Limits of large loops in the $O(n)$ model on random maps

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École polytechnique

Conference Dynamics on Random Graphs and Random Maps
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Outline

Motivation: study the large-scale geometry of

- faces of Boltzmann planar maps.
- loops in the rigid $O(n)$ model on quadrangulations.

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- loops in the rigid $O(n)$ model on quadrangulations.

1. Boltzmann maps: definitions and scaling limits

2. Geometry of large faces in Boltzmann maps

3. Applications to the rigid $O(n)$ loop model on quadrangulations

Bipartite planar map

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Figure : A **bipartite** planar map.

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Figure : A **rooted** bipartite planar map.

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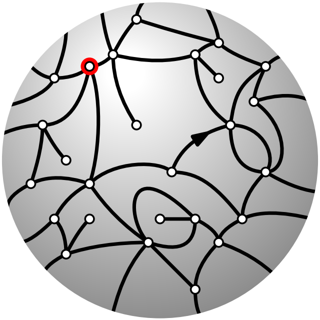
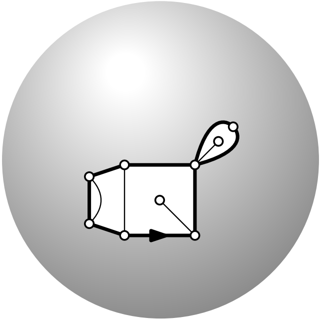


Figure : A **pointed** rooted bipartite planar map.

Map with a boundary

- Map *with a boundary*: the face on the right of the root is **external**.



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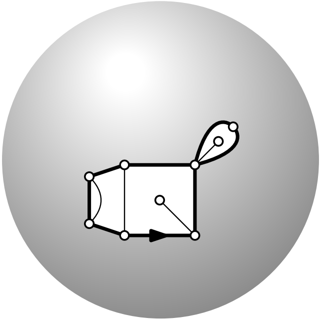

Figure : A map with a boundary of perimeter 8.

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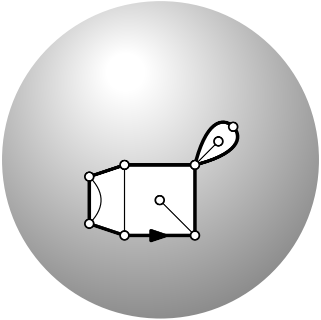

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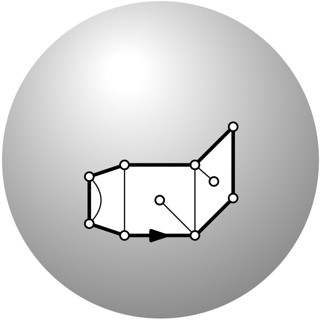

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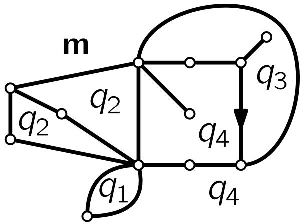

Figure : A bipartite map $\mathbf{m}$ with weight $w_{\mathsf{q}}(\mathbf{m}) = q_1 q_2^2 q_3 q_4^2$.

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(Pointed maps version: $\mathbb{P}^{\bullet}_{\mathsf{q}}$.)

Classification of weight sequences

- $\mathbb{E}^\bullet_{\mathsf{q}}(\#\mathrm{V}(M))$ = average number of vertices under $\mathbb{P}^\bullet_{\mathsf{q}}$.
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$$\begin{array}{c}\mathsf{q}\ \textbf{non-generic critical} \\ \textit{of parameter}\ \alpha \in (1,2)\end{array} \iff \mathsf{q}\ \textit{critical and}\ \mu([k,\infty)) \sim C \cdot k^{-\alpha}.$$

Enumeration

- $\mathcal{M}_k = \{$bipartite maps with a boundary of perimeter 2k$\}$

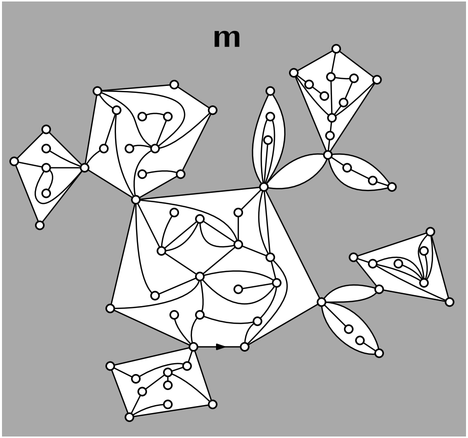

Figure : An element $\mathbf{m} \in \mathcal{M}_{19}$

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The brownian map

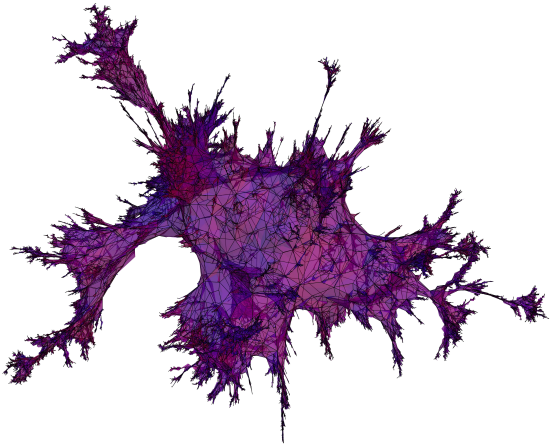

Figure : A uniform quadrangulation with 50000 faces, by Jérémie Bettinelli.

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The stable map

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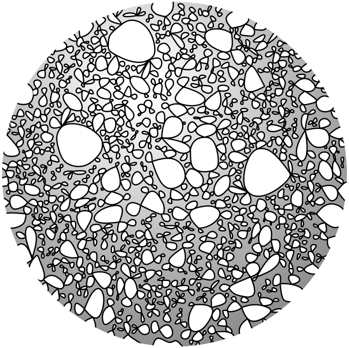

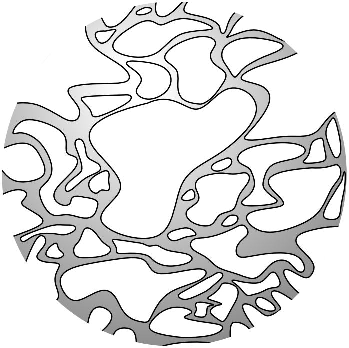

Figure : The stable map.

