On tilting theory and the radical

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ARTA VI
September 07, 2017

Notation

• Let A be a finite dimensional k-algebra over an algebraically closed field, k.

• mod A denotes the category of finitely generated right A-modules.

• ind A denotes the full subcategory of mod A which consists of the indecomposable A-modules.

• Γ_A denotes the Auslander-Reiten quiver of mod A.

Definition

For $X, Y \in \text{mod}\, A$, the **radical** of $\text{Hom}_A(X, Y)$ is defined by

$$\Re(X,Y) = \{f \in \text{Hom}_A(X,Y) \mid hfg \text{ is not an isomorphism}, \\ g : M \to X \text{ and } h : Y \to M,\ M \in \text{ind}\, A\}.$$

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Inductively, the natural powers of $\Re(X, Y)$ are defined:

$$f \in \Re^n(X,Y) \text{ if and only if } f = \sum_{i=1}^{r} h_i g_i \text{ with } M_i \in \text{mod}\, A, \\ g_i \in \Re(X, M_i) \text{ and } h_i \in \Re^{n-1}(M_i, Y).$$

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Finally, the **infinite radical** is defined:

$$\Re^{\infty}(X,Y) = \bigcap_{n \in \mathbb{N}} \Re^n(X,Y).$$

Definition

A morphism $f : X \to Y$ in mod A is said to be **irreducible** provided:

(i) f is neither a section nor a retraction and

(ii) if $f = hg$, either g is a section or h is a retraction.

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Theorem (Bautista)

Let X, Y be indecomposable modules in mod A.
A morphism $f : X \to Y$ is irreducible if and only if $f \in \Re(X, Y) \setminus \Re^2(X, Y)$.

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Definition

Let A a representation-finite algebra. The minimal lower bound m such that $\Re^m(\text{mod}\, A) = 0$ is called the **nilpotency bound** of $\Re(\text{mod}\, A)$.

Objective

The goal of our work is to establish a relationship between the radical of mod A and the radical of the module category of $\mathrm{End}_A T$, with T a tilting A-module.

Tilting theory

Definition

A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of mod A is called **torsion theory** if the following conditions are satisfied:

(a) $\text{Hom}_A(M, N) = 0$ for all $M \in \mathcal{T}$ y $N \in \mathcal{F}$.

(b) If $\text{Hom}_A(M, F) = 0$ for all $F \in \mathcal{F}$, implies $M \in \mathcal{T}$.

(c) If $\text{Hom}_A(T, N) = 0$ for all $T \in \mathcal{T}$, implies $N \in \mathcal{F}$.

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The class $\mathcal{T}$ is called the **torsion class** and the class $\mathcal{F}$ is called the **torsion-free class**.

A torsion theory $(\mathcal{T}, \mathcal{F})$ is called **splitting** if, every indecomposable A-module is either torsion or torsion-free.

Definition

An A-module T is called a ***tilting module*** *if it satisfies the following conditions:*

(T_1) $dp\, T \leq 1$.

(T_2) $Ext^1_A(T,T) = 0$.

(T_3) If $T = T_1^{(m_1)} \oplus \cdots \oplus T_t^{(m_t)}$, with $T_i \ncong T_j$ whenever $i \neq j$. Then, $t = rank K_0(A)$.

A tilting A-module T induces a torsion theory $(\mathcal{T}(T), \mathcal{F}(T))$ in mod A:

$$\mathcal{T}(T) = \{M \in \text{mod } A \mid \text{Ext}^1_A(T, M) = 0\}$$

$$\mathcal{F}(T) = \{M \in \text{mod} A \mid \text{Hom}_A(T, M) = 0\}.$$

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If $B = \text{End}_A(T)$, T also induces a torsion theory $(\mathcal{X}(T), \mathcal{Y}(T))$ in mod B:

$$\mathcal{X}(T) = \{X \in \text{mod } B \mid X \otimes_B T = 0\}$$

$$\mathcal{Y}(T) = \{Y \in \text{mod } B \mid \text{Tor}^B_1(Y, T) = 0\}.$$

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• T is said to be **separating** if the induced torsion theory $(\mathcal{T}(T), \mathcal{F}(T))$ in mod A is splitting, and
• T is said to be **splitting** if the induced torsion theory $(\mathcal{X}(T), \mathcal{Y}(T))$ in mod B is splitting.

