From peeling to random walks, an exploration of a critical percolation cluster on the infinite planar map

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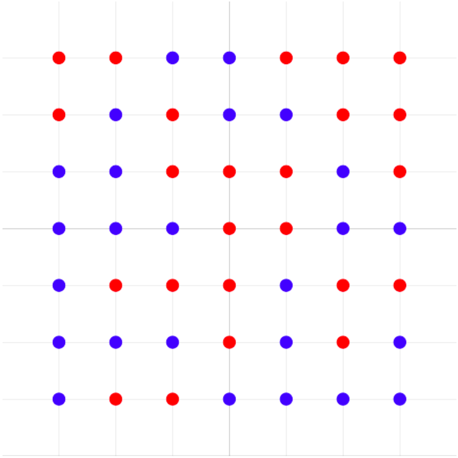
Percolation on a $\mathbb{Z}^2$

Bernoulli percolation on sites : each vertex is independently in Red with proba p and in Blue with proba $1-p$.



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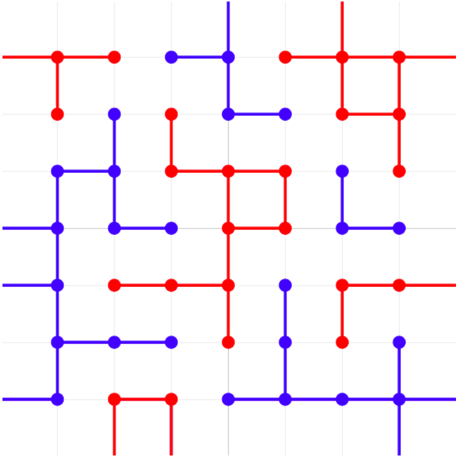


Cluster : connected components. Finite or not ?

Percolation on a $\mathbb{Z}^2$

- Phase transition :
  1. $p \leqslant \frac{1}{2}$: all Red clusters are finite
  2. $p > \frac{1}{2}$: there is a unique infinite Red cluster
- At the critical point $p = p_c = \frac{1}{2}$. Only finite Red and Blue clusters.
- Geometry of a finite cluster : $\mathcal{C}$ the cluster of 0. $|\mathcal{C}|$ its number of vertices.

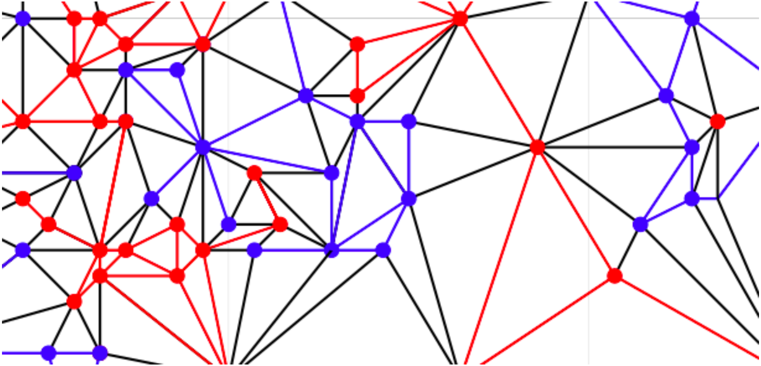
  If $p \neq p_c$, given that $\mathcal{C}$ is finite,

  $$\mathbb{P}(|\mathcal{C}| > n) \approx e^{-cn}$$

  At $p = p_c$,

  $$\mathbb{P}(|\mathcal{C}| > n) \approx n^{-\gamma}.$$

Percolation in random planar geometry

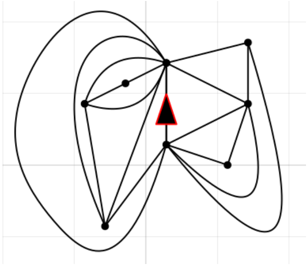


Construction of a random geometry

Uniform Infinite Planar Triangulation (UIPT) : (Schramm Angel 2001)

We begin with a finite triangulation.

$\mathcal{T}_n$: the set of finite planar rooted triangulations with n vertices (up to homeomorphism). Example with $n = 8$ vertices:



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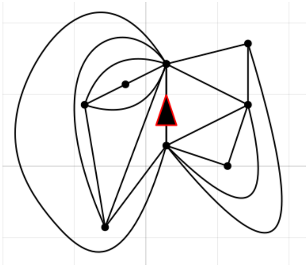

$\mathcal{T}_n$ is finite and can be enumerated (work of Tutte in the 60's).