Geometry of large faces in Boltzmann maps

Branching structure in the dense phase

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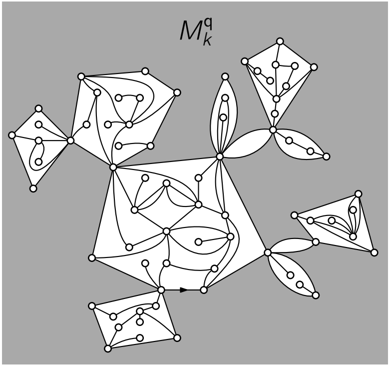

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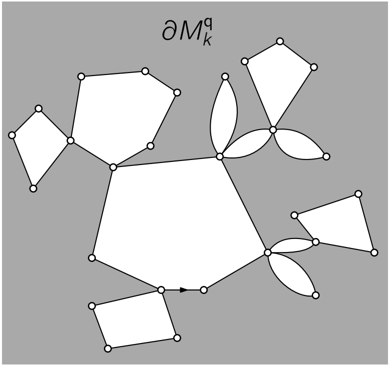

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$$\frac{C}{k^{\alpha-1/2}} \cdot \partial M_k^{\mathsf{q}} \xrightarrow[k\to\infty]{(d)} \mathscr{L}_\beta, \quad \textit{where} \quad \beta := \frac{1}{\alpha - \frac{1}{2}} \in (1, 2).$$

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- Hausdorff dimension β [Curien & Kortchemski '14]

Random stable looptree

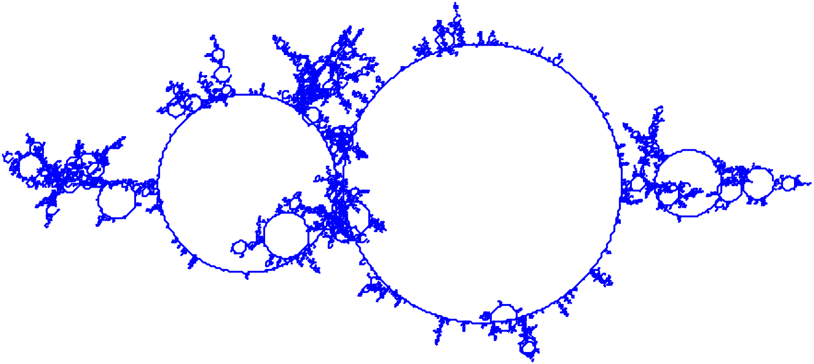

Figure : A random stable looptree ($\beta = 1.07$), by Igor Kortchemski.

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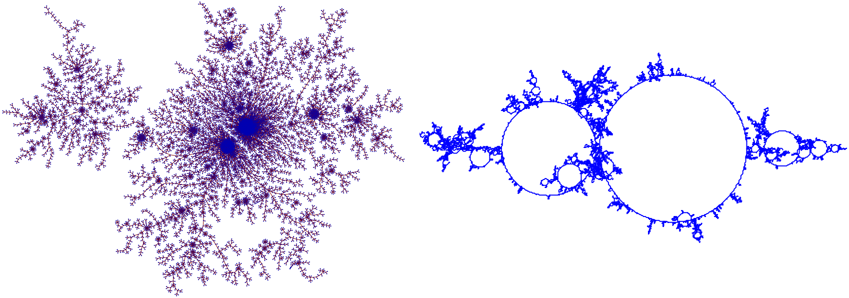

Figure : A stable tree ($\beta = 1.07$) and the associated stable looptree, by Igor Kortchemski.

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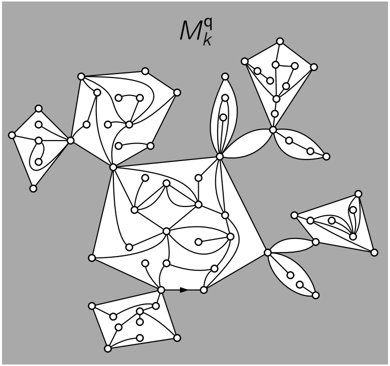

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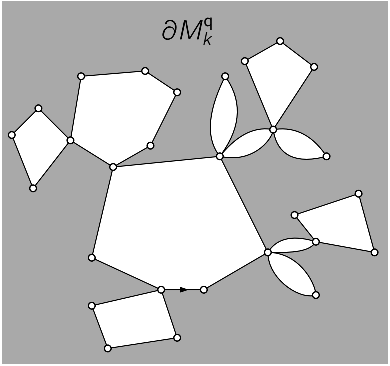

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Theorem 2

Let q *be a weight sequence of type* $\alpha \in [1,2]$.

- $\alpha \in [1, 3/2] \Longrightarrow$ *all the internal faces of* $\partial\mathbf{M}_\infty^{\mathrm{q}}$ *are finite.*
- $\alpha \in (3/2, 2] \Longrightarrow \partial\mathbf{M}_\infty^{\mathrm{q}}$ *has a unique infinite internal face.*

Phase transition

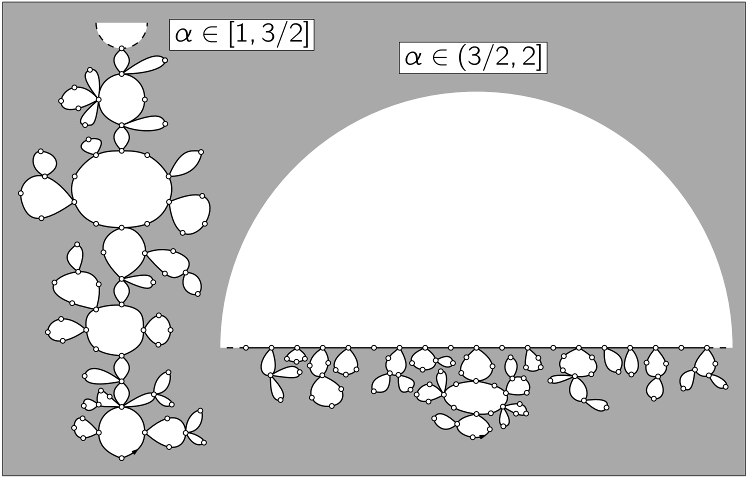
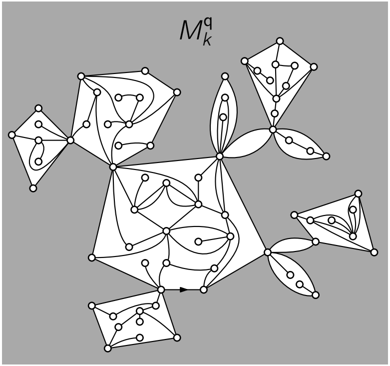


Figure : The boundary $\partial \mathbf{M}_\infty^{\mathsf{q}}$ of the infinite map $\mathbf{M}_\infty^{\mathsf{q}}$.

Sketch of proof

- M_k^{q} = map with law $\mathbb{P}_{\mathsf{q}}$ conditioned to have perimeter $2k$.



