Generic cutoff for sparse Markov chains

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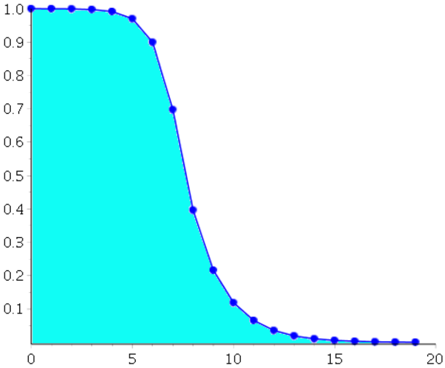
(Université Paris Diderot)



Joint work with Charles Bordenave and Pietro Caputo

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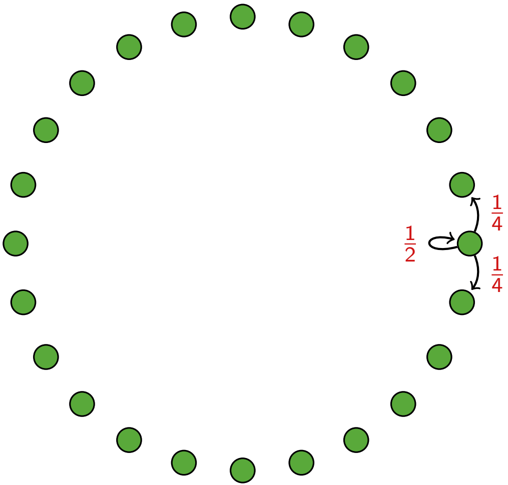
▷ **Mixing time:** $t_{\mathrm{MIX}}(\varepsilon) := \min\{t \geq 0 \colon \mathrm{d}(t) \leq \varepsilon\}$ $(0 < \varepsilon < 1)$

Lazy random walk on the cycle



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$X_t = \xi_1 + \cdots + \xi_t \mod n$, with $(\xi_t)_{t \geq 1}$ i.i.d.

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- **Corollary:** $\mathrm{d}_n(\lfloor \lambda n^2 \rfloor) \xrightarrow[n\to\infty]{} \dfrac{1}{2}\displaystyle\int_0^1 |1 - f_\lambda(u)| \,\mathrm{d}u$

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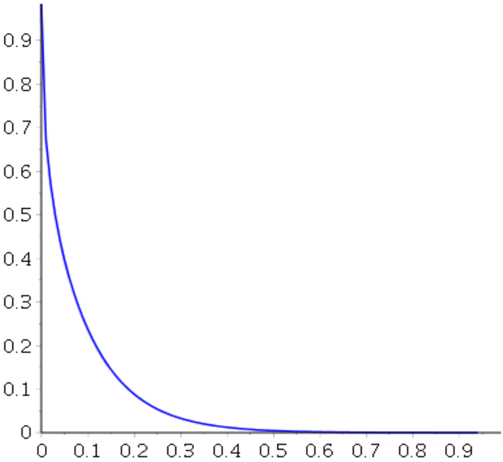

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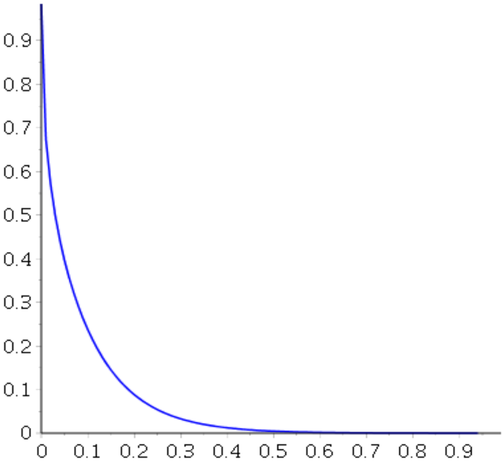
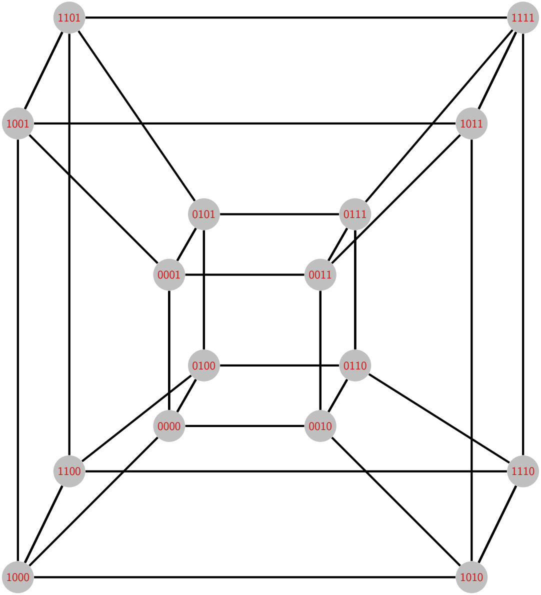


