Universality for fluctuation of the dimer model on Riemann surfaces

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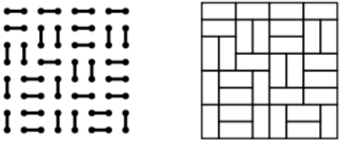
Talk plan

- First half: Overview of the project.
- Second half: Riemann surface and dimer.

The dimer model

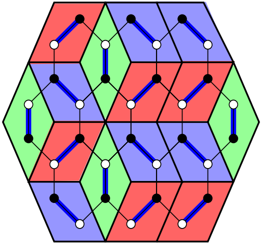
Definition

G = bipartite finite graph, planar
Dimer configuration = perfect matching on G:
each vertex incident to one edge
Dimer model: uniformly chosen configuration



On square lattice, equivalent to domino tiling.

Dimer model = random surface



- Describes a surface in $\mathbb{R}^3$.
- Boundary describes a curve in $\mathbb{R}^3$.

We pick a dimer configuration **uniformly at random** and ask questions about it's geometry.

- What is a typical surface? (Law of large numbers)
- What is the fluctuation around the typical surface?

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Parameters..

We can play with the model...

- Change the underlying graph.
- Change boundary conditions.
- Change the underlying surface (e.g. Embed the graph in torus, annulus, 2-torus etc...)

In this talk:

Our project

We concentrate on fluctuations and prove its universal behaviour in various scenarios..

Exact solvability

Let $e_i = (w_i, b_i)$.

Kasteleyn matrix K: Adjacency matrix of graph with special signs on the entries.

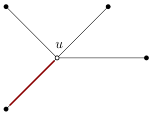
$$\text{Partition function} = \left|\det(K)\right|$$

$$P(e_1, e_2, \ldots, e_k \text{ present }) = \prod_{i=1}^{k} K(w_i, b_i) \det((K^{-1}(w_i, b_j))_{1 \leq i,j \leq k})$$

Our approach

We will **NOT** use Kasteleyn matrices in our analysis (contrary to the traditional approach.)

Height function



- Reference flow:

$$\omega_0 : \textbf{oriented edges} \mapsto \mathbb{R}, \quad \omega_0(uv) = -\omega_0(vu); \ \sum_{v \sim u} \omega(uv) = 1.$$

- Dimer flow:

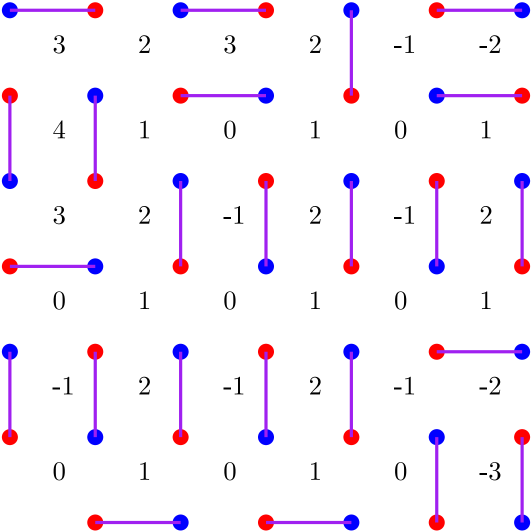
$$\omega_{\text{dim}}(uv) = 1_{\{\text{dimer in } uv, \text{ u is white}\}}; \quad \omega_0(uv) = -\omega_0(vu)$$

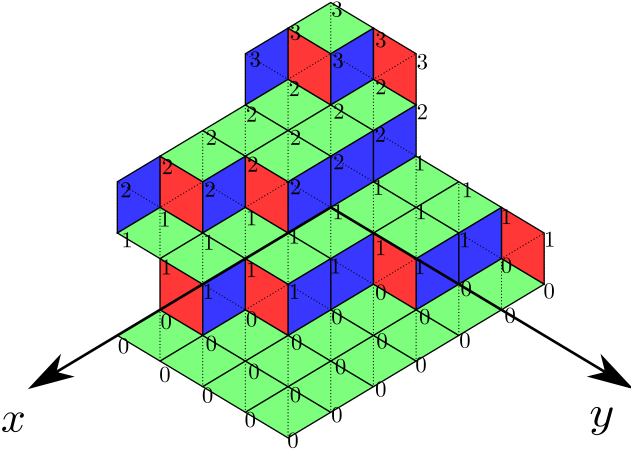
- $\omega - \omega_0$ is **divergence free:**

$$\sum_{v \sim u} (\omega - \omega_0)(uv) = 0$$

This defines a function (modulo $\mathbb{Z}$) on the faces. This is the **height function**.