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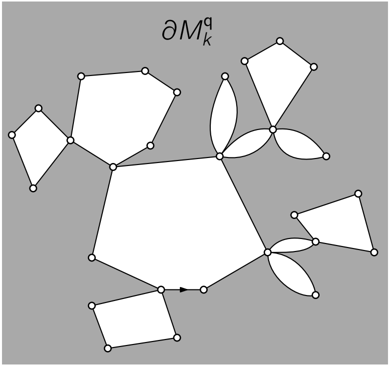

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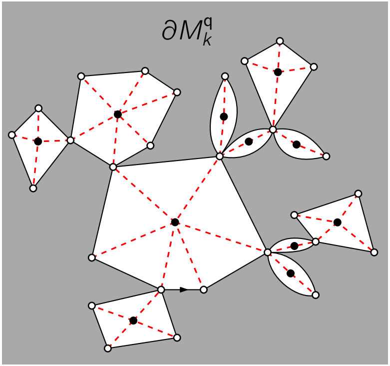

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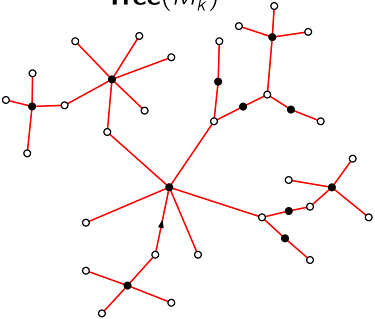
Tree(M_k^{q})



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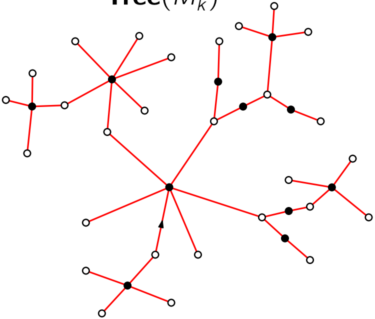

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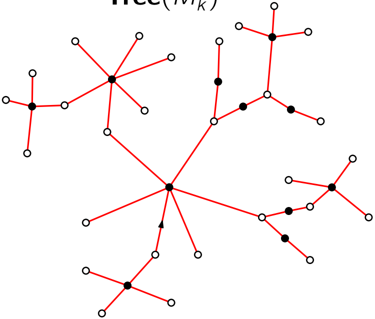

Two-type Galton-Watson tree

- $\nu_\circ$, $\nu_\bullet$ probability measures on $\mathbb{Z}_{\geq 0}$.

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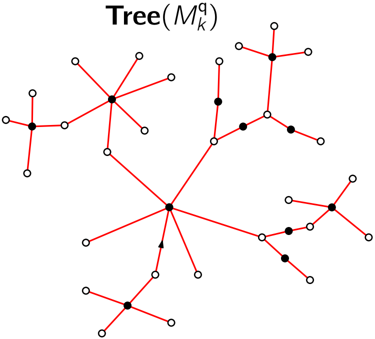

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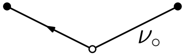

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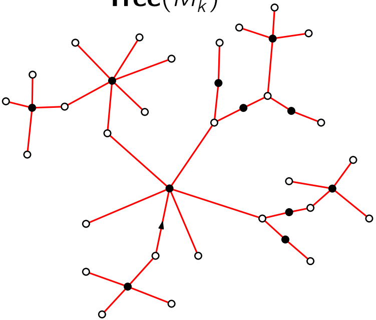

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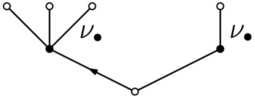

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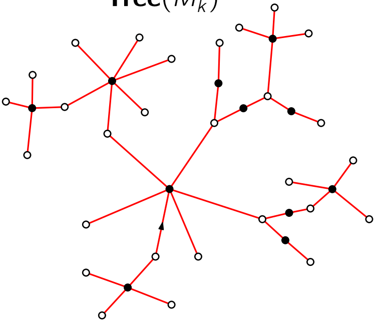

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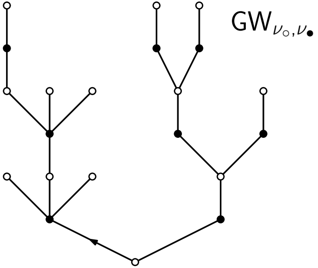

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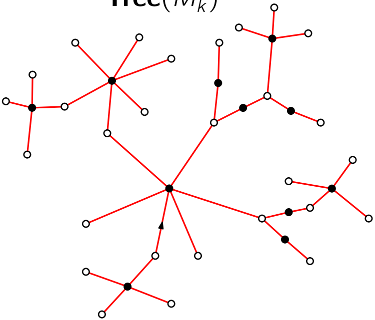

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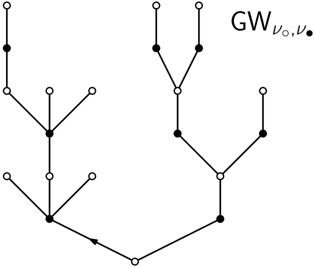

critical tree $\iff m_{\nu_\circ} m_{\nu_\bullet} = 1$.

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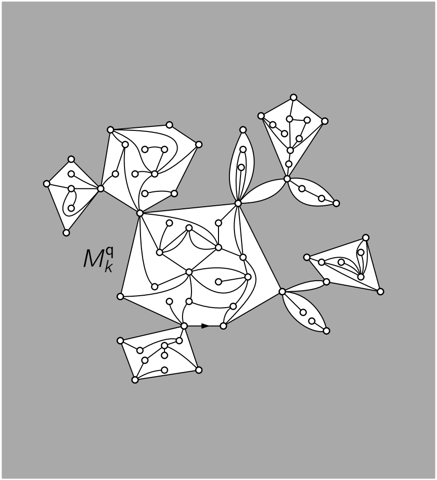

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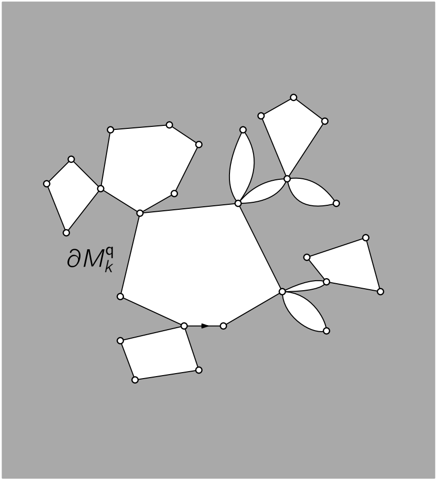

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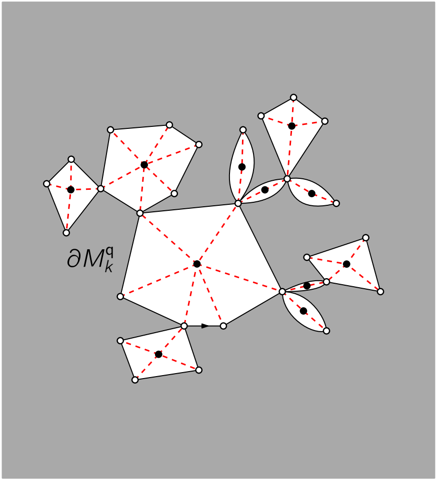

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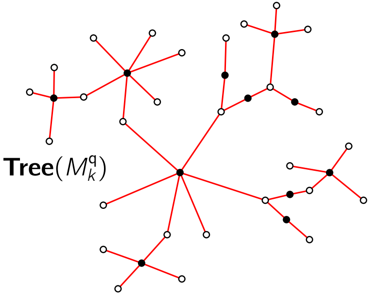

