Weighted distances in scale-free random graph models

Komjáthy Júlia

joint with E. Adriaans, E. Baroni, R.v.d. Hofstad
Eindhoven University of Technology

Dynamics on Random Graphs and Random Maps
24 Oct 2017

Information diffusion

Information diffusion

Includes

Viruses

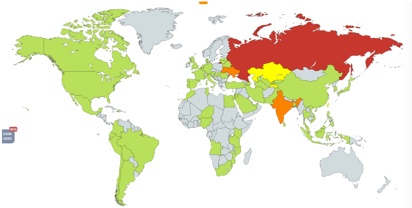

Download from
Dreamstime.com
ID 61758538
Onanong Tasanapitak | Dreamstime.com

Viruses

The spread of the Zika virus

Online viruses

Wannacry ransomware attack, May 2017

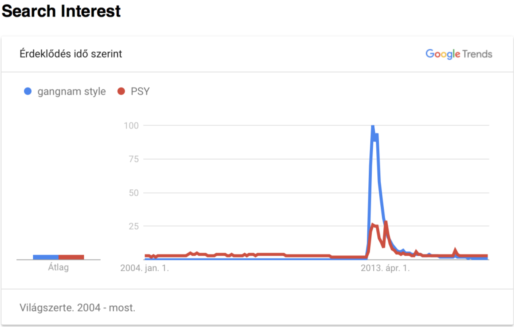


from Kasperski lab daily

Memes

Memes

Viral videos



Search intensity of Gangnam style

from knowyourmeme.com

Models

We need models!

The scale-free property

Many real-life networks have *power-law degrees*.

The scale-free property

Many real-life networks have *power-law degrees*.

Power-law paradigm

For some $\tau > 2$, the degree of a uniformly chosen vertex satisfies

$$\mathbb{P}(\deg(v) = x) \asymp \frac{C}{x^{\tau}}$$

The scale-free property

Many real-life networks have *power-law degrees*.

Power-law paradigm

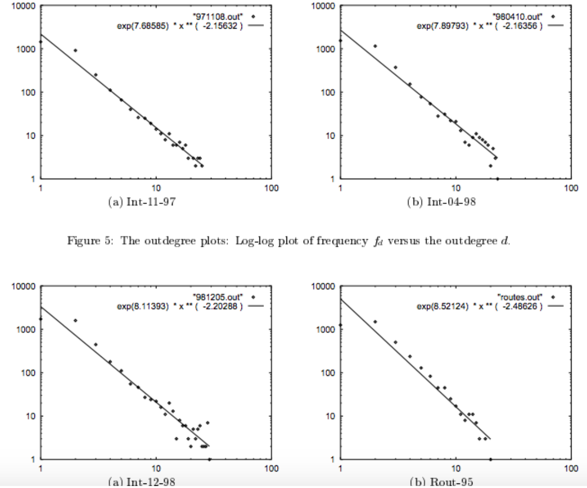
For some $\tau > 2$, the degree of a uniformly chosen vertex satisfies

$$\mathbb{P}(\deg(v) = x) \asymp \frac{C}{x^{\tau}}$$

$$\log \mathbb{P}(\deg(v) = x) \asymp \log C - \tau \log x$$

log(proportion of degree x vertices) vs $\log x$ is a straight line.

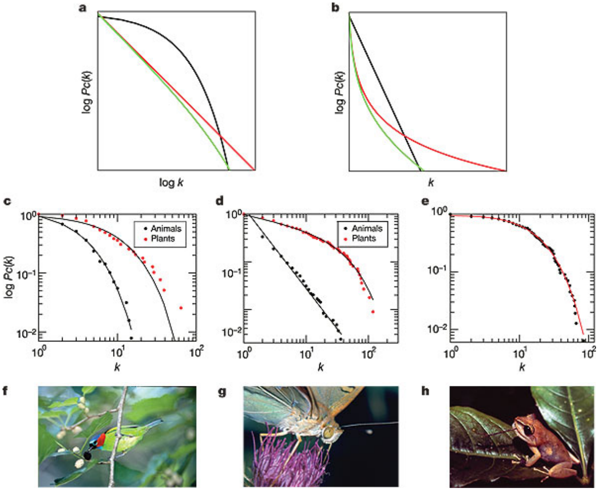
Power laws



Degree distribution of the router level internet network

from Faloutsos, Faloutsos, Faloutsos. 1999

Power laws



Degree distribution of ecological networks

from Montoya, Pimm, Polé. Nature 2006

Power laws

Note: $\tau \in (2,3)$ often!

When $\tau \in (2,3)$ then $\mathbb{V}ar_n[\deg(v)] \to \infty$ and $\mathbb{E}_n[\deg(v)] < \infty$.

Model 1.