Example: square lattice




x
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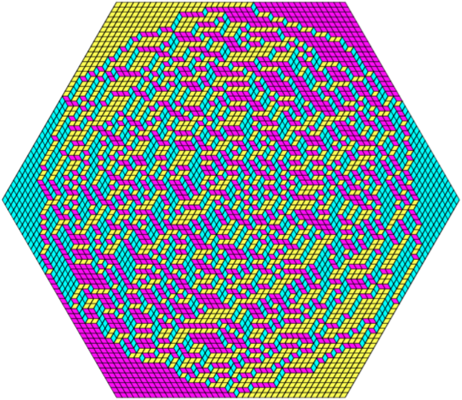
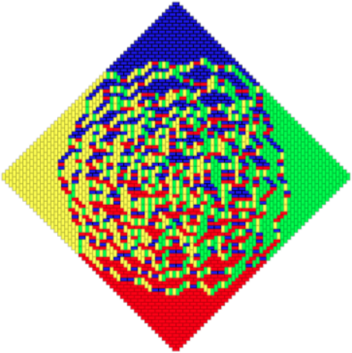
Boundary effect

Some regions can be frozen, other liquid (temperate)
Depends on boundary conditions in sensitive way
Interface between frozen / liquid = arctic circle



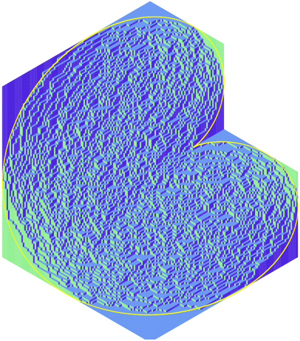
Boundary effect

Aztec diamond:



Jockusch, Propp and Shor 1996

Cardioid:



Kenyon–Okounkov–Sheffield 2006

Phases

- **Frozen face**: Fluctuations are a.s. bounded
- **Liquid phase**: Logarithmic correlations.
- **Gaseous phase**: Exponentially decaying correlation.

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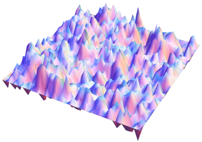
Philosophy for universality

In the liquid phase, fluctuations behave like the **Gaussian free field**.



$\int hf \sim \mathcal{N}(0, \int \nabla f \cdot \nabla f)$, $Cov(\int hf, \int hg) = \int \nabla f \cdot \nabla g$

Law of large number

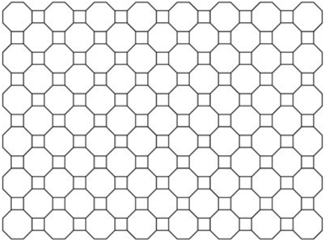
(Cohn, Kenyon, Larsen, Propp.)

The mean surface can be computed as a solution of a variational problem given the boundary conditions.

Goal: Formulate and prove universality of fluctuations under most natural conditions of

- The graph.
- The surface on which it is embedded.
- Boundary conditions.

Effect of graph can be extreme....



The square octagon lattice is in gaseous phase!

Dimers and amoeba

Kenyon, Okounkov and Sheffield proved a beautiful theorem using algebraic geometry to understand the phase diagram of the dimer model.

Effect of the surface

Height function h is no longer a function but a one form:

Hodge decomposition

We can decompose

$$h = \nabla f + \mathfrak{h}$$

where f is a function (called scalar part) and $\mathfrak{h}$ is a harmonic one form called the **instanton component**.

Past work

Except for nice graphs (like domino tiling, lozenge tiling, isoradial graphs) on the torus (works of Kenyon, Dubedat et al.), this is fairly uncharted territory.

Formulas known

For graphs embedded on genus g , there are some formulas for dimer correlations (due to Cimasoni), but its not clear how to use them.

Our approach

G: Planar bipartitle embedded graph.

Main tool

There is a mapping Φ (depends on boundary condition) such that

$$\text{height function in } G = \text{ winding of uniform spanning tree in } \Phi(G)$$

Theorem (Berestycki, Laslier, R. ;16)

If random walk converges to Brownian motion on $\Phi(G)$ (plus some mild assumptions) then fluctuation of winding of the branches of spanning tree converges to Gaussian free field.

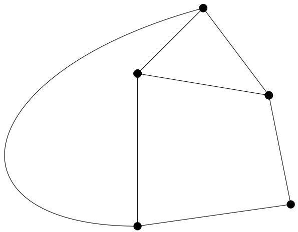
Therefore height function $\to \Phi^{-1}(GFF)$.

What can we prove?

