A generalization of quasitilted and almost hereditary algebras

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Basic facts

A is **finite dimensional algebra** over an algebraically closed field k.

mod A

$\mathcal{D}^b(\textit{mod}\, A)$

$\mathcal{H}$ is a **hereditary abelian**

(k-linear) category with split idempotents, finite-dimensional Hom-spaces,

and with tilting objects.

Generalization of quasitilted (or almost hereditary) algebras

Definition

A is said to be (m,n)-quasitilted algebra if there exists a sequence of triples $(A_i, T_i, A_{i+1} = \text{End}_{A_i} T_i)$ s.t

A_0 is a quasitilted algebra of global dimension two; and $A = A_{m+n-2}$.

each T_i is a stair splitting tilting or cotilting A_i-module (in each step i, $\text{gl.dim}A_i < \text{gl.dim}A_{i+1}$) where

$n-1$ ($m-1$) is the number of indexes i s.t, T_i is tilting (cotilting).

$(m, 1)$-almost hereditary

Lemma

Let d be a postive integer. Let A be an algebra with $\mathrm{gl.dim} A = d$, T an A-module and $B = \mathrm{End}_A T$.

(i) If T is a stair splitting tilting module, then B is $(d, 1)$-almost hereditary.

(ii) If T is a stair splitting cotilting module, then B is $(1, d)$-almost hereditary.

In particular, any (m, n)-quasitilted algebra is $(m + n - 1, 1)$-almost hereditary, or else $(1, m + n - 1)$-almost hereditary.

$(m, 1)$-almost hereditary

$\mathcal{L}_{\mathcal{C}}$ denotes the class of $X \in$ *ind* A such that every predecessor of X in *ind* A lies in $\mathcal{C}$.

Proposition (-CMP)

Let n be a positive integer. Let $\mathcal{C}$ be a torsion-free class of *mod* A such that, for all $X \in$ *ind* A,

$$X \in \mathcal{C}, \text{ or else } id_A X \leqslant n.$$

Then *ind* $A = \mathcal{L}_{\mathcal{C}} \cup \mathcal{R}_A^n$.

$(m, 1)$-almost hereditary

Proposition

Let m be a positive integer. Let A be a finite dimensional k-algebra such that $A \in \operatorname{add} \mathcal{L}_A^m$. Then

1. *gl.dim* $A \leqslant m + 1$ and, for all $X \in$ *ind* A, then *pd*$_A X \leqslant m$ or else *id*$_A X \leqslant 1$;
2. if, moreover, *gl.dim* $A = m + 1$, then A is $(m, 1)$-almost hereditary.

t-structures

$(\mathcal{W}, \mathcal{W}^{\perp}[1])$: natural t-structure in $\mathcal{D}^b(\mathcal{H})$

$(\mathcal{U}, \mathcal{U}^{\perp}[1])$ induced t-structure: $(\mathcal{T}(T), \mathcal{F}(T))$ torsion pair in $\mathcal{H}$

$\mathcal{U} = \{Z \in \mathcal{D}^b(\mathcal{H}) | \ H^0(Z) \in \mathcal{T}(T)$ and $H^i(Z) = 0$, for $i > 0\}$

heart of $(\mathcal{U}, \mathcal{U}^{\perp}[1]) \simeq$ *mod* $\mathrm{End}_{\mathcal{H}} T$

$\mathrm{End}_{\mathcal{H}}(T)^{op} = A_0$

$(\mathcal{V}, \mathcal{V}^{\perp}[1])$ induced t-structure: $(\mathcal{T}(T_0), \mathcal{F}(T_0))$ torsion pair in *mod* A_0

$\mathcal{V} = \{Z \in \mathcal{D}^b(\mathcal{H}) | \ H^0_{\mathcal{U}}(Z^{\cdot}) \in \mathcal{T}(T_0)$ and $H^i_{\mathcal{U}}(Z^{\cdot}) = 0$, for $i > 0\}$ heart of

$(\mathcal{V}, \mathcal{V}^{\perp}[1]) \simeq$ *mod* $\mathrm{End}_{A_0} T_0$

$\mathrm{End}_{A_0}(T_0)^{op} = B$

Homological properties of a $(1,2)$-quasitilted algebra

$\text{Ext}_B^{\#}(\_,\_) = 0$	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$
$\mathcal{A}_1$	≥ 2	≥ 1	≥ 1	≥ 0
$\mathcal{A}_2$	$0, \geq 3$	≥ 2	≥ 1	≥ 1
$\mathcal{A}_3$	$0, \geq 3$	$0, \geq 2$	≥ 2	≥ 1
$\mathcal{A}_4$	$0, 1, \geq 4$	$0, \geq 3$	$0, \geq 2$	≥ 2

Homological properties of a $(1,2)$-quasitilted algebra

$(add(\mathcal{A}_3 \cup \mathcal{A}_4), add(\mathcal{A}_1 \cup \mathcal{A}_2))$ is a torsion pair in *mod* B

$(add(\mathcal{A}_1 \cup \mathcal{A}_2), add(\mathcal{A}_3 \cup \mathcal{A}_4)[-1])$ is a torsion pair in *mod* A_0

$$0 \to A_1^Z \oplus A_2^Z \to Z \to A_3^Z[-1] \oplus A_4^Z[-1] \to 0$$

$Z \in \mathcal{F}[1]$ $\quad 0 \to A_1^Z \oplus A_2^Z \to Z \to A_4^Z[-1] \to 0$

$Z \in \mathcal{T}$ $\quad 0 \to A_1^Z \to Z \to A_3^Z[-1] \oplus A_4^Z[-1] \to 0$

Compatibility

Definition

(Keller-Vossieck)

Let $(\mathcal{U}, \mathcal{U}^{\perp}[1])$ and $(\mathcal{V}, \mathcal{V}^{\perp}[1])$ t-structures in a triangulated category $\mathcal{C}$. We say that $\mathcal{U}$ is compatible with $\mathcal{V}$ if $\mathcal{U}$ is stable under the truncation functors $\tau_{\mathcal{V}}^{\leq n}$, $n \in \mathbb{Z}$, that is, $\tau_{\mathcal{V}}^{\leq n}\mathcal{U} \subseteq \mathcal{U}$, for all $n \in \mathbb{Z}$.

