Critical percolation on the Hamming graph

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Joint work with **Lorenzo Federico, Remco van der Hofstad & Frank den Hollander**

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Percolation

Definition

Fix a graph $G = (\mathcal{V}, \mathcal{E})$ and $p \in [0, 1]$. Remove each edge $e \in \mathcal{E}$ independently with probability p: a product measure on $\{0, 1\}^{\mathcal{E}}$.

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Percolation on (sequences of) finite graphs.

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Example

The *Erdős-Rényi random graph:* Take $G = K_n$. Write $G(n, p)$ for the percolated graph. Study $G(n, p)$ as $n \to \infty$ (with $p = p(n) \to 0$).

The ERRG phase transition (1)

The double jump transition

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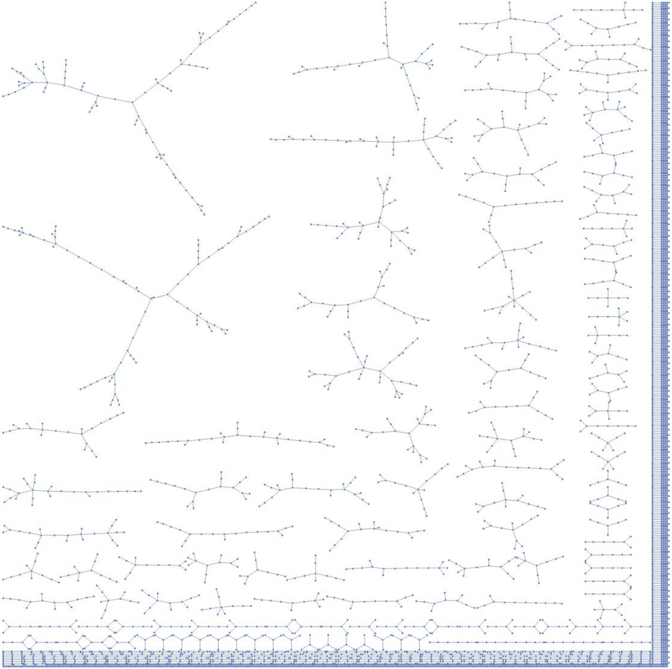
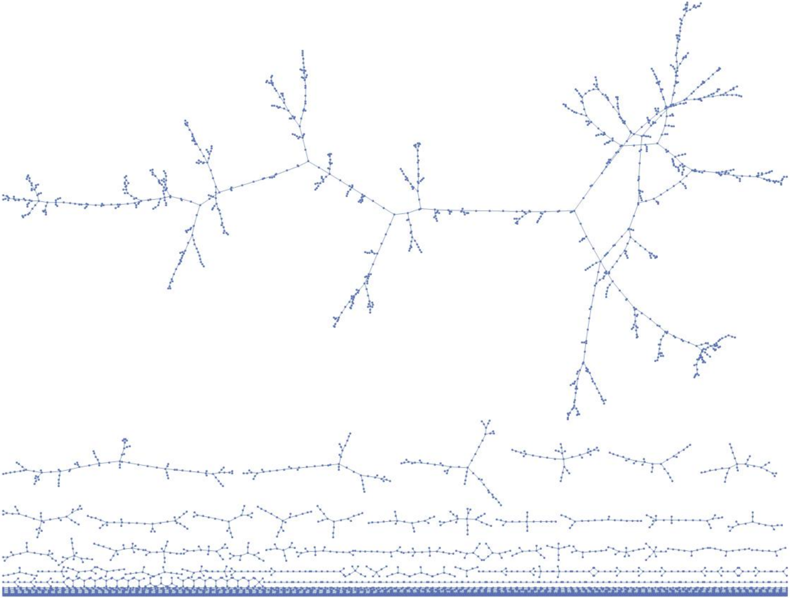
- if $p < 1/n$ then $|\mathcal{C}_j| = \Theta(\log n)$ whp *[subcricital]*
- if $p = 1/n$ then $n^{-2/3}|\mathcal{C}_j|$ is a tight random variable *[critical]*
- if $p > 1/n$ then $|\mathcal{C}_1| = \Theta(n)$ and $|\mathcal{C}_j| = \Theta(\log n)$ for $j \geq 2$ whp *[supercritical]*





The ERRG phase transition (2)

The critical window

We can zoom in on the phase transition by choosing $p = \frac{1+\varepsilon_n}{n}$ with $\varepsilon_n \to 0$. This shows a much richer structure around criticality [Bollobás '84, Łuczak '90, Janson, Knuth, Łuczak & Pittel '93, …]

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- if $\varepsilon_n n^{1/3} \to +\infty$ then $|\mathcal{C}_1| = 2\varepsilon_n n(1 + o(1))$ whp, $|\mathcal{C}_j| = 2\varepsilon_n^{-2} \log(\varepsilon_n^3 n)(1 \pm o(1))$ for $j \geq 2$ whp *[slightly supercritical]*