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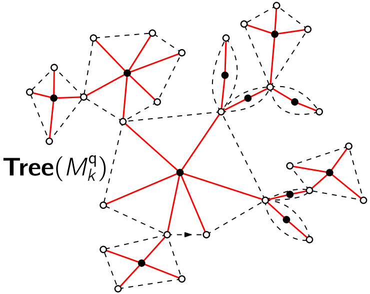

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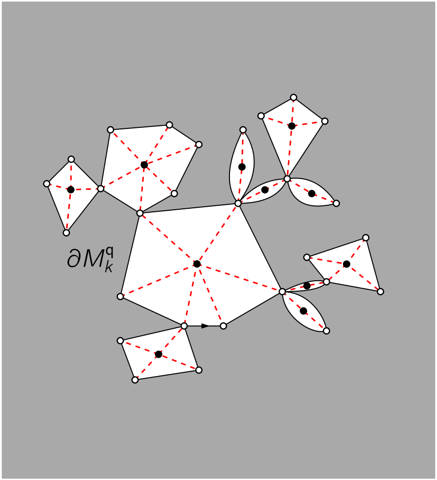

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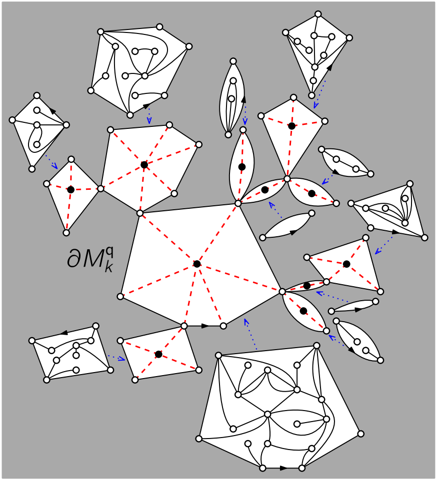

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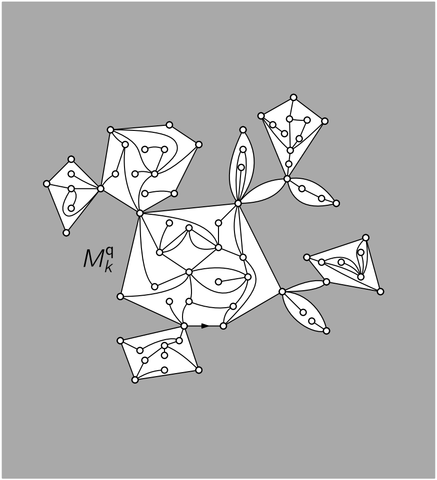

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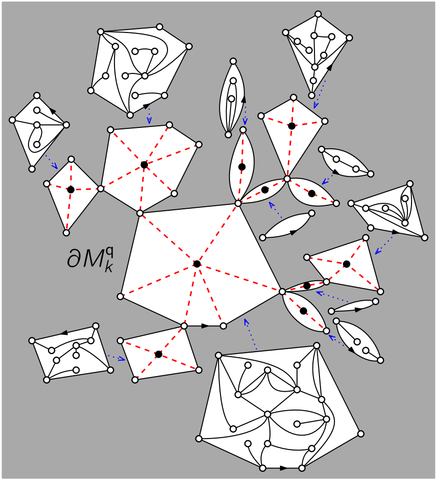

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Notation:

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$$\widehat{F}_j := \sum_{\widehat{\mathbf{m}} \in \widehat{\mathcal{M}}_j} w_{\mathrm{q}}(\widehat{\mathbf{m}}).$$

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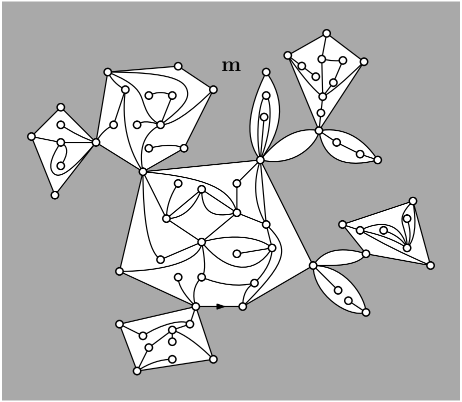

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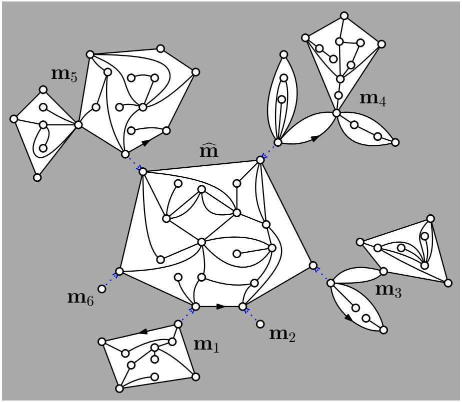

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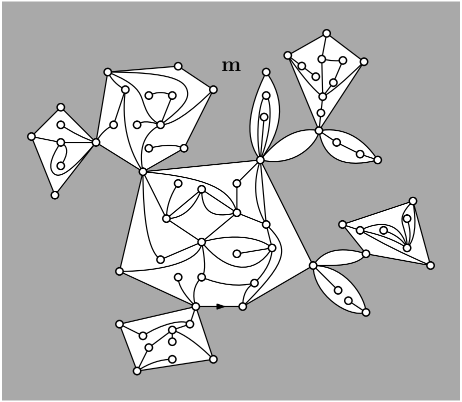

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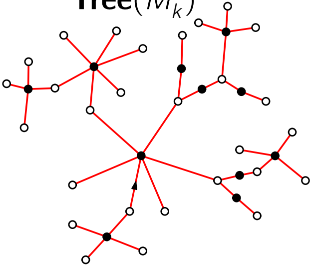
Lemma (Brézin, Itzykson, Parisi & Zuber '78)

$$F(x) = \widehat{F}(xF^2(x)).$$

Sketch of proof - branching structure

- **Tree**(M_k^{q}) = tree of law $\mathrm{GW}_{\nu_\circ,\nu_\bullet}$ having $2k+1$ vertices.

Tree(M_k^{q})



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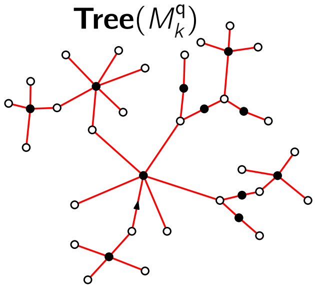
$\alpha \in (1, 3/2) \implies m_{\nu_\circ} m_{\nu_\bullet} = 1$ *and* $\nu_\bullet([k,\infty)) \sim C \cdot k^{-\beta}$.