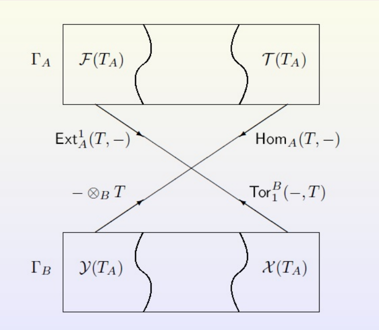
Theorem (Brenner-Butler)

Let A be an algebra, T a tilting A-module and $B = \mathrm{End}_A T$. Then:

(a) *T is a tilting B-module and $A \simeq \mathrm{End}_B T$.*

(b) (i) *The functors $\mathrm{Hom}_A(T, -)$ and $- \otimes_B T$ induce inverse equivalences between the full subcategories $\mathcal{T}(T)$ and $\mathcal{Y}(T)$.*

(ii) *The functors $\mathrm{Ext}^1_A(T, -)$ y $\mathrm{Tor}^B_1(-, T)$ induce inverse equivalences between the full subcategories $\mathcal{F}(T)$ and $\mathcal{X}(T)$.*



Theorem

Let A an algebra, T is a separating and splitting tilting $A-$module and $B = End_A T$. Let M, N be indecomposable $A-$modules of $\mathcal{T}(T)$ and $f : M \to N$ a morphism. Then,

$$f \in \Re_A^n \backslash \Re_A^{n+1} \quad \text{if and only if} \quad Hom_A(T, f) \in \Re_B^n \backslash \Re_B^{n+1}.$$

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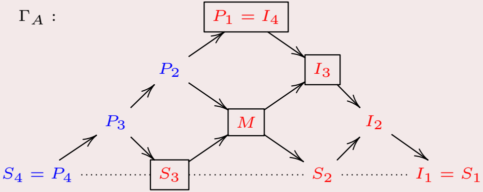
$$f \in \Re_A^n \backslash \Re_A^{n+1} \quad \text{if and only if} \quad \mathrm{Ext}_A^1(T, f) \in \Re_B^n \backslash \Re_B^{n+1}.$$

Example

Let A be the algebra $1 \to 2 \to 3 \to 4$

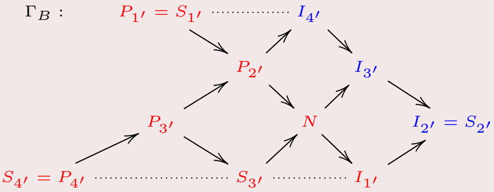
Γ_A :



$T = P_1 \oplus I_3 \oplus M \oplus S_3$

Then $B = \mathrm{End}_A T$ is the algebra $1' \leftarrow 2' \to 3' \to 4'$

Γ_B :



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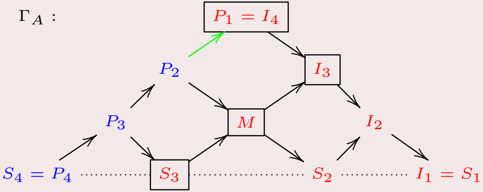

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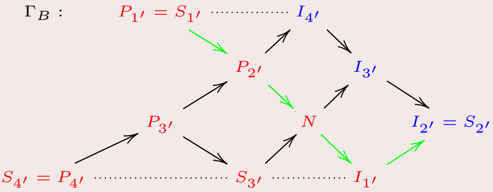

Proposition

Let A *an algebra,* T *is a separating tilting* $A-$*module and* $B = End_A T$*. Let* M, N *be indecomposable* $A-$*modules of* $\mathcal{T}(T)$ *and* $f : M \to N$ *a morphism.*

$$\text{If } f \in \Re_A^n \setminus \Re_A^{n+1} \text{ then } Hom_A(T, f) \in \Re_B^n.$$