Configuration model

The configuration model

Matches the degree sequence of the network you would like to model.

[Configuration model simulator by Robert Fitzner]

[Configuration model with power law degrees by Robert Fitzner]

Weighted Configuration model

Modeling information diffusion

Add i.i.d. weights from distribution σ to edges.

Weighted Configuration model

Modeling information diffusion

Add i.i.d. weights from distribution σ to edges.

Weighted Configuration model

Modeling information diffusion

Add i.i.d. weights from distribution σ to edges.

Spreading time = weighted distance

Edge weight = time/cost of transmission through the edge
The spreading time between two vertices u, v
= the weighted distance:

$$d_\sigma(u, v)$$

Weighted Configuration model

Modeling information diffusion

Add i.i.d. weights from distribution σ to edges.

Spreading time = weighted distance

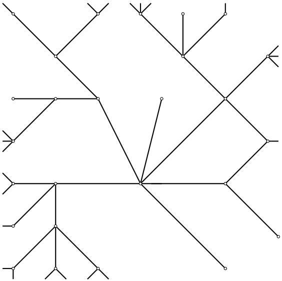
Edge weight = time/cost of transmission through the edge
The spreading time between two vertices u, v
= the weighted distance:

$$d_\sigma(u, v)$$

How does $d_\sigma(u, v)$ behave in terms of the degrees and the distribution σ?

Local weak convergence

Local neighborhoods look like random trees *with size biased degrees*.



Preliminaries

Initial stage of the spreading in the graph looks like a *random tree with*

- power law degrees, tail exponent $\alpha := \tau - 2 \in (0,1)$
- each edge has an i.i.d 'length' or 'weight'
- called *age-dependent branching process*

Age-dependent branching process

In an *age-dependent* branching process $\mathrm{BP}(X, \sigma)$

Age-dependent branching process

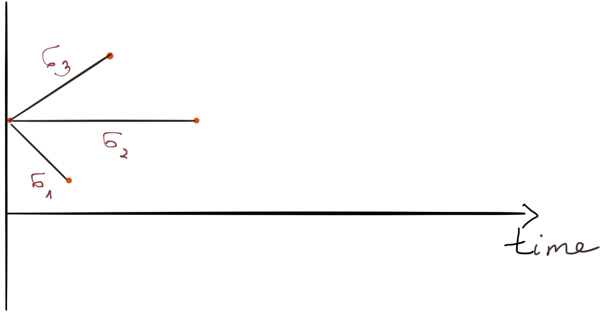
In an *age-dependent* branching process $\mathrm{BP}(X, \sigma)$

- root is born at time 0,

Age-dependent branching process

In an *age-dependent* branching process $\mathrm{BP}(X, \sigma)$

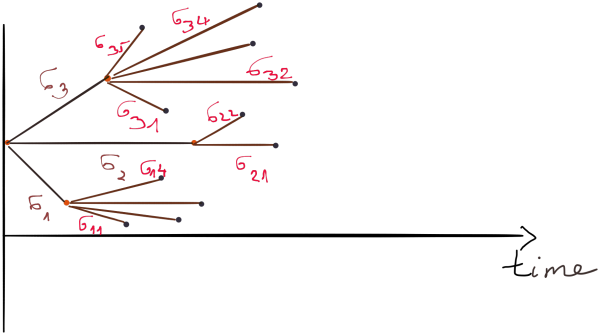
- root is born at time 0,
- the number of children of each individual is i.i.d. $\sim X$,



Age-dependent branching process

In an *age-dependent* branching process $\mathrm{BP}(X, \sigma)$

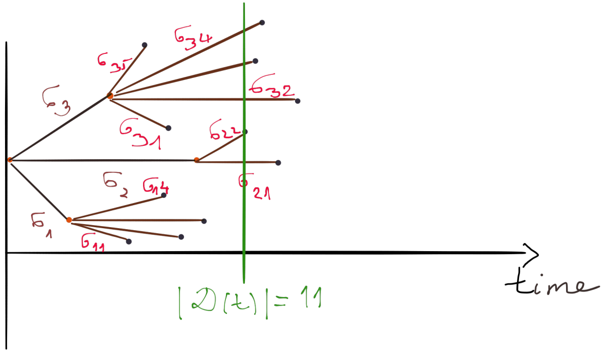
- root is born at time 0,
- the number of children of each individual is i.i.d. $\sim X$,
- birth-times of children are i.i.d. $\sim \sigma$.



Age-dependent branching process

In an *age-dependent* branching process $\mathrm{BP}(X, \sigma)$

- root is born at time 0,
- the number of children of each individual is i.i.d. $\sim X$,
- birth-times of children are i.i.d. $\sim \sigma$.



Explosion?

Definition

$\mathcal{D}(t) =$ population born by time t.