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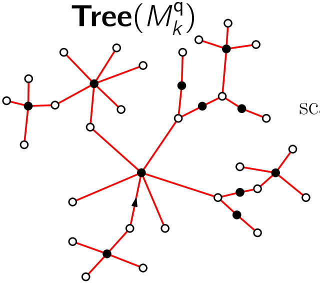
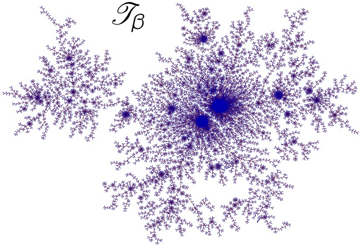


$$\xrightarrow[k \to \infty]{\text{scaling} \cdot k^{-(1-1/\beta)}}$$



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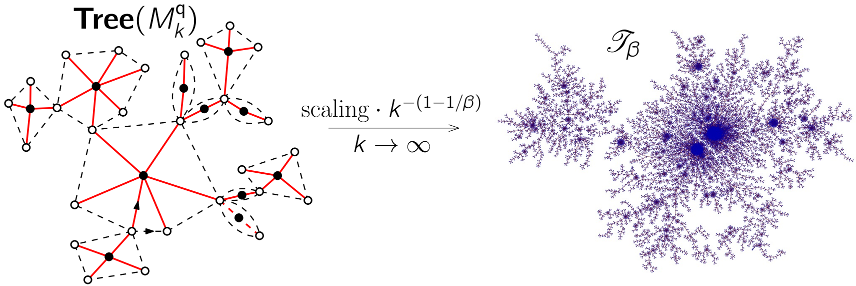

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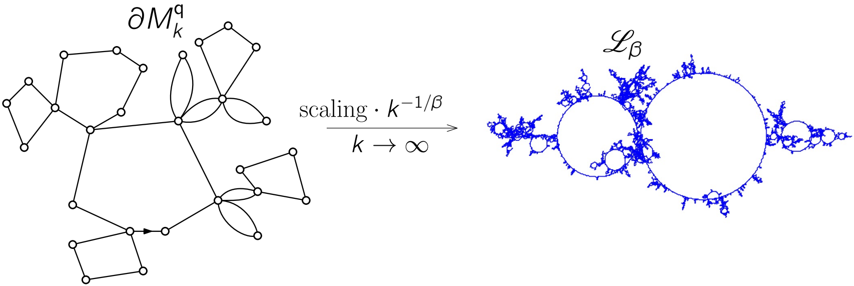

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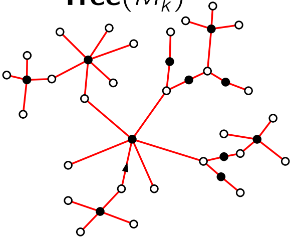

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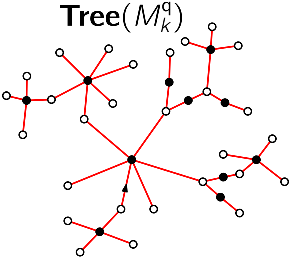

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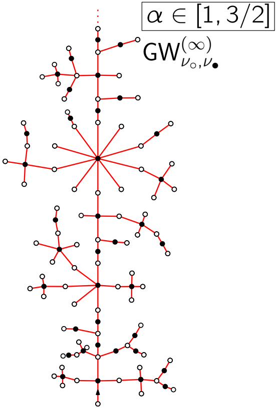

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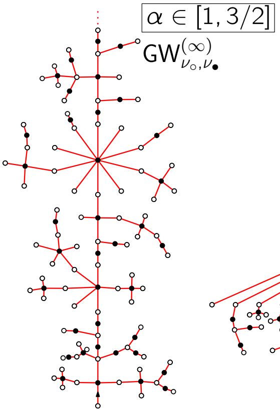

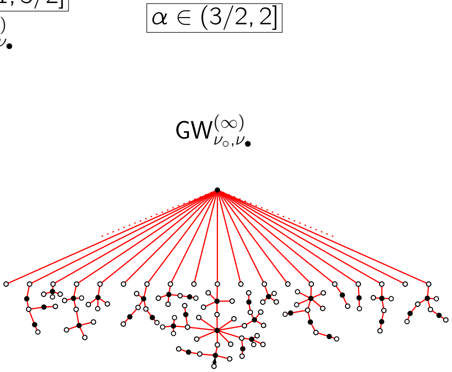

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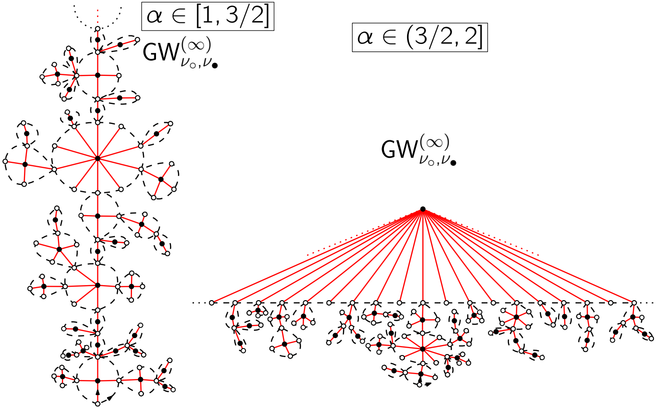

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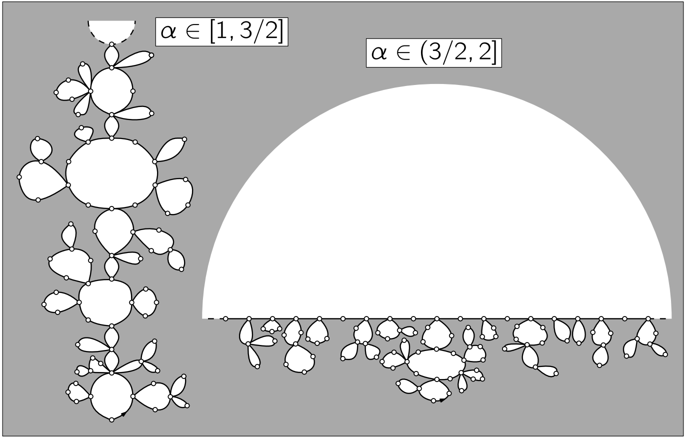

Sketch of proof - phase transition

- **Tree**(M_k^{q}) = tree of law $\mathrm{GW}_{\nu_\circ,\nu_\bullet}$ having $2k+1$ vertices.