Number of planar triangulation with n vertices and a border of size m :

$$\frac{2^{n+1}(2m+1)!(2m+3n)!}{(m!)^2 n!(2m+2n+2)!}.$$

Construction of the UIPT: n goes to ∞

Local topology.

- If (G, ρ) is a rooted graph with root ρ, we define $B_G(R)$ the ball centered in ρ of radius R (for the graph distance) : the sub-graph of vertices at distance $\leqslant R$ of ρ.

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- For $R > 0$ fixed, if t_n is a planar triangulation chosen uniformly in $\mathcal{T}_n$ then when n goes to infinity the law of $B_{t_n}(R)$ has a limit in law and these laws are compatible for different values of R.

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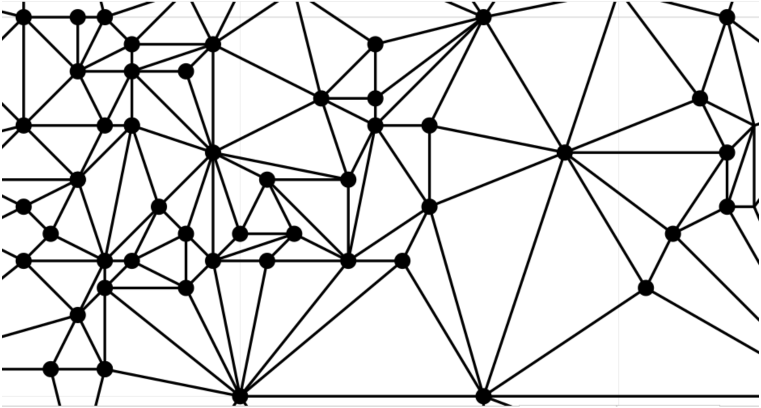
- Angel Schramm 01 : There exists t_∞ an inifinite planar rooted triangulation which is the "local" limit of the t_n :

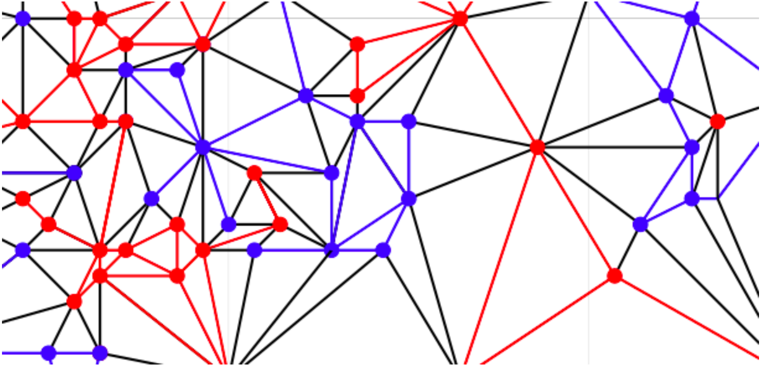
$$t_n \overset{n \to +\infty}{\longrightarrow} t_\infty$$

(i.e. for all $R > 0$, $B_{t_n}(R) \xrightarrow{\mathcal{L}} B_{t_\infty}(R)$.)

UIPT : Uniform Infinite Planar Triangulation

A "natural" model of random planar geometry.





Percolation on the UIPT

Bernoulli percolation on sites : each vertex is independently in Red with proba p and in Blue with proba $1-p$.

Cluster : connected components. Finite or not ?

Theorem (Angel 03) Similar to the euclidian case : Phase transition :

1. $p \leqslant \frac{1}{2}$: all Red clusters are finite
2. $p > \frac{1}{2}$: there is a unique infinite Red cluster