▷ Convergence to stationarity occurs **gradually** on timescale $\Theta(n^2)$

Lazy random walk on the hypercube



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- **Corollary:** $\mathrm{d}_n\left(\left\lfloor \frac{n \ln n}{2} + \lambda n \right\rfloor\right) \xrightarrow[n\to\infty]{} \frac{1}{2\pi} \int_{-\frac{e^{-\lambda}}{2}}^{+\frac{e^{-\lambda}}{2}} e^{-\frac{u^2}{2}} \, \mathrm{d}u$

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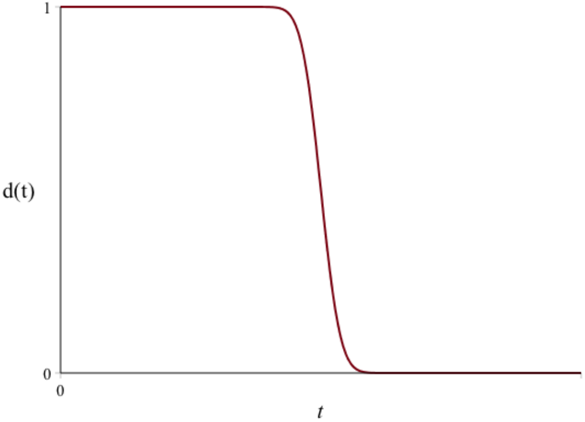

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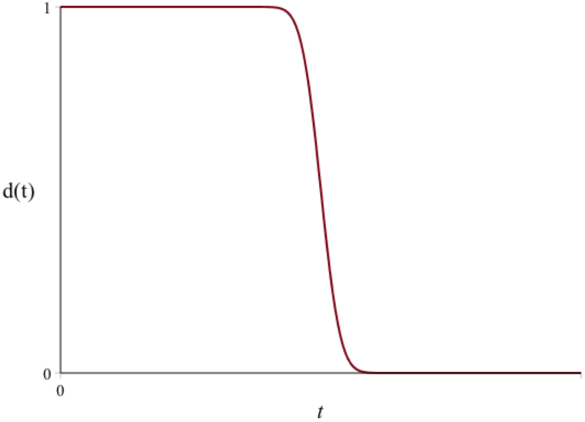

▷ Convergence to stationarity occurs **abruptly** at $t \approx \frac{n \log n}{2}$

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A sequence of Markov chains (indexed by n) exhibits **cutoff** if

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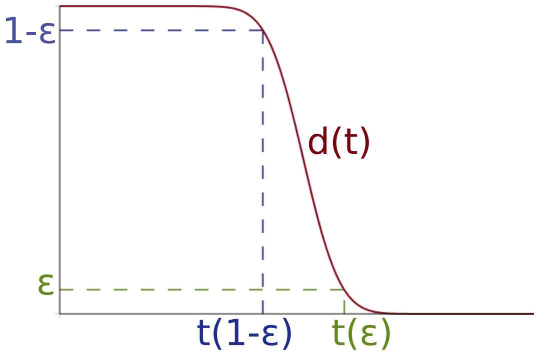

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- *It occurs in all the examples we can explicitly calculate, but we know no general result which says that the phenomenon must happen for all "reasonable" shuffling methods.*

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"Almost every *reasonable* chain should exhibit cutoff"

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- **Corollary:** complete mixing in two steps only, no cutoff !

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3. **Exchangeable**: swaps within a row preserve the law of P_n.

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More generally: may allow for arbitrary degrees $(d_{n,i})$, provided

$$\frac{1}{n}\sum_{i=1}^{n} \log d_{n,i} \;=\; \mathcal{O}(1) \quad \text{and} \quad 2 \leq d_{n,i} \ll \sqrt{\log n}.$$

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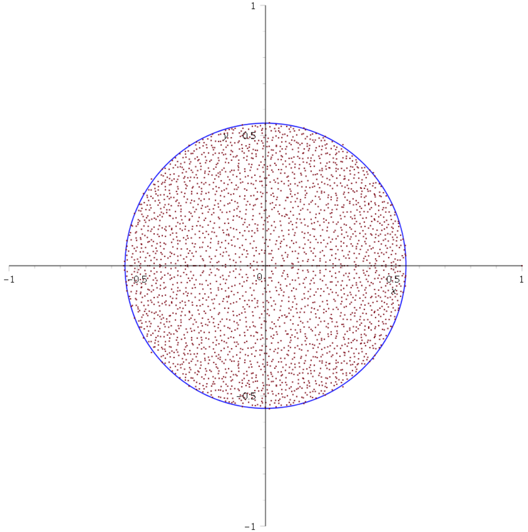
Corollary: with high probability, P_n exhibits cutoff at time

$$t_n := \frac{\log n}{\psi(1)-\psi(1-\alpha)} \quad \text{where} \quad \psi = \frac{\Gamma'}{\Gamma}.$$

Eigenvalues of P_n



Thank you for your attention !