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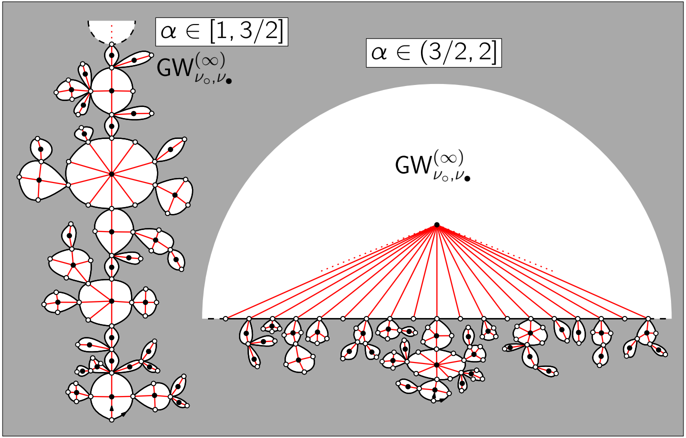
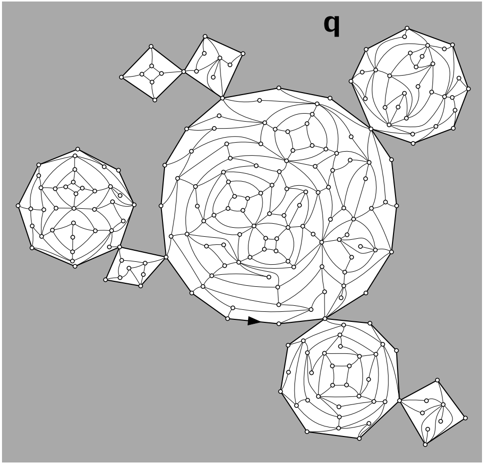


Applications to the rigid $O(n)$ loop model on quadrangulations

Loop-decorated quadrangulations

- $\mathbf{q}$ = quadrangulation with a boundary.



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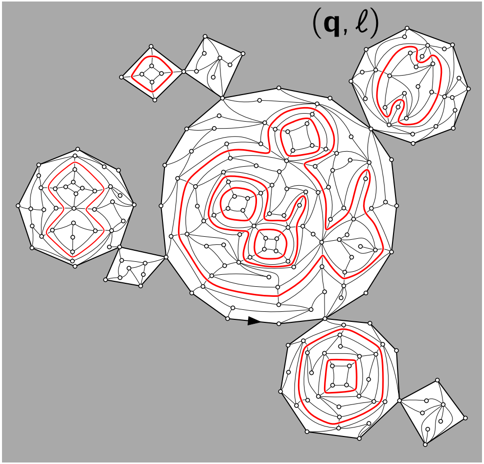
- $\boldsymbol{\ell}$ = rigid loop configuration on $\mathbf{q}$.



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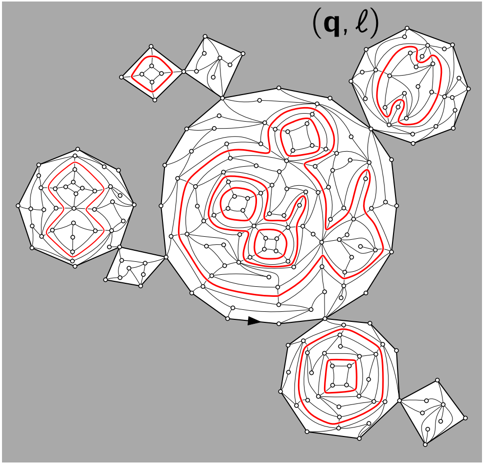

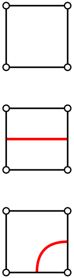

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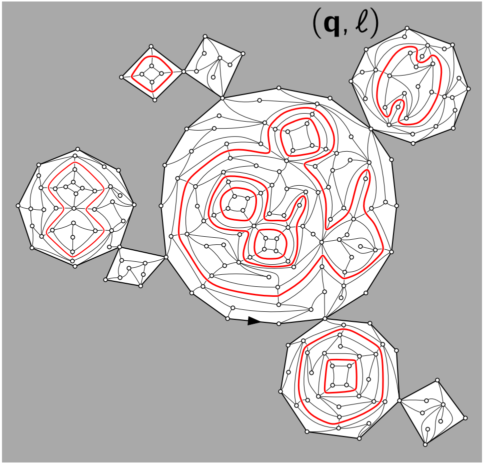
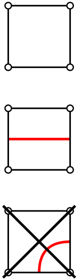
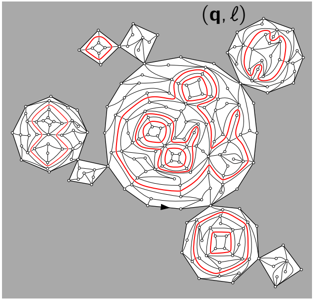




Rigid $O(n)$ loop model on quadrangulations

- $(\mathbf{q}, \ell) \in \mathcal{O}_k$ = loop-decorated quadrangulation **with perimeter** $2k$.



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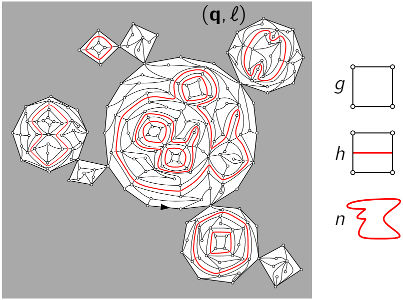
- $(\mathbf{q}, \boldsymbol{\ell}) \in \mathcal{O}_k$ = loop-decorated quadrangulation with perimeter $2k$.
- Weight of $(\mathbf{q}, \boldsymbol{\ell})$: $W_{(n;g,h)}(\mathbf{q}, \boldsymbol{\ell}) = g^{\#\square}\ h^{\#\boxminus}\ n^{\#}$. $(n \in (0,2))$



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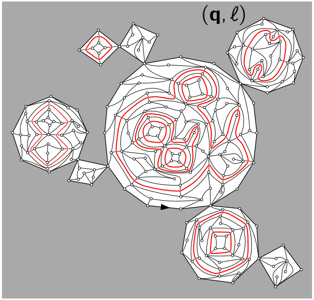
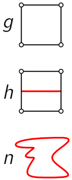
Definition

$(n; g, h)$ **admissible** $\iff G_k := \sum_{(\mathbf{q},\boldsymbol{\ell}) \in \mathcal{O}_k} W_{(n;g,h)}(\mathbf{q}, \boldsymbol{\ell}) < \infty.$





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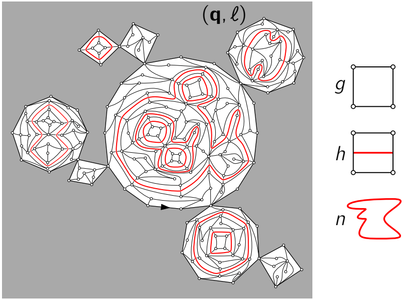
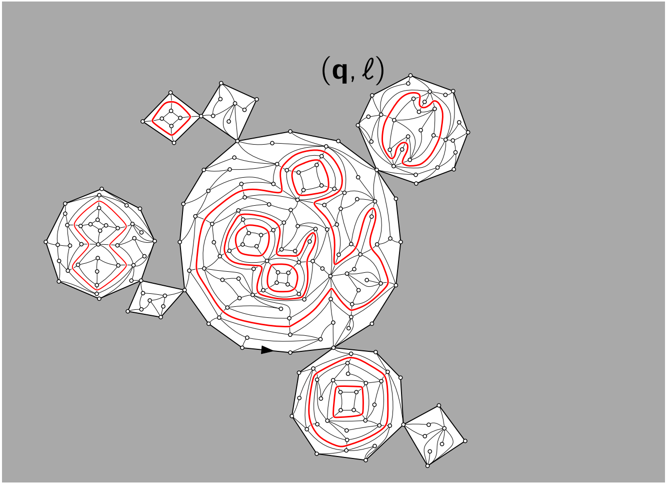
$O(n)$ measure on the set $\mathcal{O}_k$: $\mathbf{P}^{(k)}_{(n;g,h)}(\mathbf{q}, \ell) := \dfrac{W_{(n;g,h)}(\mathbf{q}, \ell)}{G_k}.$



The gasket decomposition

- $(\mathbf{q}, \ell) =$ (rigid) loop-decorated quadrangulation.