Explosion?

Definition

$\mathcal{D}(t)$ = population born by time t.
A branching process is *explosive* if for some $t > 0$,

$$\mathbb{P}(|\mathcal{D}(t)| = \infty) > 0.$$

Explosion?

Definition

$\mathcal{D}(t) =$ population born by time t.
A branching process is *explosive* if for some $t > 0$,

$$\mathbb{P}(|\mathcal{D}(t)| = \infty) > 0.$$

Otherwise *conservative*.

Explosion?

Definition

$\mathcal{D}(t) =$ population born by time t.
A branching process is *explosive* if for some $t > 0$,

$$\mathbb{P}(|\mathcal{D}(t)| = \infty) > 0.$$

Otherwise *conservative*.

Explosive vs conservative

When is a branching process $\mathrm{BP}(X, \sigma)$ explosive?

Explosion of BPs

Theorem (Amini, Devroye, Griffith, Olver & K)

Assume for x large enough and some $\varepsilon > 0$

$$\frac{1}{x^{\varepsilon}} > \mathbb{P}(X > x) > \frac{1}{x^{1-\varepsilon}}. \qquad \textbf{(PL)}$$

Explosion of BPs

Theorem (Amini, Devroye, Griffith, Olver & K)

Assume for x large enough and some $\varepsilon > 0$

$$\frac{1}{x^{\varepsilon}} > \mathbb{P}(X > x) > \frac{1}{x^{1-\varepsilon}}. \qquad \textbf{(PL)}$$

The branching process $\mathrm{BP}(X, \sigma)$ is explosive if and only if for some $K > 0$

$$\int_K^{\infty} F_\sigma^{(-1)}\left(\mathrm{e}^{-\mathrm{e}^z}\right)\mathrm{d}z < \infty \qquad \textbf{(I)}$$

where $F_\sigma^{(-1)}$ is the inverse of the distribution function $F_\sigma(x) = \mathbb{P}(\sigma \le x)$.

Explosion of BPs

Theorem (Amini, Devroye, Griffith, Olver & K)

Assume for x large enough and some $\varepsilon > 0$

$$\frac{1}{x^{\varepsilon}} > \mathbb{P}(X > x) > \frac{1}{x^{1-\varepsilon}}. \qquad \textbf{(PL)}$$

The branching process $\mathrm{BP}(X, \sigma)$ is explosive if and only if for some $K > 0$

$$\int_K^{\infty} F_{\sigma}^{(-1)}\left(\mathrm{e}^{-\mathrm{e}^{z}}\right)\mathrm{d}z < \infty \qquad \textbf{(I)}$$

where $F_{\sigma}^{(-1)}$ is the inverse of the distribution function $F_{\sigma}(x) = \mathbb{P}(\sigma \leq x)$.

Corollary

If a distribution σ explodes for one X satisfying **(PL)** *then it explodes for all X satisfying* **(PL)** *(including all power laws $\tau \in (2, 3)$).*

Explosive σ-s

Examples

Flatness of distribution function F_σ at the origin matters.

Explosive σ-s

Examples

Flatness of distribution function F_σ at the origin matters.

- Exponential, Gamma, Uniform, etc.

Explosive σ-s

Examples

Flatness of distribution function F_σ at the origin matters.

- Exponential, Gamma, Uniform, etc.
- $F_\sigma(t) = \exp\{-1/t^\beta\}, \beta > 0$

Explosive σ-s

Examples

Flatness of distribution function F_σ at the origin matters.

- Exponential, Gamma, Uniform, etc.
- $F_\sigma(t) = \exp\{-1/t^\beta\}, \beta > 0$
- Boundary case: $F_\sigma(t) = \exp\{-\exp\{\frac{1}{t^\beta}\}\}$. Explosive for $\beta < 1$, conservative for $\beta \geq 1$.

Back to the configuration model

Explosive weights on the configuration model

Theorem (Baroni, van der Hofstad, K)

Consider the configuration model with

Explosive weights on the configuration model

Theorem (Baroni, van der Hofstad, K)

Consider the configuration model with

- *power-law degree distribution with exponent* $\tau \in (2, 3)$

Explosive weights on the configuration model

Theorem (Baroni, van der Hofstad, K)

Consider the configuration model with

- *power-law degree distribution with exponent* $\tau \in (2, 3)$
- *i.i.d. edge-weights from distribution* σ.

Explosive weights on the configuration model

Theorem (Baroni, van der Hofstad, K)

Consider the configuration model with

- *power-law degree distribution with exponent* $\tau \in (2,3)$
- *i.i.d. edge-weights from distribution* σ.