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- if $\varepsilon_n n^{1/3} \to -\infty$ then $|\mathcal{C}_j| = 2\varepsilon_n^{-2} \log(\varepsilon_n^3 n)(1 \pm o(1))$ whp *[slightly subcritical]*
- if $\varepsilon_n n^{1/3} \to \theta \in \mathbb{R}$ then **Aldous' scaling limit** *[the critical window]*
- if $\varepsilon_n n^{1/3} \to +\infty$ then $|\mathcal{C}_1| = 2\varepsilon_n n(1 + o(1))$ whp, $|\mathcal{C}_j| = 2\varepsilon_n^{-2} \log(\varepsilon_n^3 n)(1 \pm o(1))$ for $j \geq 2$ whp *[slightly supercritical]*

Cluster sizes of the critical ERRG

Theorem [Aldous '97]

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$$B^{\theta}(t) := B(t) + \theta t - \frac{t^2}{2} \qquad \text{(BM w/ parabolic drift)}$$

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and

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Consider $G(n, \frac{1+\varepsilon_n}{n})$ with $\varepsilon_n n^{1/3} \to \theta$. Then,

$$\left(\frac{|\mathcal{C}_i|}{n^{2/3}} \right)_{i \ge 1} \xrightarrow{\text{d}} (\gamma_i(\theta))_{i \ge 1}$$

About the proof

A graph exploration algorithm

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- If $\exists$ an active vertex: move token to an active vertex. Call it v. **Go to (2)**
- If $\nexists$ an active vertex: **Go to (1)** [explored a component]

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Define the stochastic process

$$S_0 = 0, \qquad S_i = S_{i-1} - \quad 1 \quad + \quad X_i$$

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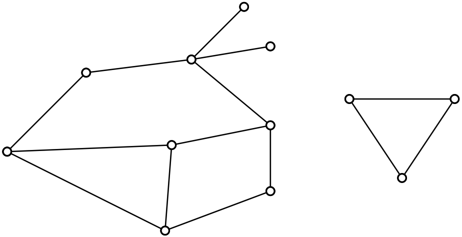
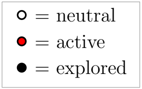
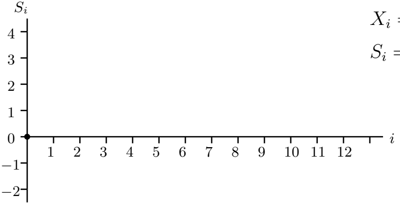
$$S_0 = 0, \qquad S_i = S_{i-1} - 1 + \underset{\substack{\uparrow \\ \text{new active vertices}}}{X_i}$$





i	X_i	S_i
0	0	0



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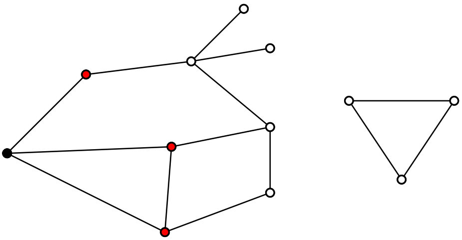

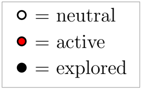

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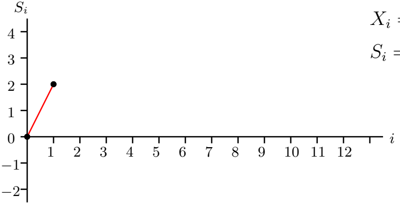
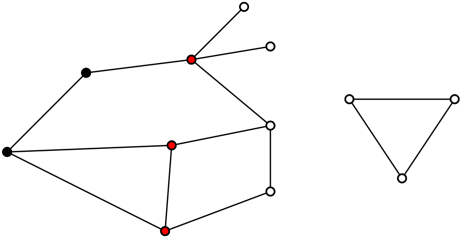
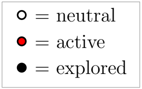
i	X_i	S_i
0	0	0
1	3	2







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0	0	0
1	3	2
2	1	2

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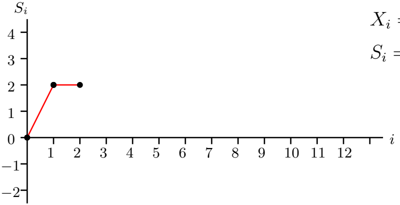
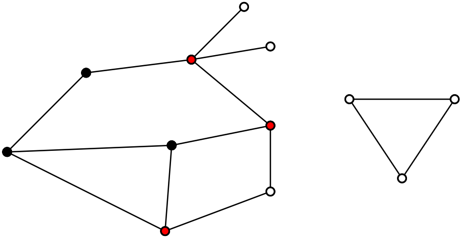
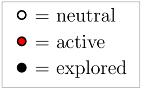
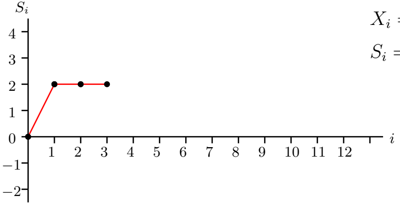