- GFF fluctuations for Temperleyan graphs satisfying CLT(https://arxiv.org/abs/1603.09740). Previously known for isoradial graphs (Kenyon, Li, Dubedat.)
- GFF fluctuation for hexagonal lattice with planar boundary condition of any slope (https://arxiv.org/abs/1603.09740)
- Convergence of height forms for Temperleyan graphs on Riemann surfaces satisfying CLT (**this talk!**)
- GFF fluctuation for non-flat boundary condition on the hexagonal lattice (ongoing work.)
- ......

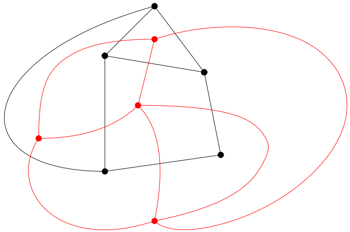
Temperleyan graph (Temperley; Kenyon, Propp, Wilson)

- A graph G.



Temperleyan graph (Temperley; Kenyon, Propp, Wilson)

- A graph G.
- Its dual.



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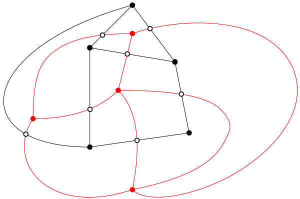

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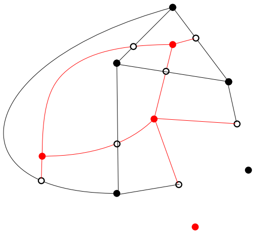

Temperleyan graph (Temperley; Kenyon, Propp, Wilson)

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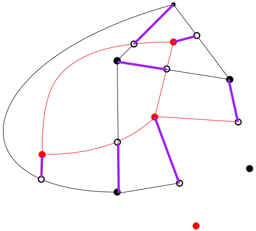
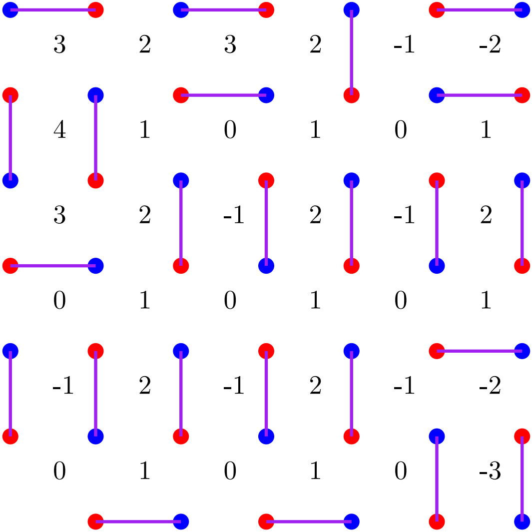
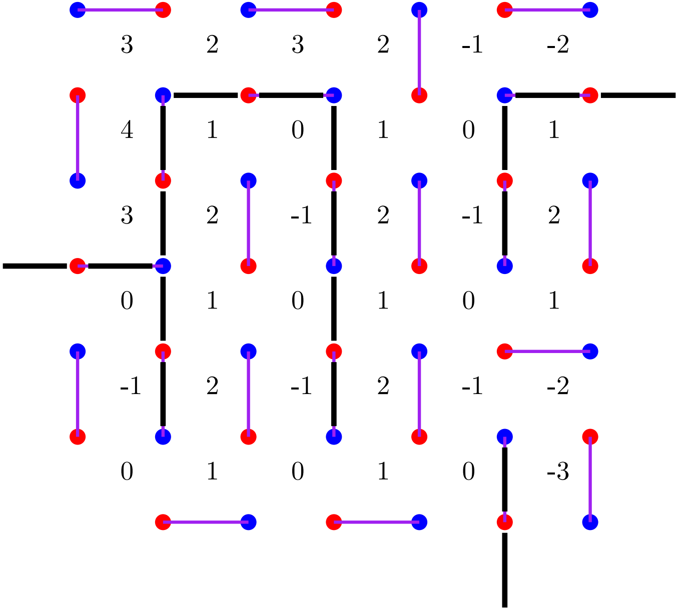
- Dimers