If the branching process $\mathrm{BP}(D^\star, \sigma)$ *is explosive,*

$$d_\sigma(u,v) \xrightarrow{d} V^{(1)} + V^{(2)}.$$

$V^{(1)}, V^{(2)}$ *explosion times of two copies of* $\mathrm{BP}(D^\star, \sigma)$,
$D^\star$=*size biased degree*

Explosive weights on the configuration model

Theorem (Baroni, van der Hofstad, K)

Consider the configuration model with

- *power-law degree distribution with exponent* $\tau \in (2,3)$
- *i.i.d. edge-weights from distribution* σ.

If the branching process $\mathrm{BP}(D^\star, \sigma)$ *is explosive,*

$$d_\sigma(u,v) \xrightarrow{d} V^{(1)} + V^{(2)}.$$

$V^{(1)}, V^{(2)}$ *explosion times of two copies of* $\mathrm{BP}(D^\star, \sigma)$,
$D^\star$*=size biased degree*
Otherwise $d_\sigma(u,v) \to \infty$.

Explosive weights on the configuration model

Theorem (Baroni, van der Hofstad, K)

Consider the configuration model with

- *power-law degree distribution with exponent* $\tau \in (2,3)$
- *i.i.d. edge-weights from distribution* σ.

If the branching process $\mathrm{BP}(D^\star, \sigma)$ *is explosive,*

$$d_\sigma(u,v) \xrightarrow{d} V^{(1)} + V^{(2)}.$$

$V^{(1)}, V^{(2)}$ *explosion times of two copies of* $\mathrm{BP}(D^\star, \sigma)$*,*
$D^\star$*=size biased degree*
Otherwise $d_\sigma(u,v) \to \infty$.

This was first shown for $\sigma \sim \mathrm{Exp}(1)$ by Bhamidi, Hofstad, Hooghiemstra.

Conservative weights on the configuration model

Theorem (Adriaans, K)

Consider the configuration model with

Conservative weights on the configuration model

Theorem (Adriaans, K)

Consider the configuration model with

- *power-law degree distribution with exponent* $\tau \in (2, 3)$

Conservative weights on the configuration model

Theorem (Adriaans, K)

Consider the configuration model with

- *power-law degree distribution with exponent* $\tau \in (2, 3)$
- *i.i.d. edge-weights from distribution* σ.

Conservative weights on the configuration model

Theorem (Adriaans, K)

Consider the configuration model with

- *power-law degree distribution with exponent* $\tau \in (2,3)$
- *i.i.d. edge-weights from distribution* σ.

If $\mathrm{BP}(D^\star, \sigma)$ *is conservative, then for all* $\varepsilon > 0$,

$$\frac{d_\sigma(u,v)}{} \xrightarrow{\mathbb{P}} 1.$$

Conservative weights on the configuration model

Theorem (Adriaans, K)

Consider the configuration model with

- *power-law degree distribution with exponent* $\tau \in (2,3)$
- *i.i.d. edge-weights from distribution* σ.

If $\mathrm{BP}(D^\star, \sigma)$ *is conservative, then for all* $\varepsilon > 0$,

$$\frac{d_\sigma(u,v)}{2\sum_{k=1}^{\log\log n/|\log(\tau-2)|}} \xrightarrow{\mathbb{P}} 1.$$

Gives back the main term for $\sigma \equiv 1$.

Conservative weights on the configuration model

Theorem (Adriaans, K)

Consider the configuration model with

- *power-law degree distribution with exponent* $\tau \in (2,3)$
- *i.i.d. edge-weights from distribution* σ.

If $\mathrm{BP}(D^\star, \sigma)$ *is conservative, then for all* $\varepsilon > 0$,

$$\frac{d_\sigma(u,v)}{2\sum_{k=1}^{\log\log n/|\log(\tau-2)|} F_\sigma^{(-1)}\Big(\exp\big(-(\frac{1}{\tau-2})^k\big)\Big)} \xrightarrow{\mathbb{P}} 1.$$

Gives back the main term for $\sigma \equiv 1$.

When $\tau > 3$

$\tau \in (2,3)$

Dichotomy: bounded average distance for explosive weight distributions, non-bounded average distance for conservative weight distributions

Theorem (Bhamidi, Hofstad, Hooghiemstra)

Universally, for all σ that have a density,

$$d_\sigma(u,v) = \frac{1}{\lambda}\log n + \textit{tight},$$

where λ is the Malthusian parameter (exponential growth rate) of $\mathrm{BP}(D^\star, \sigma)$.

A model with geometry

Scale-free percolation (SFP)

- Vertex set is $\mathbf{Z}^d$, collection of vertex-weights $(W_z)_{z\in\mathbf{Z}^d}$.
- nearest neighbor edges are present
- Conditionally on $(W_z)_{z\in\mathbf{Z}^d}$, edges are present independently. With $\alpha > d$, $\lambda > 0$,

$$\mathbb{P}((x,y) \text{ open}) = 1 - \exp\big(-\lambda W_x W_y / \|x-y\|^\alpha\big)$$

- add i.i.d. edge-weights σ to all open edges.