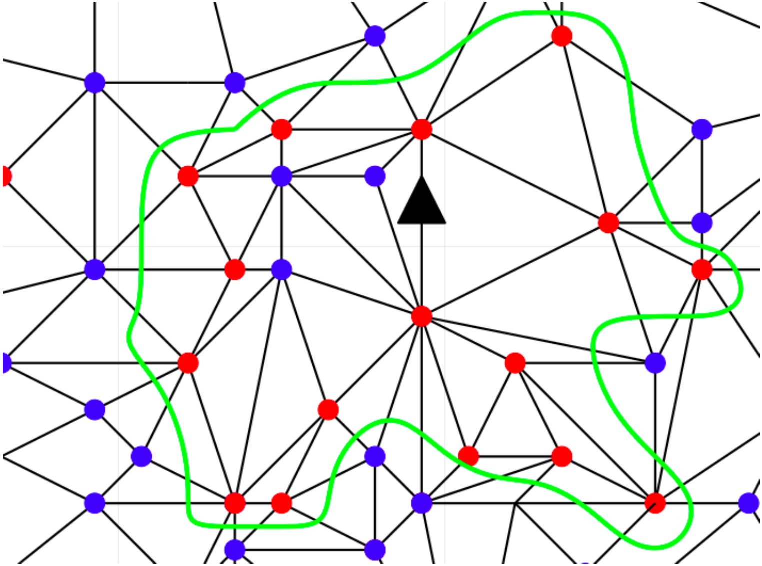
At the critical point $p = p_c = \frac{1}{2}$. Only finite Red and Blue clusters.

$\mathcal{C}$ the cluster of 0. What is the geometry of $\mathcal{C}$?

A Hull $\mathcal{H}$ with $|\mathcal{H}| = 19$ and $|\partial\mathcal{H}| = 44$

Volume $= |\mathcal{H}| =$ number of vertices

Perimeter $= |\partial\mathcal{H}| =$ length of the green interface



Results

Theorem (Angel Curien 2015) : In the UIHPT (closely related model in the half-plane instead of the plane), at the critical parameter:

$$\mathbb{P}(|\mathcal{H}| > n) = \frac{1}{n^{1/4+o(1)}} \text{ and } \mathbb{P}(|\partial\mathcal{H}|) \asymp \frac{1}{n^{1/3}}.$$

($u_n \asymp v_n$ means that for two constants $0 < c, C < \infty$ $cu_n \leqslant v_n \leqslant Cu_n$ for sufficiently large n)

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$$\mathbb{P}(|\mathcal{H}| > n) \asymp \frac{1}{n^{1/8}} \text{ and } \mathbb{P}(|\partial\mathcal{H}|) \asymp \frac{1}{n^{1/6}}.$$

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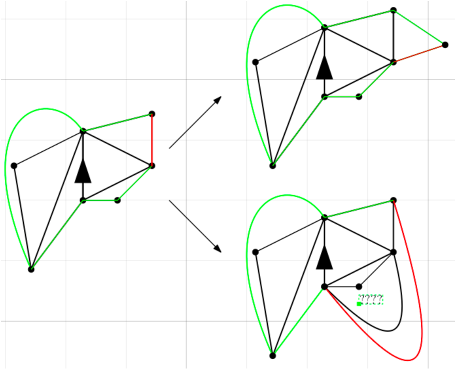
Conjecture : The number of Red vertices connected to the origin ($|\mathcal{C}| \leqslant |\mathcal{H}|$) has a tail distribution in $n^{-\frac{1}{7}}$.

Exploration of the UIPT : "peeling" (Watabiki 95 Angel 03)

We forget about the percolation part for a while. Discovery step by step of the triangulation.

S_n = size of the border of the discovered region after n steps. The border is in green with a distinguished red edge.

One step : discover the triangle on the other side of the red edge (here $S_n = 6$) :



Evolution of the border

- S_n is a Markov chain ! What we discover only depend on the size of the border, not on the particular geometry of what is inside.
- The law of S_n does not depend on the strategy chosen to discover new triangles (the choice of the red edge)
- 2 possibilities when $S_n = m$
  - $S_{n+1} = m + 1$ if we discover a new vertex. This happens with probability $p_1 h(m+1)/h(m)$.
  - $S_{n+1} = m - k$ with $k \geqslant 1$ when the new triangle is attached to the border at distance k by its third vertex. This happens with proba $p_{-k} h(m-k)/h(m)$.

  with explicit $p = (p_k)_{k \leqslant 1}$ ($p_{-k} \sim ck^{-5/2}$) and h is a p-harmonic null on $]-\infty; 1]$. (Everything is explicit and can be computed by taking limits in the combinatorial formulas of Tutte).

