k[x]$ - homomorphism
- morphisms: pairs $(h_1, h_2) : (N_\alpha, N_\beta, f) \to (N_\gamma, N_\delta, g)$ such that $h_1 : N_\alpha \to N_\gamma$, $h_2 : N_\beta \to N_\delta$ and the following diagram is commutative

$$\begin{array}{ccc} N_\alpha & \xrightarrow{f} & N_\beta \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} \\ N_\gamma & \xrightarrow{g} & N_\delta \end{array}$$

The category S is wild.

LR-tableaux and extensions

Theorem (Green, Klein)

There exists a short exact sequence of nilpotent $k[x]$-modules

$$\eta \quad : \quad 0 \longrightarrow N_\alpha \xrightarrow{f} N_\beta \longrightarrow N_\gamma \longrightarrow 0$$

if and only if there exists an LR-tableau Γ of type (α, β, γ).

If $\Gamma = [\gamma^{(0)}, \ldots, \gamma^{(s)}]$, then

$$N_{\gamma^{(i)}} \cong N_\beta / x^i f(N_\alpha)$$

for all i.
We say that f is of type Γ.

$\mathcal{S}_1 = \mathcal{S}_1(k)$ - full subcategory of $\mathcal{S}$ consisting of triples (N_α, N_β, f) such that $\alpha_1 \leq 1$

Invariant subspaces of nilpotent linear operators

$\mathcal{S}_1 = \mathcal{S}_1(k)$ - full subcategory of $\mathcal{S}$ consisting of triples (N_α, N_β, f) such that $\alpha_1 \leq 1$

the category $\mathcal{S}_1$ has discrete representation type

Invariant subspaces of nilpotent linear operators

$\mathcal{S}_1 = \mathcal{S}_1(k)$ - full subcategory of $\mathcal{S}$ consisting of triples (N_α, N_β, f) such that $\alpha_1 \leq 1$

the category $\mathcal{S}_1$ has discrete representation type

indecomposables:

- $P_0^m = (0, N_{(m)}, 0)$, for all $m \in \mathbb{N}$
- $P_1^m = (N_{(1)}, N_{(m)}, f)$, for all $m \in \mathbb{N}$, $f(1) = x^{m-1}$

Invariant subspaces of nilpotent linear operators

$\mathcal{S}_1 = \mathcal{S}_1(k)$ - full subcategory of $\mathcal{S}$ consisting of triples (N_α, N_β, f) such that $\alpha_1 \leq 1$

the category $\mathcal{S}_1$ has discrete representation type

indecomposables:

- $P_0^m = (0, N_{(m)}, 0)$, for all $m \in \mathbb{N}$
- $P_1^m = (N_{(1)}, N_{(m)}, f)$, for all $m \in \mathbb{N}$, $f(1) = x^{m-1}$

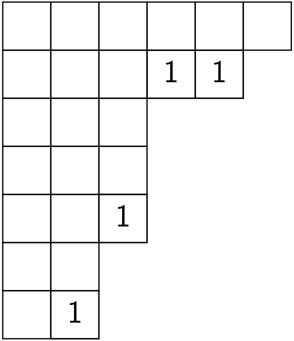
$$\{\text{objects of } \mathcal{S}_1\}/_{\simeq} \xleftrightarrow{1-1} LR_1$$

LR_1 – the set of Littlewood-Richardson tableaux with entries 1

The bijection

$$N(\Gamma) = P_0^7 \oplus P_1^7 \oplus P_1^5 \oplus P_1^2 \oplus P_1^2 \oplus P_0^1$$

$\Gamma =$



Hall polynomials

Theorem (S. Kasjan - J. K., 2017)

Let $\Gamma, \Sigma, \Delta \in LR_1$. There exists a polynomial $\varphi^{\Delta}_{\Gamma,\Sigma} \in \mathbb{Q}[T]$ such that for any finite field k:

$$\varphi^{\Delta}_{\Gamma,\Sigma}(\#k) = F^{\Delta}_{\Gamma,\Sigma}(k),$$

where

$$F^{\Delta}_{\Gamma,\Sigma} = \#\{U \subseteq N(\Delta) \;\; ; \;\; U \simeq N(\Gamma) \text{ and } N(\Delta)/U \simeq N(\Sigma)\}$$

Hall polynomials

Theorem (S. Kasjan - J. K., 2017)

