Scaling limit of dynamical percolation on critical Erdös-Rényi random graphs

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Erdös-Rényi random graphs

- n vertices
- E_n: set of edges of the complete graph
- For $p \in [0,1]$, state of edge e in $\mathcal{G}(n,p)$: present with prob. p
- edges of $G(n,p)$ are independent

The birth of a giant connected component (giant = of size $\Theta(n)$):

- $p = c/n$, $c < 1 \implies$ no giant component
- $p = c/n$, $c > 1 \implies \exists$ a unique giant component
- Critical window: $p = p(n,\lambda) = \frac{1}{n} + \frac{\lambda}{n^{4/3}}$, $\lambda \in \mathbb{R}$

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Background: convergence of rescaled components, λ fixed

$\mathcal{G}^{n,\lambda} := (\mathcal{C}_1, \mathcal{C}_2, \dots) =$ sequence of rescaled connected components listed in decreasing order of size, viewed as measured metric spaces:

measure $=$ counting measure on vertices $\times n^{-2/3}$
distance $=$ graph distance $\times n^{-1/3}$

Theorem (Aldous'97 + Addario-Berry, Broutin & Goldschmidt'10)

$(\mathcal{G}^{n,\lambda})$ converges in distribution ($n \to \infty$) to a random sequence of measured metric spaces (which are $\mathbb{R}$-graphs) $\mathcal{G}_\lambda$.

Topology: product of Gromov-Hausdorff-Prokhorov

$$d_{GHP}((X,\mu,d),(X',\mu',d')) := \inf_{\delta \text{ on } X \sqcup X'} \{\delta_H(X,X') \vee \delta_P(\mu,\mu')\}$$

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Dynamical percolation, coalescence and fragmentation on $\mathcal{G}(n, p)$

Fix some intensity γ_n

- Coalescence: create the edge at rate $\gamma_n p$
- Fragmentation: kill the edge at rate $\gamma_n(1-p)$
- Dynamical Percolation: perform coalescence and fragmentation independently

Background: multiplicative coalescent

$x^{n,\lambda}$:= sequence of masses in $\mathcal{G}^{n,\lambda}$

Under coalescence, connected components i and j, of masses $x_i n^{2/3}$ and $x_j n^{2/3}$ merge at rate

$$\gamma_n p \cdot x_i n^{2/3} \cdot x_j n^{2/3}$$

Choose intensity $\gamma_n = n^{-1/3}$:

$$i \text{ and } j \text{ merge at rate } \simeq x_i x_j$$

$\rightarrow$ multiplicative coalescent

Theorem (Aldous '97)

Multiplicative coalescent has the Feller property on $\ell^2(\mathbb{N})$, and $(x^{n,\lambda})_{\lambda\in\mathbb{R}}$ converges as $(n \to +\infty)$ to some version of the multiplicative coalescent (in the sense of fidi).

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Dynamical percolation, coalescence and fragmentation on measured $\mathbb{R}$-graphs

- Coalescence on a collection of measured metric spaces (X, μ, d): in $\mathrm{Coal}(X, \mathcal{P}_t^+)$, identify points of $\mathcal{P}_t^+$ where $\mathcal{P}^+$ is a Poisson process of intensity $\frac{1}{2}\mu^{\otimes 2} \times \mathrm{leb}_{\mathbb{R}^+}$ on $X^2 \times \mathbb{R}_+$.
- Fragmentation on a collection of $\mathbb{R}$-graphs (X, d): in $\mathrm{Frag}(X, \mathcal{P}_t^-)$, cut points of $\mathcal{P}_t^-$ where $\mathcal{P}^-$ is a Poisson random set of intensity $\ell_X \times \mathrm{leb}_{\mathbb{R}_+}$ on $X \times \mathbb{R}_+$.
- Dynamical percolation on a collection of measured $\mathbb{R}$-graphs (X, μ, d): perform coalescence and fragmentation, simultaneously and independently (if possible !).