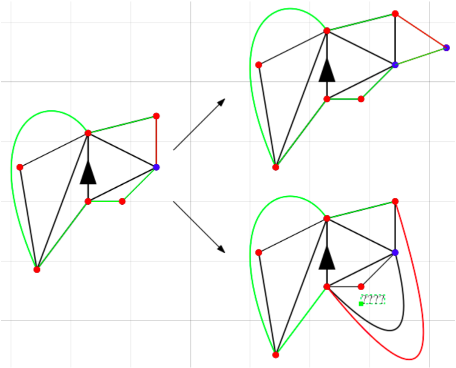
Consequences

- $$\mathbb{P}(S_{n+1} = m + k | S_n = m) = p_{-k} \frac{h(m+k)}{h(m)},$$

  This is a Doob h-transform.
- Corollary : $S = (S_n)_{n \geqslant 1}$ has the same law than a random walk on $\mathbb{Z}$ with steps of law given by $p = (p_k)_{k \leqslant 1}$ conditioned to stay $\geqslant 2$.
- Consequences : $p_{-k} \sim ck^{-5/2}$ is in the domain of attraction of a 3/2-stable law and thus $S_n = O(n^{\frac{2}{3}})$

Peeling : now with percolation

We want to explore the Red cluster of the origin at $p = p_c = \frac{1}{2}$. It is possible to choose a strategy in such a way that on the border the Red vertices are on one side and the Blue on the other.



Now we follow the evolution by looking at the two processes. $\mathbf{R}_n$ is the number of Red vertices on the border and $\mathbf{B}_n$ the number of Blue ones.

Peeling : now with percolation

- $$(r_{n+1}, b_{n+1}) = (r_n, b_n) + (S_{n+1} - S_n)\begin{pmatrix} \eta_n \\ 1 - \eta_n \end{pmatrix}$$

  with $\eta_n = 1$ if we discover a new Red vertex or if the new triangle attach itself in the Red direction. ($\eta_n = 1$ otherwise).
- $r_n + b_n = S_n$ and luckily, since $p = p_c = \frac{1}{2}$, η_n is a sequence of i.i.d. Bernoulli $1/2$ independent of S.

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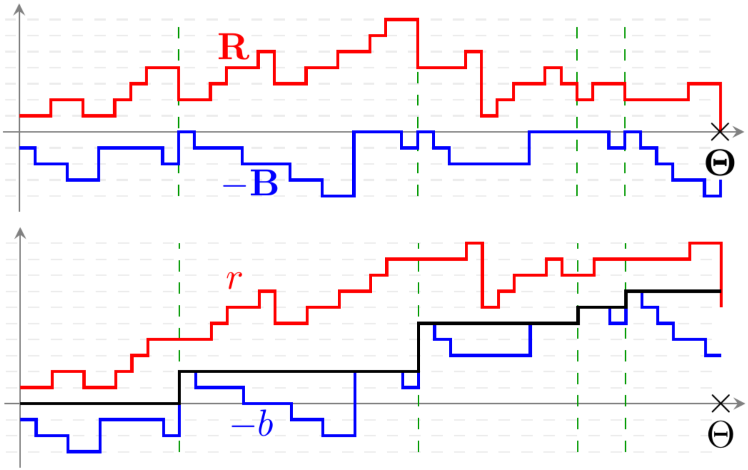
- $r_n + b_n = S_n$ and luckily, since $p = p_c = \frac{1}{2}$, η_n is a sequence of i.i.d. Bernoulli $1/2$ independent of S.
- But r_n, b_n defined this way is not the number of Red,Blue vertices on the border. Still, while we are still in the cluster of the origin :
  $$\mathbf{R}_n = r_n + \inf_{k \leqslant n} b_k$$
  $$\mathbf{B}_n = b_n - \inf_{k \leqslant n} b_k$$
  The exploration of the cluster stops if $\mathbf{R}_n \leqslant 0$ i.e. at Θ the first time n such that $r_n + \inf_{k \leqslant n} b_k \leqslant 0$.

Encoding of the peeling bytwo processes

Above : evolution of the real number of Red/Blue sites $\mathbf{R}_n, \mathbf{B}_n$.



Below : the processes r et $-b$.