Theorem (Deijfen, Hofstad)

Consider SFP with $\alpha > d$, and $\gamma := \alpha(\tau-1)/d > 1$. Then the degree distribution satisfies

$$\mathbb{P}(D_0 > x) = \ell(x)/x^\gamma$$

for some slowly varying function $\ell(x)$.

Scale free percolation

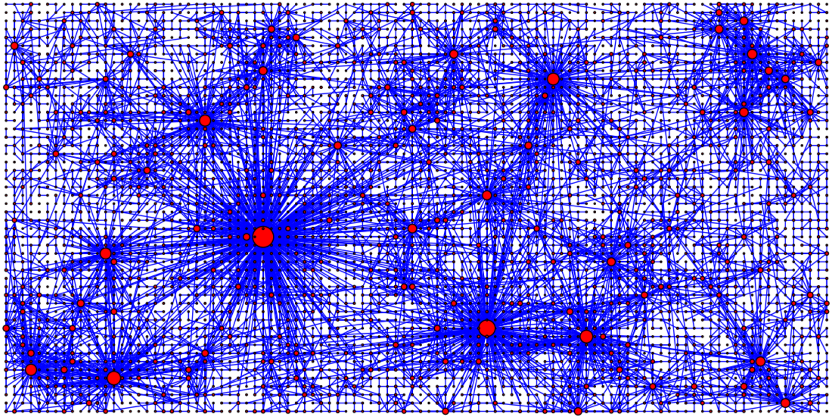


Figure: Scale free percolation with $\alpha = 3.9, \tau = 1.95, \lambda = 0.1$. Simulation by B. Lodewijks

Weighted distances for SFP

Theorem (Hofstad, K)

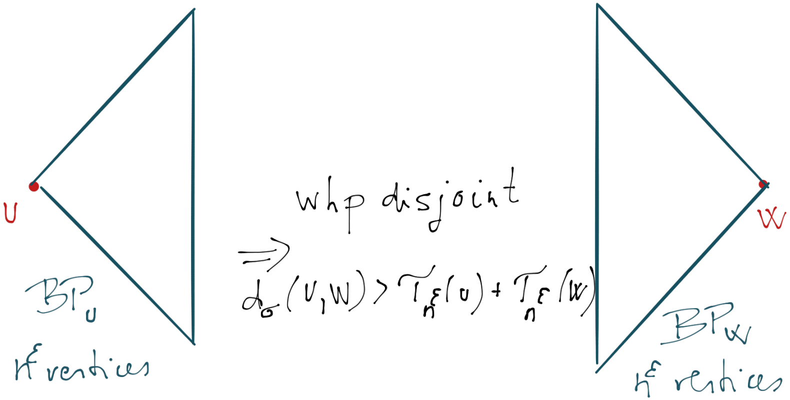
Consider SFP with $\alpha > d, \gamma = \alpha(\tau-1)/d \in (1,2)$ *and conservative edge-weights* σ*. Fix an arbitrary unit vector* $\underline{e}$*. Then, as* $m \to \infty$*,*

$$\frac{\mathrm{d}_\sigma(0, \lfloor m\underline{e} \rfloor)}{2 \sum_{k=1}^{\lfloor \log\log m / |\log(\gamma-1)| \rfloor} F_\sigma^{(-1)}\left(\exp(-\frac{1}{(\gamma-1)^k})\right)} \xrightarrow{\mathbb{P}} 1.$$