Let $\Gamma, \Sigma, \Delta \in LR_1$. *There exists a polynomial* $\varphi^{\Delta}_{\Gamma,\Sigma} \in \mathbb{Q}[T]$ *such that for any finite field* k*:*

$$\varphi^{\Delta}_{\Gamma,\Sigma}(\#k) = F^{\Delta}_{\Gamma,\Sigma}(k),$$

where

$$F^{\Delta}_{\Gamma,\Sigma} = \#\{U \subseteq N(\Delta) \;\; ; \;\; U \simeq N(\Gamma) \text{ and } N(\Delta)/U \simeq N(\Sigma)\}$$

Conjecture

$$\varphi^{\Delta}_{\Gamma,\Sigma} \in \mathbb{Z}[T]$$

Example

$$\#k = q$$

$$P_0^1 = (0, N_{(1)}, 0) \quad , \quad P_1^1 = (N_{(1)}, N_{(1)}, 1)$$

$$P_1^2 = (N_{(1)}, N_{(2)}, 1 \mapsto x)$$

$$F_{P_1^1, P_0^1}^{P_1^2} = 1$$

$$F_{P_1^1, P_0^1}^{P_1^1 \oplus P_0^1} = 1 \qquad \mathrm{Hom}(P_1^1, P_0^1) = 0$$

$$F_{P_0^1, P_1^1}^{P_1^1 \oplus P_0^1} = q$$

Hall algebras

$\mathcal{H}_q(\mathcal{N})$ – the Hall $\mathbb{Q}$-algebra associated with the category $\mathcal{N}$:

Hall algebras

$\mathcal{H}_q(\mathcal{N})$ – the Hall $\mathbb{Q}$-algebra associated with the category $\mathcal{N}$:

- $\mathbb{Q}$-linear basis: $\{u_\alpha\}_{\alpha\in\mathcal{P}}$, $\mathcal{P}$ – the set of all partitions
- $u_\alpha \cdot u_\gamma = \sum_{\beta\in\mathcal{P}} \varphi^{\beta}_{\alpha,\gamma}(q) \cdot u_\beta$

Hall algebras

$\mathcal{H}_q(\mathcal{N})$ – the Hall $\mathbb{Q}$-algebra associated with the category $\mathcal{N}$:

- $\mathbb{Q}$-linear basis: $\{u_\alpha\}_{\alpha\in\mathcal{P}}$, $\mathcal{P}$ – the set of all partitions
- $u_\alpha \cdot u_\gamma = \sum_{\beta\in\mathcal{P}} \varphi^{\beta}_{\alpha,\gamma}(q) \cdot u_\beta$

$\mathcal{H}_q(\mathcal{S}_1)$ – the Hall $\mathbb{Q}$-algebra associated with the category $\mathcal{S}_1$:

Hall algebras

$\mathcal{H}_q(\mathcal{N})$ – the Hall $\mathbb{Q}$-algebra associated with the category $\mathcal{N}$:

- $\mathbb{Q}$-linear basis: $\{u_\alpha\}_{\alpha\in\mathcal{P}}$, $\mathcal{P}$ – the set of all partitions
- $u_\alpha \cdot u_\gamma = \sum_{\beta\in\mathcal{P}} \varphi^{\beta}_{\alpha,\gamma}(q)\cdot u_\beta$

$\mathcal{H}_q(\mathcal{S}_1)$ – the Hall $\mathbb{Q}$-algebra associated with the category $\mathcal{S}_1$:

- $\mathbb{Q}$-linear basis: $\{u_\Gamma\}_{\Gamma\in LR_1}$, LR_1 – the set of LR-tableaux with entries 1
- $u_\Gamma \cdot u_\Sigma = \sum_{\Delta\in LR_1} \varphi^{\Delta}_{\Gamma,\Sigma}(q)\cdot u_\Delta$

Hall algebras

$\mathcal{H}_q(\mathcal{N})$ – the Hall $\mathbb{Q}$-algebra associated with the category $\mathcal{N}$:

- $\mathbb{Q}$-linear basis: $\{u_\alpha\}_{\alpha\in\mathcal{P}}$, $\mathcal{P}$ – the set of all partitions
- $u_\alpha \cdot u_\gamma = \sum_{\beta\in\mathcal{P}} \varphi^{\beta}_{\alpha,\gamma}(q)\cdot u_\beta$

$\mathcal{H}_q(\mathcal{S}_1)$ – the Hall $\mathbb{Q}$-algebra associated with the category $\mathcal{S}_1$:

- $\mathbb{Q}$-linear basis: $\{u_\Gamma\}_{\Gamma\in LR_1}$, LR_1 – the set of LR-tableaux with entries 1
- $u_\Gamma \cdot u_\Sigma = \sum_{\Delta\in LR_1} \varphi^{\Delta}_{\Gamma,\Sigma}(q)\cdot u_\Delta$

Both are associative algebras with unit u_0.