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Moreover, if $Hom_A(T, f) \notin \Re_B^{n+1}$, then there exists

$$F(M) \xrightarrow{\widetilde{f}_1} \widetilde{X}_1 \xrightarrow{\widetilde{f}_2} \dots \xrightarrow{\widetilde{f}_{n-1}} \widetilde{X}_{n-1} \xrightarrow{\widetilde{f}_n} F(N)$$

with $\widetilde{f}_i$ irreducible and $\widetilde{X}_i \in \mathcal{Y}(T)$, for all i.

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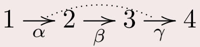

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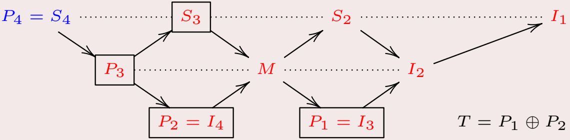

$T = P_1 \oplus P_2 \oplus P_3 \oplus S_3$

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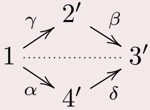

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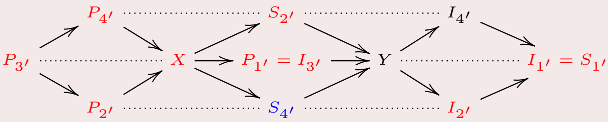

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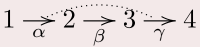

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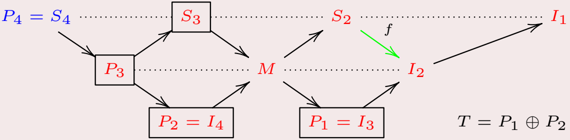

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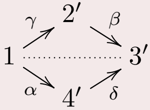

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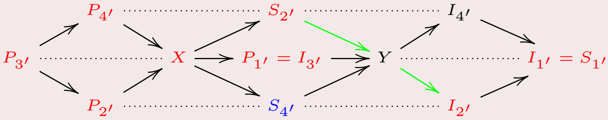

$$F(f) \in R^2$$

The APR tilting module

Let $A = KQ/I$ be an algebra and let S_a be a simple projective non-injective module (correspondig to the sink $a \in Q_0$).
Then, the module

$$T[a] = \tau^{-1}(S_a) \oplus (\bigoplus_{b \neq a} P_b)$$

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Moreover, $T[a]$ is a separating module, where

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Definition

We say that $T[a]$ is a **free APR-tilting module** if the sink $a \in Q_0$ is free, that is, it is not the terminal point of a generating relation on Q.

Proposition

Let A be an algebra, let T be an APR tilting A-module and $B = \text{End}_A T$. Then the following statement hold.

(i) If A is of infinite representation type, then B is of infinite representation type.

(ii) Let A be a representation-finite algebra. If there exist a natural number m such that $\Re^m(\text{mod}\, B) = 0$ then $\Re^m(\text{mod}\, A) = 0$.

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If T is a free APR-tilting A-module, then

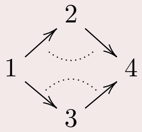
(i) A is representation-finite if and only if B is representation-finite.

(ii) Let A be a representation finite algebra. Then for any a natural number m,

$$\Re^m(\text{mod}\, B) = 0 \text{ if and only if } \Re^m(\text{mod}\, A) = 0.$$

Example

Let A be an algebra given by the bound quiver



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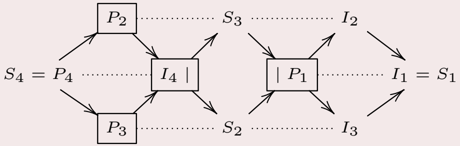

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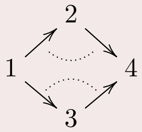

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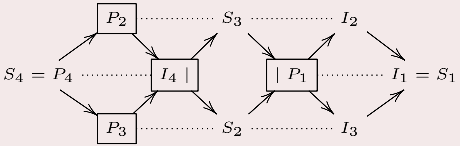
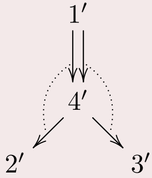


$T[4] = P_1 \oplus P_2 \oplus P_3 \oplus \tau^{-1} P_4$

Then $B = \mathrm{End}_A T$ is the algebra given by the bound quiver



B is of infinite representation type.