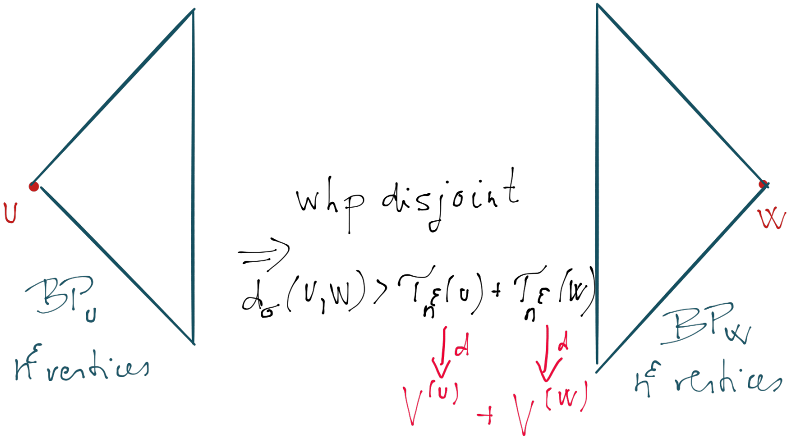
Picture-proof of explosion for CM

Picture-proof of explosion for CM

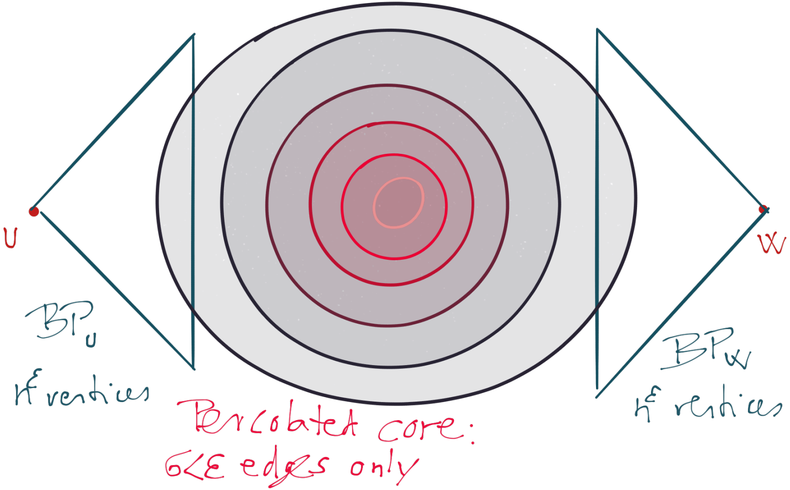
Picture-proof of explosion for CM



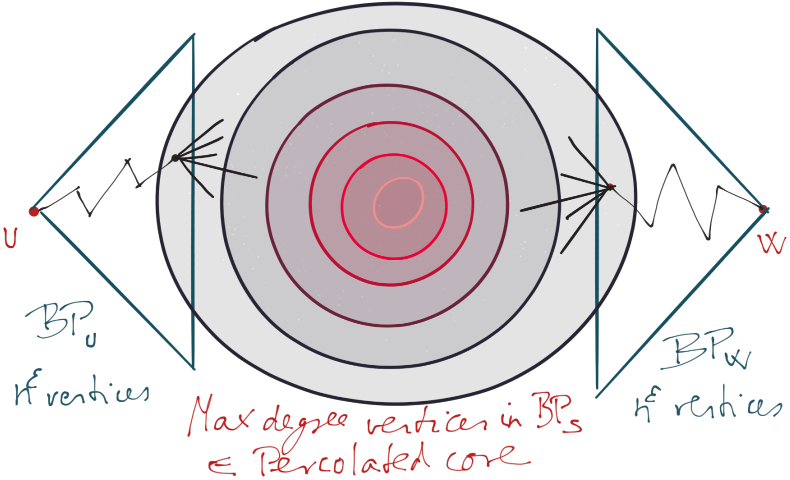
Picture-proof of explosion for CM



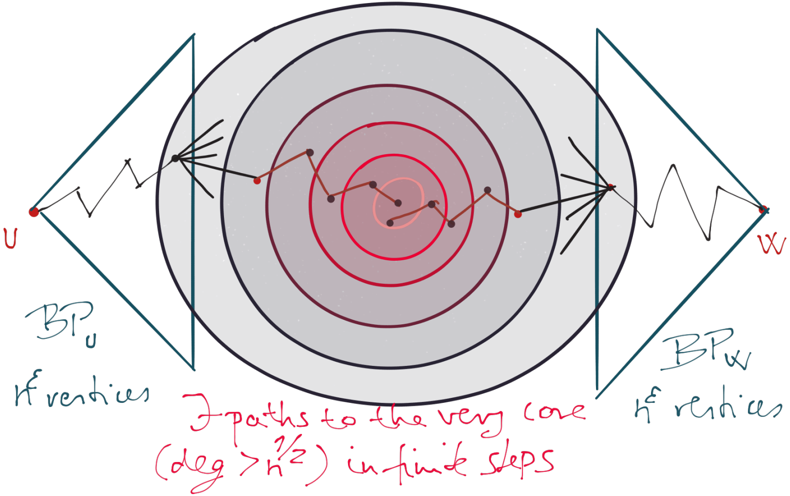
Picture-proof of explosion for CM



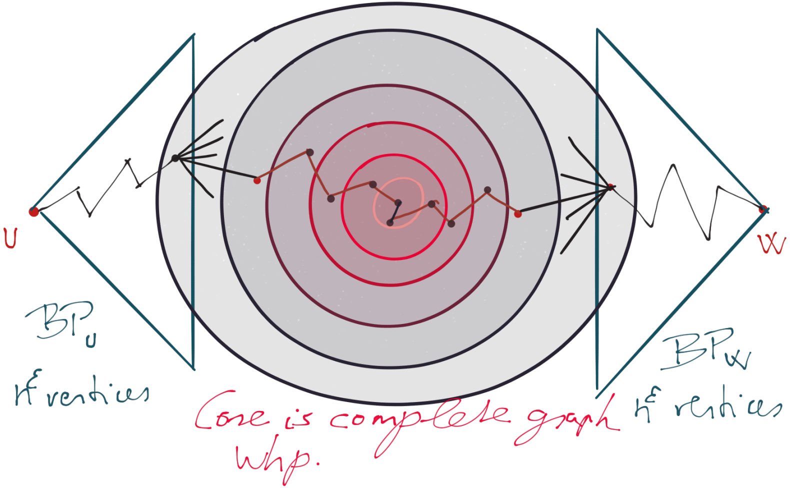
Picture-proof of explosion for CM



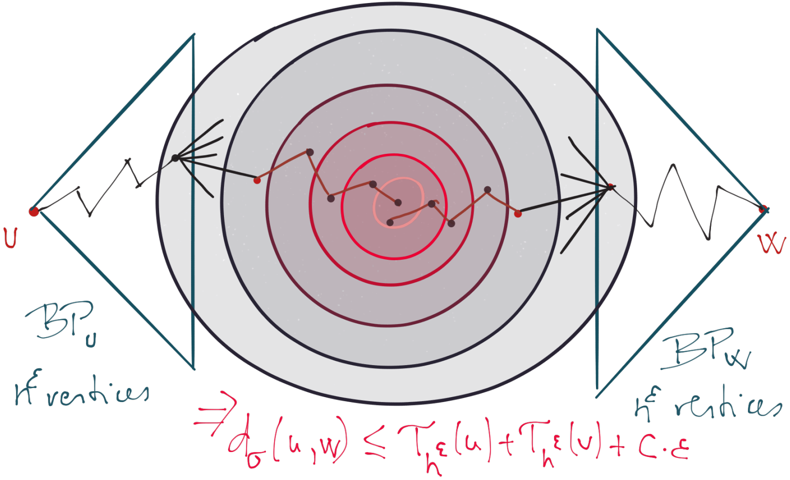
Picture-proof of explosion for CM



Picture-proof of explosion for CM



Picture-proof of explosion for CM



Proof for conservative edge-weights

Step 1: Coupling

Couple the initial stages of the spreading by two independent BPs, one started at u, one at v, until generation M_n for some small $M_n = o(\log n)$.

Proof for conservative edge-weights

Step 1: Coupling

Couple the initial stages of the spreading by two independent BPs, one started at u, one at v, until generation M_n for some small $M_n = o(\log n)$.

Step 2: Degree-dependent percolation

Percolate the whole graph: independently, for each edge e,

$$\mathbb{1}_{e \text{ is kept}} = \mathbb{1}_{\sigma_e \le \mathrm{tr}(d_1, d_2)}$$

e connects vertices with degrees d_1, d_2,
$\mathrm{tr}(d_1, d_2)$ some well-chosen threshold function.

Proof for conservative edge-weights

Step 1: Coupling