Hall algebras

$\mathcal{H}_q(\mathcal{N})$ – the Hall $\mathbb{Q}$-algebra associated with the category $\mathcal{N}$:

- $\mathbb{Q}$-linear basis: $\{u_\alpha\}_{\alpha\in\mathcal{P}}$, $\mathcal{P}$ – the set of all partitions
- $u_\alpha \cdot u_\gamma = \sum_{\beta\in\mathcal{P}} \varphi^{\beta}_{\alpha,\gamma}(q)\cdot u_\beta$

$\mathcal{H}_q(\mathcal{S}_1)$ – the Hall $\mathbb{Q}$-algebra associated with the category $\mathcal{S}_1$:

- $\mathbb{Q}$-linear basis: $\{u_\Gamma\}_{\Gamma\in LR_1}$, LR_1 – the set of LR-tableaux with entries 1
- $u_\Gamma \cdot u_\Sigma = \sum_{\Delta\in LR_1} \varphi^{\Delta}_{\Gamma,\Sigma}(q)\cdot u_\Delta$

Both are associative algebras with unit u_0.

$\mathcal{H}_q(\mathcal{N}) \subseteq \mathcal{H}_q(\mathcal{S}_1) \qquad u_\alpha \mapsto u_{P_0^{|\alpha|}} \qquad |\alpha| = \alpha_1 + \ldots + \alpha_n$

Hall algebras

$\mathcal{H}_q(\mathcal{N})$ – the Hall $\mathbb{Q}$-algebra associated with the category $\mathcal{N}$:

- $\mathbb{Q}$-linear basis: $\{u_\alpha\}_{\alpha\in\mathcal{P}}$, $\mathcal{P}$ – the set of all partitions
- $u_\alpha \cdot u_\gamma = \sum_{\beta\in\mathcal{P}} \varphi^\beta_{\alpha,\gamma}(q)\cdot u_\beta$

$\mathcal{H}_q(\mathcal{S}_1)$ – the Hall $\mathbb{Q}$-algebra associated with the category $\mathcal{S}_1$:

- $\mathbb{Q}$-linear basis: $\{u_\Gamma\}_{\Gamma\in LR_1}$, LR_1 – the set of LR-tableaux with entries 1
- $u_\Gamma \cdot u_\Sigma = \sum_{\Delta\in LR_1} \varphi^\Delta_{\Gamma,\Sigma}(q)\cdot u_\Delta$

Both are associative algebras with unit u_0.

$\mathcal{H}_q(\mathcal{N}) \subseteq \mathcal{H}_q(\mathcal{S}_1) \qquad u_\alpha \mapsto u_{P_0^{|\alpha|}} \qquad |\alpha| = \alpha_1 + \ldots + \alpha_n$

$\mathcal{H}_q(\mathcal{N})$ is commutative, $\mathcal{H}_q(\mathcal{S}_1)$ is non-commutative

Example

$$P_0^1 = (0, N_{(1)}, 0) \quad , \quad P_1^1 = (N_{(1)}, N_{(1)}, 1)$$

$$P_1^2 = (N_{(1)}, N_{(2)}, 1 \mapsto x)$$

$$u_{P_0^1} \cdot u_{P_1^1} = q \cdot u_{P_1^1 \oplus P_0^1}$$

$$u_{P_1^1} \cdot u_{P_0^1} = u_{P_1^1 \oplus P_0^1} + u_{P_1^2}$$

Hall algebras and generic extensions

Theorem

$$\mathcal{H}_0(\mathcal{N}) \simeq \mathbb{Q}\mathcal{M}(\mathcal{N}),$$

where $\mathbb{Q}\mathcal{M}(\mathcal{N})$ *is the* $\mathbb{Q}$*-algebra generated by the monoid* $\mathcal{M}(\mathcal{N})$ *of generic extensions in* $\mathcal{N}$*.*

Hall algebras and generic extensions

Theorem

$$\mathcal{H}_0(\mathcal{N}) \simeq \mathbb{Q}\mathcal{M}(\mathcal{N}),$$

where $\mathbb{Q}\mathcal{M}(\mathcal{N})$ is the $\mathbb{Q}$-algebra generated by the monoid $\mathcal{M}(\mathcal{N})$ of generic extensions in $\mathcal{N}$.