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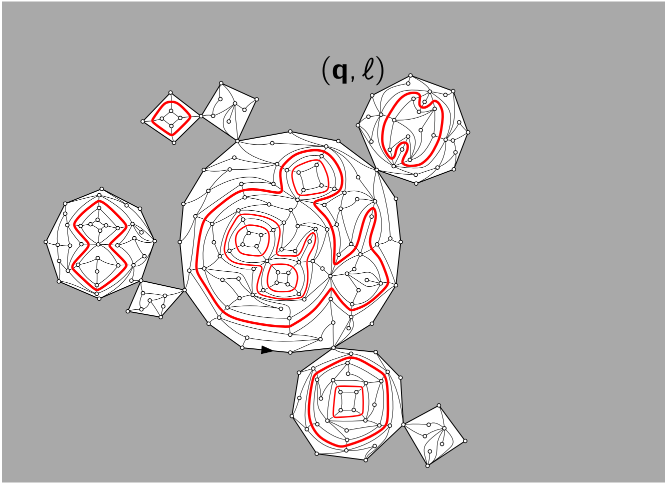

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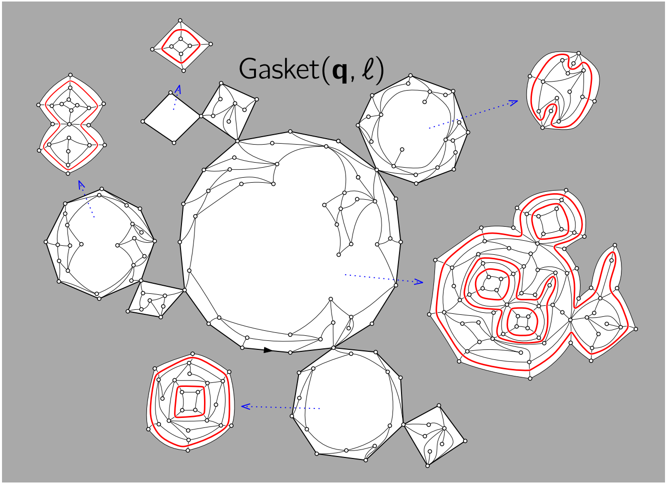

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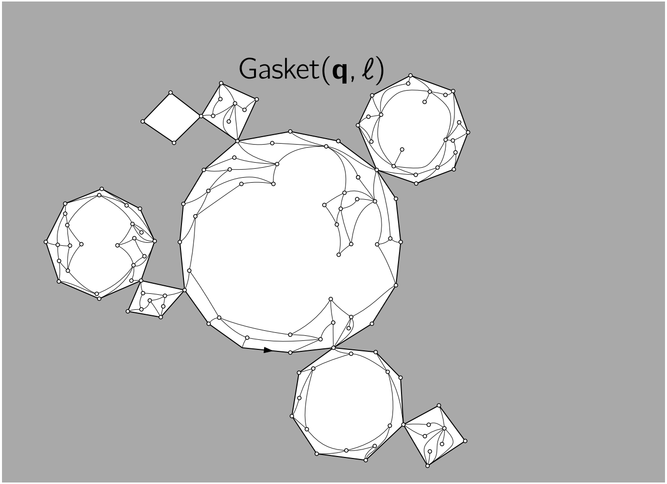

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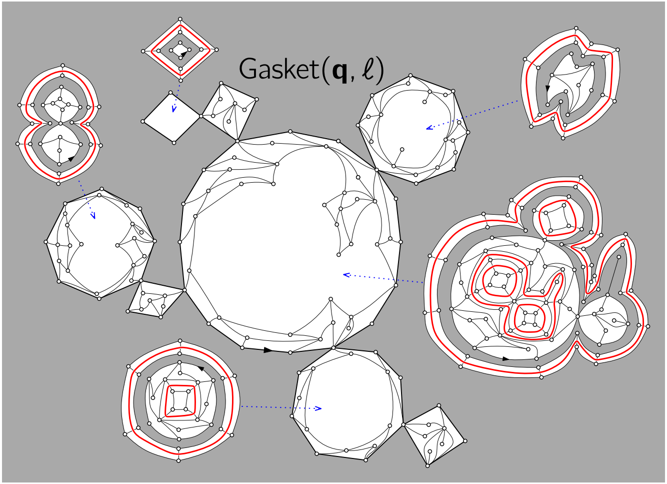

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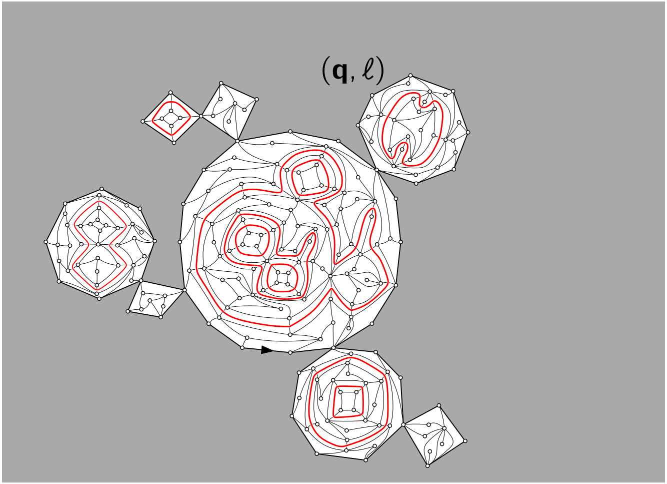

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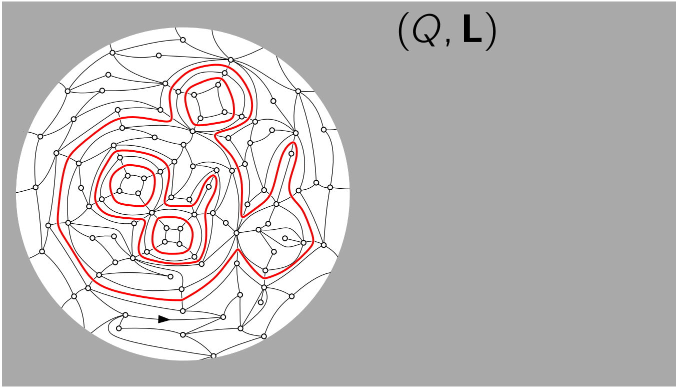