Main result

Choose intensity $\gamma_n = n^{-1/3}$

Theorem (R. 2017+)

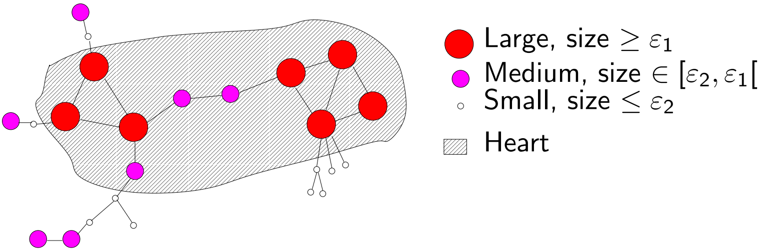
The dynamical percolation (resp. coalescence, resp. fragmentation) process on $\mathcal{G}(n, n^{-1} + \lambda n^{-4/3})$ converges in distribution ($n \to \infty$) to the dynamical percolation (resp. coalescence, resp. fragmentation) process on the limit $\mathcal{G}_\lambda$

Perspectives (with N. Frilet)

- Scaling limit of dynamical percolation on other random graphs in the basin of attraction of $\mathcal{G}_\lambda$:
  - configuration model
  - IRG
- Show that dynamical percolation on $\mathcal{G}_\lambda$ is mixing
  $\Rightarrow$ Noise-sensitivity of large components

Structure for the multiplicative coalescent

For every $\varepsilon > 0$, there exists $\varepsilon_2 \leq \varepsilon_1 \leq \varepsilon$ such that with probability larger than $1 - \varepsilon$, every component $\mathcal{C}$ of $Coal(x, t)$ of size at least ε has the following structure



Crown:=$\mathcal{C} \setminus Heart$, $mass(Crown) < \varepsilon_1$

Almost-Feller property for coalescence

Suppose:

- $d_{GHP}(X^{(n)}, X^{(\infty)}) + \|sizes(X^{(n)}) - sizes(X^{(\infty)})\|_2 \xrightarrow[n\infty]{} 0$
- (extra-condition) $\forall \varepsilon > 0$,

$$\limsup_{n\infty} \mathbb{P}(diam(Coal(X^{(n)}_{<\varepsilon_1}, t)) > \varepsilon) \xrightarrow[\varepsilon_1 \to 0]{} 0$$

Then $(Coal(X^{(n)}, t))_{t \geq 0}$ converges to $(Coal(X^{(\infty)}, t))_{t \geq 0}$ in distribution.

Main steps of AdBrGo'10, λ fixed

- BM with parabolic drift $B_t^\lambda := B_t + \lambda t - \frac{t^2}{2}$,
- Reflect it above past minima: $h_t^{\infty,\lambda} := 2(B_t^\lambda - \min_{0\leq s\leq t} B_s^\lambda)$,
- The rescaled height $h^{n,\lambda}$ of the exploration process converges in distribution to $h^{\infty,\lambda}$ (Marckert & Mokkadem '03 + AdBrGo'10).
- Additional edges $\simeq$ Poisson point process of intensity $1/2$ below $h^{\infty,\lambda}$

Control of the diameter of the crown for $\mathcal{G}(n, p)$

- In depth-first order, a component of the crown is explored in at most two intervals
- $w_n :=$ the modulus of continuity of $4h^{n,\lambda+t}$. Then, w.h.p

  $$\text{diameter of the crown } \leq w_n(\varepsilon_1)$$

  which goes to zero when ε_1 goes to zero, uniformly in n

Scaling limit for discrete fragmentation on $\mathbb{R}$-graphs

- New distance, with surplus
- Feller property
- Close to Addario-Berry, Broutin, Goldschmidt and Miermont 2017. Uses Evans, Pitman, Winter to couple Poisson processes

$\Rightarrow$ limit of discrete fragmentation and discrete dynamical percolation

Fragmentation vs coalescence

On $\mathcal{G}_\lambda$, fragmentation is the time-reversal of coalescence.

$$(\mathcal{G}_\lambda, \mathrm{Coal}(\mathcal{G}_\lambda, t)) \overset{(d)}{=} (\mathrm{Frag}(\mathcal{G}_{\lambda+t}, t), \mathcal{G}_{\lambda+t})$$

Proof: true for finite n + convergence

Noise sensitivity

$N_\varepsilon(G^{n,\lambda})$: refresh each edge with probability ε.
A graph property A_n is ε_n-noise sensitive if

$$\mathrm{Cor}(\mathbb{1}_{A_n}(G^{n,\lambda}), \mathbb{1}_{A_n}(N_\varepsilon(G^{n,\lambda}))) \xrightarrow[n\infty]{} 0$$

and ε_n-noise stable if

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Suppose A_n is a sequence of properties which can be "seen" in the scaling limit

- Main result $\Rightarrow$ ε_n-noise stability of A_n when $\varepsilon_n \ll n^{-1/3}$
- Mixing of dyn. perc. (to be proved) $\Rightarrow$ ε_n-noise sensitivity of A_n when $\varepsilon_n \gg n^{-1/3}$

Example: having a complex component, being planar

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The End

Thanks !!!