Couple the initial stages of the spreading by two independent BPs, one started at u, one at v, until generation M_n for some small $M_n = o(\log n)$.

Step 2: Degree-dependent percolation

Percolate the whole graph: independently, for each edge e,

$$\mathbb{1}_{e \text{ is kept}} = \mathbb{1}_{\sigma_e \leq \mathrm{tr}(d_1, d_2)}$$

e connects vertices with degrees d_1, d_2,
$\mathrm{tr}(d_1, d_2)$ some well-chosen threshold function.

Step 3: High-deg vertices $\widetilde{u}, \widetilde{v}$

Find two vertices $\widetilde{u}$, $\widetilde{v}$ with high enough percolated degree $\deg_p(\widetilde{u}), \deg_p(\widetilde{v}) > K_n = K_n(M_n)$ in the two BPs

Degree-dependent percolation

$$\mathbb{1}_{e \text{ is kept}} = \mathbb{1}_{\sigma_e \le \mathrm{tr}(d_1, d_2)}$$

Degree-dependent percolation

$$\mathbb{1}_{e \text{ is kept}} = \mathbb{1}_{\sigma_e \le \mathrm{tr}(d_1, d_2)}$$

Janson's argument

When $\mathbb{P}(e \text{ is kept}) = p(d_1)p(d_2)$, for some $p(\cdot)$, percolated graph can be looked at as a (new) configuration model

Degree-dependent percolation

$$\mathbb{1}_{e \text{ is kept}} = \mathbb{1}_{\sigma_e \leq \mathrm{tr}(d_1, d_2)}$$

Janson's argument

When $\mathbb{P}(e \text{ is kept}) = p(d_1)p(d_2)$, for some $p(\cdot)$, percolated graph can be looked at as a (new) configuration model