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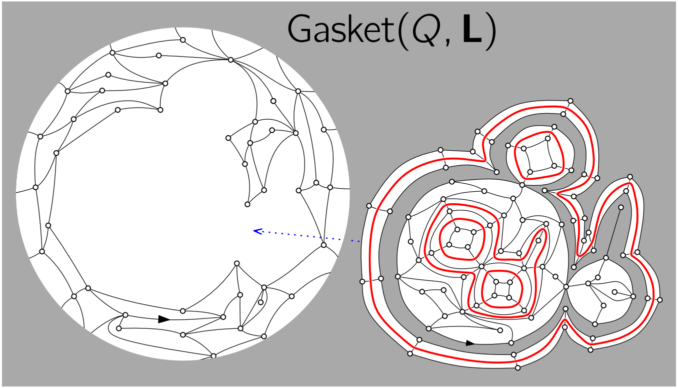

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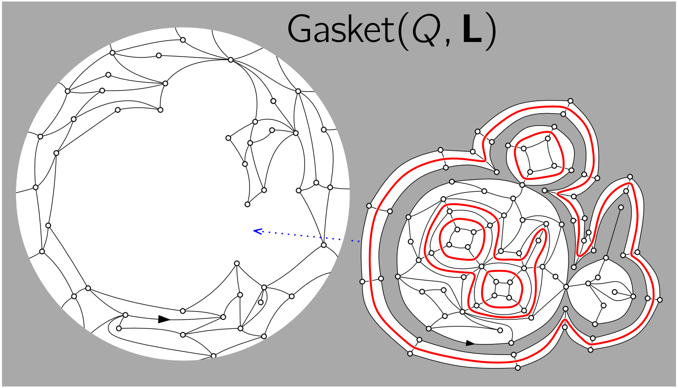

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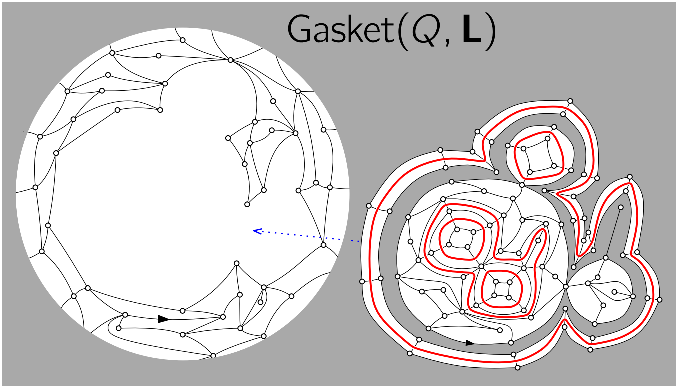
$$\text{Then, } L_k \overset{(d)}{=} \partial Q_k \ \left(\text{where } (Q_k, \mathbf{L}_k) \text{ has law } \mathbf{P}^{(k)}_{(n;g,h)}\right).$$



Phase diagram

- $(Q_k, \mathbf{L}_k)$ = loop-decorated quadrangulation with law $\mathbf{P}^{(k)}_{(n;g,h)}$.

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Theorem (Borot, Bouttier & Guitter '12, Budd & Chen)

Gasket$(Q_k, \mathbf{L}_k)$ *is a Boltzmann map with law* $\mathbb{P}_{\mathsf{q}}$ *conditioned to have perimeter* $2k$.

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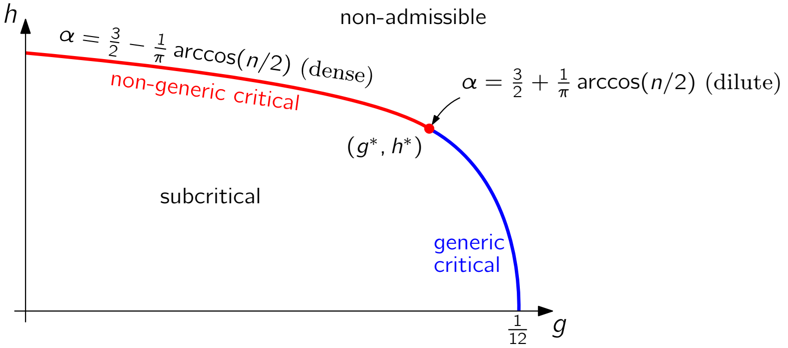

Scaling limits of large loops

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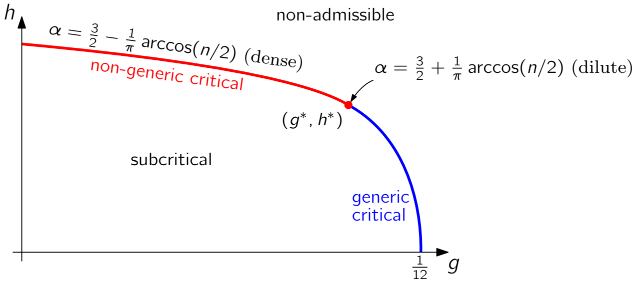

Theorem 6

Let $n \in (0,2)$ and (g,h) in the dense phase.

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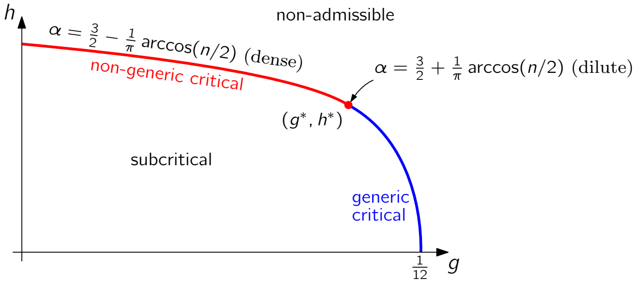


Theorem 6

Let $n \in (0,2)$ and (g,h) in the dense phase.
In the Gromov-Hausdorff sense,

$$\frac{C}{k^{1/\beta}} \cdot \partial Q_k \xrightarrow[k\to\infty]{(d)} \mathscr{L}_\beta, \quad \textit{where} \;\; \beta := \left(1 - \frac{1}{\pi}\arccos\left(\frac{n}{2}\right)\right)^{-1}.$$

Thank you!

Connection with other models

- $\mathcal{C}_k$ = critical site percolation cluster of perimeter $2k$ in the **UIPT**.

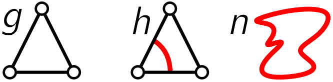
Theorem (Curien & Kortchemski '14)

In the Gromov-Hausdorff sense,

$$\frac{C}{k^{2/3}} \cdot \partial\mathcal{C}_k \xrightarrow[k\to\infty]{(d)} \mathscr{L}_{3/2}.$$

[Borot, Bouttier & Guitter '12]:

$O(n)$ loop model on triangulations



For $n = 1$, site Ising model on Boltzmann triangulations.

- $g = h \longleftrightarrow$ critical site percolation on Boltzmann triangulations.
  $\hookrightarrow$ dense phase: $\alpha = 7/6$ and $\beta = 3/2$.
- $(g^*, h^*) \longleftrightarrow$ phase transition of the Ising model.
  $\hookrightarrow$ dilute phase: $\alpha = 11/6$.