This is also true for representations of Dynkin quivers (M. Reineke).

Hall algebras and generic extensions

Theorem

$$\mathcal{H}_0(\mathcal{N}) \simeq \mathbb{Q}\mathcal{M}(\mathcal{N}),$$

where $\mathbb{Q}\mathcal{M}(\mathcal{N})$ *is the* $\mathbb{Q}$*-algebra generated by the monoid* $\mathcal{M}(\mathcal{N})$ *of generic extensions in* $\mathcal{N}$*.*

This is also true for representations of Dynkin quivers (M. Reineke).

Questions

1. Is the same true for $\mathcal{H}_0(\mathcal{S}_1)$?
2. Do there exist generic extensions in the category $\mathcal{S}_1$?

Generic extensions

Definition

Let $M, N \in \mathcal{S}_1(k)$ (k-arbitrary). If there exists exactly one (up to isomorphism) extension $X \in \mathcal{S}_1$ of M by N with the minimal dimension of endomorphism ring $\mathrm{End}_{\mathcal{S}_1}(X)$, then we call X **the generic extension** of M by N and denote it by $X = M * N$.

Generic extensions

Definition

Let $M, N \in \mathcal{S}_1(k)$ (k-arbitrary). If there exists exactly one (up to isomorphism) extension $X \in \mathcal{S}_1$ of M by N with the minimal dimension of endomorphism ring $\mathrm{End}_{\mathcal{S}_1}(X)$, then we call X **the generic extension** of M by N and denote it by $X = M * N$.

Theorem (M. Kaniecki, J. K., 2017)

1. *For arbitrary objects $M, N \in \mathcal{S}_1$, there exists the generic extension $M * N$.*
2. *For arbitrary objects $M, N, U \in \mathcal{S}_1$, we have $(M * N) * U = M * (N * U)$.*

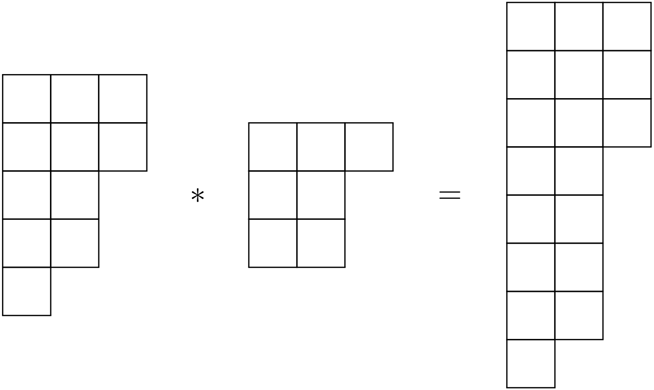
About the proof

1. Given $M, N \in \mathcal{S}_1$ we described a combinatorial algorithm that computes an extnesion X of M by N.
2. We proved that X (constructed by this algorithm) is the generic extension of M by N:
   1. the degeneration order $\leq_{deg}$ in $\mathcal{S}_1$ was used,
   2. the equivalence of orders $\leq_{box}, \leq_{ext}, \leq_{deg}, \leq_{hom}, \leq_{dom}$ was proved and applied.