3 2 3 2 -1 -2
4 1 0 1 0 1
3 2 -1 2 -1 2
0 1 0 1 0 1
-1 2 -1 2 -1 -2
0 1 0 1 0 -3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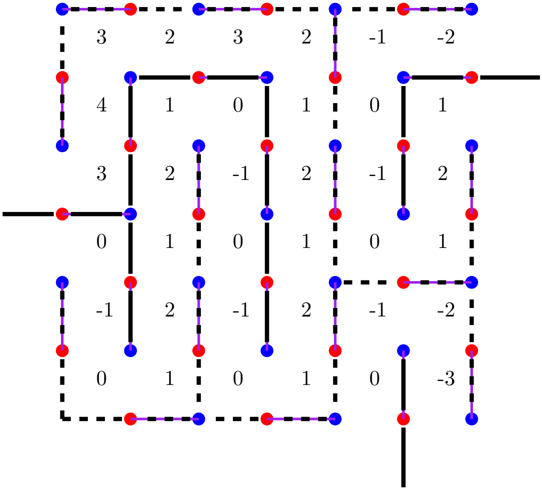
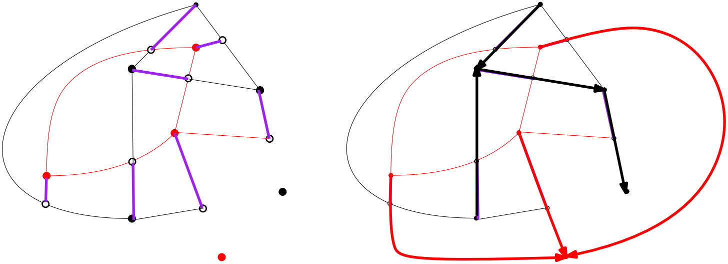


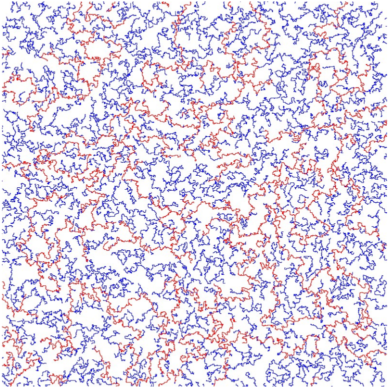
Temperleyan bijection, generalization due Kenyon, Propp, Wilson

Dimer on Temperleyan graph = spanning tree, dual tree pair.



Theorem (Lawler,Schramm, Werner; Yadin Yehudayoff)

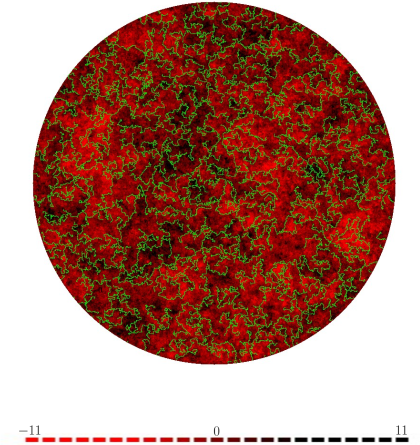
Uniform spanning trees converge if Random walk $\to$ Brownian motion.



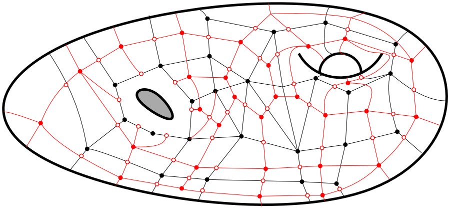
Warning!

Winding is not a continuous function of a curve with Hausdorff topology.

Winding of UST $\to$ GFF



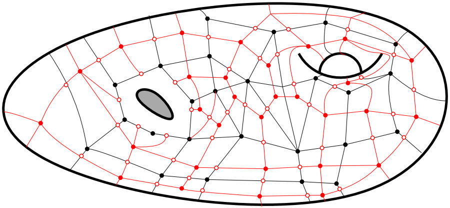
Temperleyan graphs



$$V - E + F = 2 - 2g - b$$

g : genus ; b: boundary components

Temperleyan graphs



$$V - E + F = 2 - 2g - b$$

g : genus ; b: boundary components

In superimposed graph, dimer configuration exists only if $V - E + F = 0$

Temperleyan graphs on Riemann surface

If not simply connected,

$$V - E + F = 2 - 2g - b \leq 0$$

Therefore we need to remove $2g + b - 2$ white vertices from superimposed graph. We call it a **Temperleyan graph**.

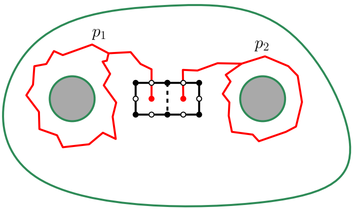
Temperleyan graphs on Riemann surface

If not simply connected,

$$V - E + F = 2 - 2g - b \leq 0$$

Therefore we need to remove $2g + b - 2$ white vertices from superimposed graph. We call it a **Temperleyan graph**. Note $2 - 2g - b = 0$ for torus and annulus.

Temperleyan graph on a surface



Lemma

Dimer configuration exists on a Temperleyan graph.