Let $c > 0$, $\eta < 1$, define

$$p(d) := \exp(-c(\log d)^\eta)$$

Threshold function

Keep each edge independently with probability $p(d_1)p(d_2)$, i.e.,

$$\mathrm{tr}(d_1, d_2) := F_\sigma^{(-1)}\left(\exp\left(-c(\log d_1)^\eta - c(\log d_2)^\eta\right)\right)$$

Degree-dependent percolation

$$\mathbb{1}_{e \text{ is kept}} = \mathbb{1}_{\sigma_e \leq \mathrm{tr}(d_1, d_2)}$$

Janson's argument

When $\mathbb{P}(e \text{ is kept}) = p(d_1)p(d_2)$, for some $p(\cdot)$, percolated graph can be looked at as a (new) configuration model

Let $c > 0$, $\eta < 1$, define

$$p(d) := \exp(-c(\log d)^{\eta})$$

Threshold function

Keep each edge independently with probability $p(d_1)p(d_2)$, i.e.,

$$\mathrm{tr}(d_1, d_2) := F_{\sigma}^{(-1)}\left(\exp\left(-c(\log d_1)^{\eta} - c(\log d_2)^{\eta}\right)\right)$$

Power law preserving $\eta < 1$

For $\eta < 1$, the new degree distribution has the same power-law exponent τ.

Proof continued

Step 4: Layers

In the percolated subgraph, whp there is a nested layering starting with degree K_n with the property that a vertex in layer i is connected to at least one vertex in layer $i+1$, and the degrees $\deg v$ in layer i is $\approx K_n^{1/(\tau-2)^i}$.

Proof continued

Step 4: Layers

In the percolated subgraph, whp there is a nested layering starting with degree K_n with the property that a vertex in layer i is connected to at least one vertex in layer $i+1$, and the degrees $\deg v$ in layer i is $\approx K_n^{1/(\tau-2)^i}$.

Step 5: Combine

$\widetilde{u}$, $\widetilde{v}$ falls into layer 1. Thus whp

$$d_L(u,v) \le d_L(u,\widetilde{u}) + d_L(v,\widetilde{v}) + 2 \sum_{i=1}^{\#\text{ layers}} \mathrm{tr}(K_n^{1/(\tau-2)^i}, K_n^{1/(\tau-2)^{i+1}}).$$

Proof continued

$$d_L(u,v) \le d_L(u,\widetilde{u}) + d_L(v,\widetilde{v}) + 2 \sum_{i=1}^{\#\text{ layers}} \mathrm{tr}\big(K_n^{1/(\tau-2)^i}, K_n^{1/(\tau-2)^{i+1}}\big).$$

use

$$\mathrm{tr}(d_1, d_2) := F_\sigma^{(-1)}\left(\exp\left(-c(\log d_1)^\eta - c(\log d_2)^\eta\right)\right)$$

Proof continued

$$d_L(u,v) \le d_L(u,\widetilde{u}) + d_L(v,\widetilde{v}) + 2 \sum_{i=1}^{\#\text{ layers}} \mathrm{tr}\big(K_n^{1/(\tau-2)^i}, K_n^{1/(\tau-2)^{i+1}}\big).$$

use

$$\mathrm{tr}(d_1, d_2) := F_\sigma^{(-1)}\left(\exp\left(-c(\log d_1)^\eta - c(\log d_2)^\eta\right)\right)$$

Step 6: Last term is

$$\frac{2}{\eta} \sum_{i=1}^{\eta \log\log n/|\log(\tau-2)|} F_\sigma^{-1}\left(\exp\{-(\tau-2)^{-i}\}\right).$$

$\eta < 1$, so summation boundary is fine, choose $1/\eta < 1 + \varepsilon/3$.

Proof continued

$$d_L(u,v) \le d_L(u,\widetilde{u}) + d_L(v,\widetilde{v}) + 2(1+\varepsilon/3) \sum_{i=1}^{\log\log n/|\log(\tau-2)|} F_\sigma^{-1}\left(\exp\{-(\tau-2)^{-i}\}\right)$$

First two terms: If we choose $\widetilde{u}$ independently of the edge weights σ_e,

$$d_L(u,\widetilde{u}) \overset{d}{\le} \sum_{i=1}^{M_n} \sigma_i$$

Choose $M_n \to \infty$ so that

$$\lim_{n\to\infty} \mathbb{P}\left(\sum_{i=1}^{M_n} \sigma_i \ge \varepsilon/3 \sum_{i=1}^{\log\log n/|\log(\tau-2)|} F_\sigma^{-1}\left(\exp\{-(\tau-2)^{-i}\}\right)\right) = 1$$

This is always possible, M_n (also) depends on the tail of σ.

Thank you for the attention!