Iterated tilted algebras

Definition

Let Δ be a finite connected quiver without oriented cycles. An algebra A is called **iterated tilted algebra** of type Δ if there exist a sequence of algebras $A = A_0, A_1, \ldots, A_r = k\Delta$ and a sequence $T^{(i)}$, $0 \leq i < r$ of separating tilting A_i-modules such that $A_{i+1} = \mathrm{End}_{A_i} T^{(i)}$, for each i.

Theorem (Happel)

If A is iterated tilted of type Δ, where Δ is a Dynkin quiver, then A may be transformed to an hereditary algebra of Dynkin type by a finite sequences of APR-tilting modules.

Theorem

Let Δ be a quiver of Dynkin type and let A be an iterated tilted algebra of type Δ. Then the following statement hold.

(a) *If* $\overline{\Delta} = A_n$, *then* $\Re^n(\mathit{mod}\, A) = 0$ *for* $n \geq 1$.
(b) *If* $\overline{\Delta} = D_n$, *then* $\Re^{2n-3}(\mathit{mod}\, A) = 0$ *for* $n \geq 4$.
(c) *If* $\overline{\Delta} = E_6$, *then* $\Re^{11}(\mathit{mod}\, A) = 0$.
(d) *If* $\overline{\Delta} = E_7$, *then* $\Re^{17}(\mathit{mod}\, A) = 0$.
(e) *If* $\overline{\Delta} = E_8$, *then* $\Re^{29}(\mathit{mod}\, A) = 0$.

Example

Let A and $\widetilde{A}$ be the iterated tilted algebras of type A_5 given by the bound quivers

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4 \xrightarrow{\delta} 5$$

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We prove that $\Re^5(\text{mod}\, A) = 0 = \Re^5(\text{mod}\, \widetilde{A})$.

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$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4 \xrightarrow{\delta} 5 \qquad (\alpha\beta = 0,\ \gamma\delta = 0)$$

We prove that $\Re^5(\text{mod}\, A) = 0 = \Re^5(\text{mod}\, \widetilde{A})$.

Moreover we have that $\Re^4(\text{mod}\, A) \neq 0$, but $\Re^4(\text{mod}\, \widetilde{A}) = 0$.

Theorem (Chaio)

Let $A \cong kQ/I$ be a representation-finite algebra. For each vertex $a \in (Q_A)_0$ we consider

$$r_a = \ell(P_a \rightsquigarrow S_a \rightsquigarrow I_a)$$

Then,

$$\Re^m(\text{mod } A) \neq 0 \text{ and } \Re^{m+1}(\text{mod } A) = 0,$$

where $$m = \max\{r_a\}_{a \in Q_0}.$$

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where $$m = \max\{r_a\}_{a \in Q_0}.$$

We denote by

$$R_0 = \{u \in Q_0 \mid r_u = m\}$$

Proposition

Let $A \cong kQ/I$ be an algebra. Let T be a separating tilting $A-$module and $B =$ End$_A T$.

Proposition

Let $A \cong kQ/I$ *be an algebra. Let* T *be a separating tilting* $A-$*module and* $B = End_A T$.

(i) *If* A *is of infinite representation type, then* B *is of infinite representation type.*

(ii) *Let* A *be a representation-finite algebra and* $P_u \in addT$, *for some* $u \in R_0 \subset Q_0$. *If there exist a natural number* m *such that* $\Re^m(mod\, B) = 0$ *then* $\Re^m(mod\, A) = 0$.

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Thank you for your attention!