Proposition

$\mathcal{W}$ is compatible with $\mathcal{U}$.

$\mathcal{U}$ is compatible with $\mathcal{V}$.

Compatibility

Remark $\mathcal{W}$ is compatible with $\mathcal{V}$ if and only if, for each $C \in \mathcal{W}$, $A_1 = 0$, where

$$0 \to A_1 \oplus A_2 \to \mathcal{H}^1_{\mathcal{U}}(C) \to A_3[-1] \oplus A_4[-1] \to 0$$

is the short exact sequence relatively to $(\mathcal{T}(T_0), \mathcal{F}(T_0))$.

Compatibility

Corollary

Let B be a $(1,2)$-quasitilted algebra, $X \in \mathcal{F}(T_0)[1]$ and let

$$0 \to A_1 \oplus A_2 \to X \to A_4[-1] \to 0$$

be the canonical exact sequence for X relatively to $(\mathcal{T}(T_0), \mathcal{F}(T_0))$.

Then $\mathcal{W}$ is compatible with $\mathcal{V}$ if and only if $A_1 = 0$.

Compatibility - $(1,2)$-quasitilted algebra

Corollary

If $\mathcal{W}$ is compatible with $\mathcal{V}$ then:

H_1 either $\mathrm{Hom}_{\mathcal{D}^b(B)}(A_4[-1], A_1[1]) = 0$ or $\mathrm{Hom}_{\mathcal{D}^b(B)}(A_4[-1], A_2[1]) = 0$.

H_2 *Ker* $(A_2 \to A_4) = \oplus K_i \Rightarrow K_i \notin \mathcal{A}_1$ and

$Ker(A_3 \oplus A_4 \to C_4) = \oplus L_i \Rightarrow L_i \notin \mathcal{A}_1$.

Compatibility

$$\mathcal{X} = \{Z \in \textit{mod}\, A_1;\ A_1^Z = 0 \text{ and } A_3^Z[-1] = 0\}$$

$$\mathcal{Y} = \{Z \in \textit{mod}\, A_1;\ A_2^Z = 0 \text{ and the directs summands of } Z \text{ do not belongs to } \mathcal{A}_4[-1]\}.$$

Compatibly

Lemma

Seja $\mathcal{X} = \{Z \in \textit{mod}\, A;\ A_1^Z = 0 \text{ e } A_3^Z[-1] = 0\}$*. Então* $\textit{id}_A\ \mathcal{X} \leq 1$*.*

Lemma

Seja $\mathcal{Y} = \{Z \in \textit{mod}\, A;\ A_2^Z = 0 \text{ e } Z \text{ não admite somando direto em } \mathcal{A}_4[-1]\}$*.*
Então $\textit{pd}_A\ \mathcal{Y} \leq 1$*.*

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Lemma

Let B be an $(1,2)$-quasitilted algebra, $w : A_4'[-1] \to A_1[1]$ a right minimal add $\mathcal{A}_4[-1]$-approximation of $A_1[1] \in$ add $\mathcal{A}_1[1]$ and $w' : A_4[-1] \to A_1'[1]$ a left minimal add $\mathcal{A}_1[1]$-approximation of $A_4[-1] \in$ add $\mathcal{A}_4[-1]$.

(i) If $\gamma : B_4[-1] \to A_1[1]$, with $B_4[-1] \in$ add $\mathcal{A}_4[-1]$, is such that cone$(\gamma)[-1] \in \mathcal{Y}(T_0)$, then a morphism $\delta : B_4[-1] \to A_4'[-1]$ that satisfies $w\delta = \gamma$ is such that Hom$(_, \delta)$ is injective in $\mathcal{A}_4[-1]$.

(ii) If $\gamma' : A_4[-1] \to B_1[1]$, with $B_1 \in$ add $\mathcal{A}_1$, is a morphism such that cone$(\gamma')[-1] \in \mathcal{Y}(T_0)$, then a morphism $\delta' : A_1'[1] \to B_1[1]$ that satisfies $\delta' w' = \gamma'$ is such that Hom$(\delta', _)$ is surjective in $\mathcal{A}_1[2]$.

Torsion pair given by a tilting

$\mathcal{T}$ is a cogenerator for $\mathcal{A}$

$\mathcal{T} = Fac\ T$

$Ext^i(T, X) = 0$ for $X \in \mathcal{T}$ and $i > 0$

If $Z \in \mathcal{T}$ satisfies $Ext^i(Z, X) = 0$ for all $X \in \mathcal{T}$ and $i > 0$, then $Z \in addT$

If $Ext^i(T, X) = 0$ for $i \geq 0$ and $X \in \mathcal{A}$, then $X = 0$.

Obrigado!