Proof.

Using pair of pants decomposition, enough to prove for annulus and pair of pants. For pair of pants, we decompose into annuli as above. For annulus, its easy.

□

Theorem (Berestycki, Laslier, R. '17)

Let M be a Riemann surface with finitely many holes and handles. Let G^δ be a Temperleyan graph embedded on M **so that CLT holds** *as $\delta \to 0$ and (*) holds. If M is* **torus or annulus**, *then both scalar field and instanton component converges, is universal and conformally invariant.*

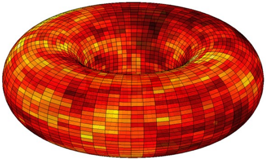
If (****) holds, same is true in general.

Using the result of Dubedat for square lattice on flat torus,

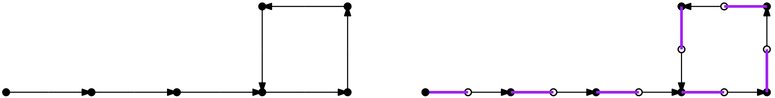
Corollary

If M = flat torus, height 1-form converges to a **compactified GFF** *(GFF on torus + Gaussian type instanton component).*

Answers a question of Dubedát and Gheissari.



Extending Temperleyan bijection



Observation

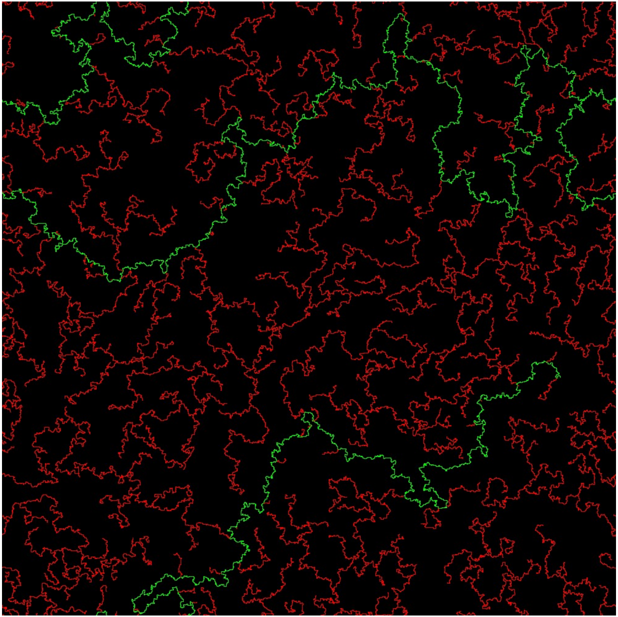
Can only find such an orientation iff every component has at most a single cycle.

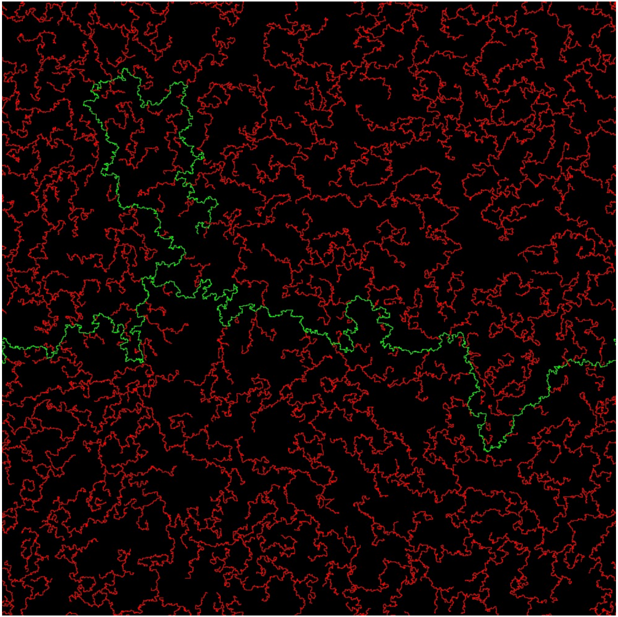
Oriented cycle rooted spanning forest

Definition: CRSF
Oriented (wired) Cycle Rooted Spanning Forest

Oriented subgraph T of G:
– $\forall v \notin \partial G$, unique outgoing edge (no outgoing edge for $v \in \partial G$).
– Every cycle is non-contractible.

A given (nonboundary) component has unique cycle; branches flowing toward it.





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On general surfaces, Temperleyan CRSF's are singular with respect to CRSF's near the special points (where a white vertex is removed). (****) condition is to prove convergence near these special points.

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Related to the work of Kassel-Kenyon.

Work in progress

Convergence of special branches.

Thanks for listening!