Generic extensions - classical case

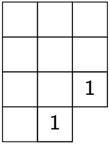
In the category $\mathcal{N}$: $N_\alpha * N_\gamma = N_{\alpha+\gamma}$

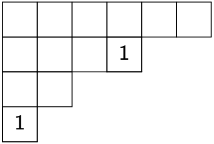
$\alpha = (5, 4, 2)$ $\gamma = (3, 3, 1)$ $\alpha + \gamma = (8, 7, 3)$

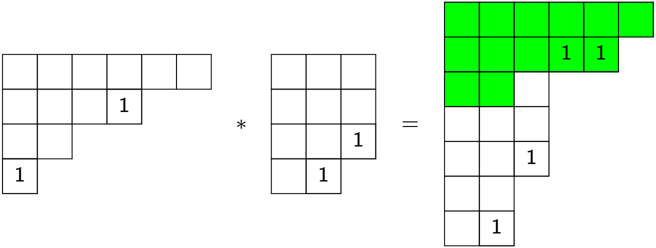


Generic extensions

$$N(\Gamma) = P_0^4 \oplus P_1^4 \oplus P_1^3 \quad N(\Sigma) = P_1^4 \oplus P_0^3 \oplus P_0^2 \oplus P_1^2 \oplus P_0^1 \oplus P_0^1$$

$\Gamma =$ 

$\Sigma =$ 



Algorithm

Input. $X, Y \in \mathcal{S}_1$.
Output. The generic extension $Z = Y * X$.

1. set $n = 0$
2. for any $i = 1, \ldots, \min\{\overline{\beta_1^X}, \overline{\beta_1^Y}\}$, do
   1. put $\gamma_i^Z = \gamma_i^X + \gamma_i^Y$
   2. put $\beta_i^Z = \beta_i^X + \gamma_i^Y$
   3. if $\beta_i^Y \neq \gamma_i^Y$, then put $n = n + 1$
3. if $\overline{\beta_1^X} > \min\{\overline{\beta_1^X}, \overline{\beta_1^Y}\}$, then for $i = \min\{\overline{\beta_1^X}, \overline{\beta_1^Y}\} + 1, \ldots, \overline{\beta_1^X}$ put

   $$\gamma_Z^i = \gamma_X^i \quad \text{and} \quad \beta_Z^i = \beta_X^i,$$

   else for $i = \min\{\overline{\beta_1^X}, \overline{\beta_1^Y}\} + 1, \ldots, \overline{\beta_1^Y}$ we set

   $$\gamma_i^Z = \gamma_i^Y \text{ and } \beta_i^Z = \beta_i^Y + \mathbf{1}\{\gamma_i^Y = \beta_i^Y \text{ and } n > 0\} \text{ and } n = n - \mathbf{1}\{\gamma_i^Y = \beta_i^Y \text{ and } n > 0\},$$

   where by $\mathbf{1}\{X\}$ we denote the characteristic function of a set X.
4. We set

   $$\beta_Z = \beta_Z \cup \alpha$$

   where $\alpha = (1, 1, \ldots, 1)$ is a partition with n copies of 1.

$\mathcal{M}(\mathcal{S}_1) = (\{[M]\ ;\ M \in obj(\mathcal{S}_1)\}, *)$ - the monoid of generic extensions of the category $\mathcal{S}_1$

The algebra generated by generic extensions

$\mathcal{M}(\mathcal{S}_1) = (\{[M]\ ;\ M \in obj(\mathcal{S}_1)\}, *)$ - the monoid of generic extensions of the category $\mathcal{S}_1$

$\mathbb{Q}\mathcal{M}(\mathcal{S}_1)$ – the $\mathbb{Q}$-algebra generated by $\mathcal{M}(\mathcal{S}_1)$

The algebra generated by generic extensions

$\mathcal{M}(\mathcal{S}_1) = (\{[M]\ ;\ M \in obj(\mathcal{S}_1)\}, *)$ - the monoid of generic extensions of the category $\mathcal{S}_1$

$\mathbb{Q}\mathcal{M}(\mathcal{S}_1)$ – the $\mathbb{Q}$-algebra generated by $\mathcal{M}(\mathcal{S}_1)$

Question: $\mathbb{Q}\mathcal{M}(\mathcal{S}_1) \simeq \mathcal{H}_0(\mathcal{S}_1)$

The algebra generated by generic extensions

$\mathcal{M}(\mathcal{S}_1) = (\{[M] \; ; \; M \in obj(\mathcal{S}_1)\}, *)$ - the monoid of generic extensions of the category $\mathcal{S}_1$

$\mathbb{Q}\mathcal{M}(\mathcal{S}_1)$ – the $\mathbb{Q}$-algebra generated by $\mathcal{M}(\mathcal{S}_1)$

Question: $\mathbb{Q}\mathcal{M}(\mathcal{S}_1) \simeq \mathcal{H}_0(\mathcal{S}_1)$

Answer ...