Proof I

- On one hand $(\mathbf{R}_n, \mathbf{B}_n)$ the number of Red/Blue vertices on the boudary of the peeling process. A two-dimensional Markov process.
- On the other hand (r, b) a couple a random walks condioned to satisfy $r + b \geqslant 2$ (i.e. at all time $r_n + b_n \geqslant 2$). Should be simpler to study.

$$(r_{n+1}, b_{n+1}) = (r_n, b_n) + X_{n+1} \left(\begin{array}{c} \eta_n \\ 1 - \eta_n \end{array} \right)$$

  Under the law $\mathbb{P}(.|r + b \geqslant 2)$
- The condition $r + b \geqslant 2$ is the only thing which correlates r and b. But while we are still in the exploration of the Red cluster of the origin, one should expect to see more Red vertices than Blue ones and thus $r \geqslant 0$ should imply $r + b \geqslant 2$. We can study the law of (r, b) under the probability measure $\mathbb{P}(.|r \geqslant 0)$ instead of $\mathbb{P}(.|r + b \geqslant 2)$. We lose some multiplicative constants in this approximation

Proof I

- X and Y i.i.d. with law p which is $3/2$-stable:

$$r_{n+1} = r_n + X_{n+1}$$

$$b_{n+1} = b_n + Y_{n+1}$$

Under the law $\mathbb{P}(.|r \geqslant 0)$. What is the order of the time spend in $\mathcal{C}$? Key quantity:

$$\Theta = \inf\{n | r_n \leqslant \sup_{k \leqslant n} -b_k\}$$

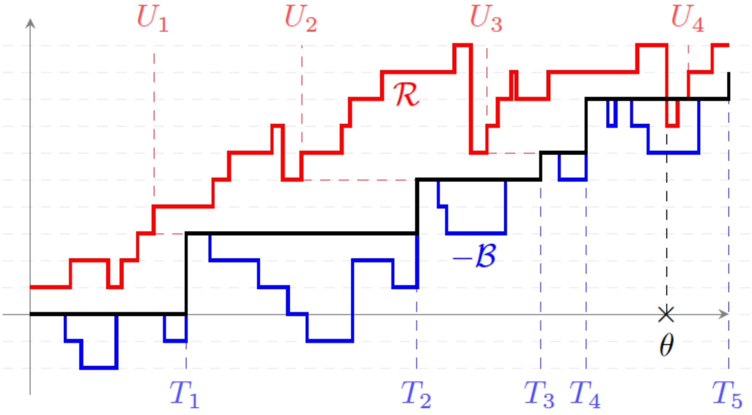
- If r visits x after $-b$ visits x then we are after Θ.

$$T_i = i\text{-th new maximum of } -b$$

$$U_i = \text{last time } r \text{ visits } -b_{T_i}$$

While $U_i < T_i$ we are still in $\mathcal{C}$.

Symmetry



Proof II

- (r, b) independent random walks with the same law under the condition $r \geqslant 0$

$$T_i = i\text{-th new maximum of } -b$$

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- U and T are two random walks (in the domain of $1/3$ stable laws) and miracle of Tanaka (which transform first passage time of a walk in last passage time of the same walk conditioned to saty positive) : they have the same law
- $(U_i - T_i)_{i \geqslant 0}$ is a symmetric random walk. While is is positive we are still exploring the cluster. Universal estimate : the probability that a symmetric random walk stays positive k steps is of order $k^{-1/2}$
- If $T_{n^{1/3}} + U_{n^{1/3}} > 2n$ (which happens with reasonnable probability) and $T_j - U_j$ has stayed positive (proba $(n^{1/3})^{-\frac{1}{2}} = n^{-1/6}$) then $\Theta > n$. Vysotsky : these two events are independent.

Proof, the end

- $$\mathbb{P}(\Theta > n) \asymp n^{-\frac{1}{6}}$$
- Curien Le Gall 16 If V_n is the number of vertices in the peeled region up to time n then typically V_n is of order $n^{4/3}$.
- Thus, heuritically,

$$\mathbb{P}(|\mathcal{H}| > n) \asymp \mathbb{P}(V_\Theta > n) \asymp \mathbb{P}(\Theta^{4/3} > n) \asymp (n^{-\frac{1}{6}})^{3/4} = n^{-1/8}$$