i	X_i	S_i
0	0	0
1	3	2
2	1	2
3	1	2



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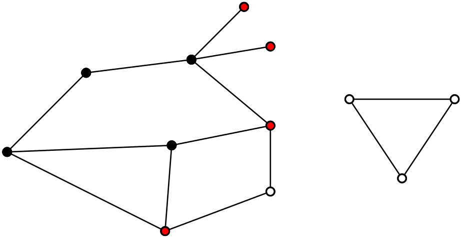
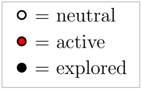
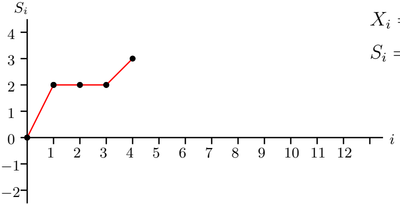




i	X_i	S_i
0	0	0
1	3	2
2	1	2
3	1	2
4	2	3



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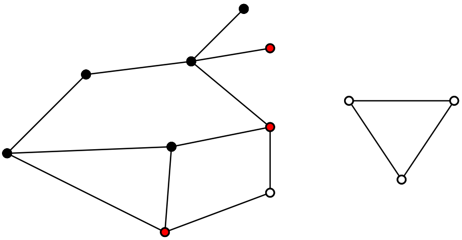

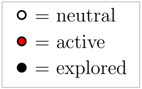

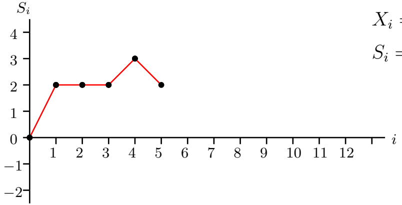

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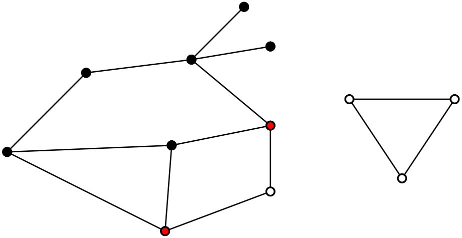
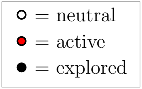
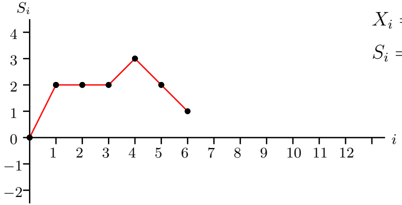
i	X_i	S_i
0	0	0
1	3	2
2	1	2
3	1	2
4	2	3
5	0	2





i	X_i	S_i
0	0	0
1	3	2
2	1	2
3	1	2
4	2	3
5	0	2
6	0	1



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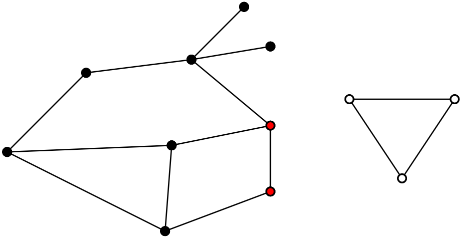

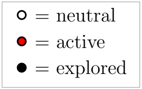

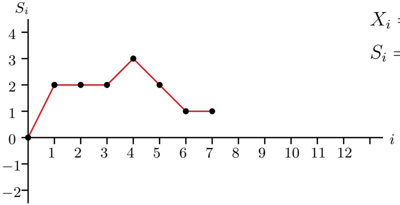

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0	0	0
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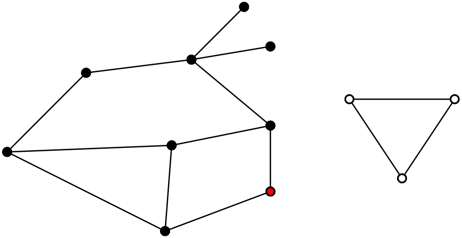
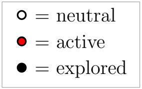
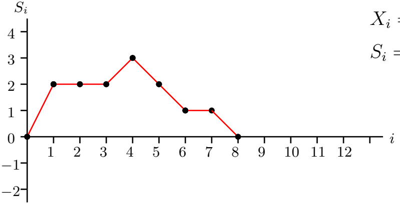
2	1	2
3	1	2
4	2	3
5	0	2
6	0	1
7	1	1







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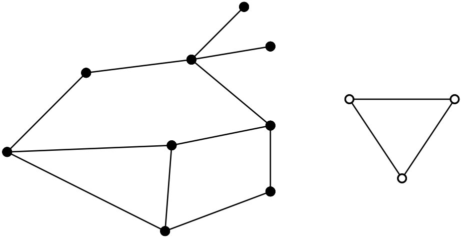
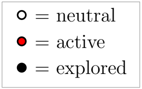
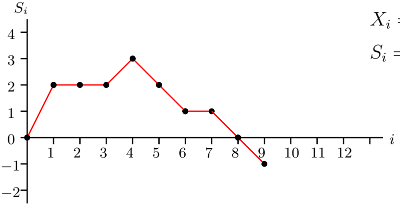
0	0	0
1	3	2
2	1	2
3	1	2
4	2	3
5	0	2
6	0	1
7	1	1
8	0	0







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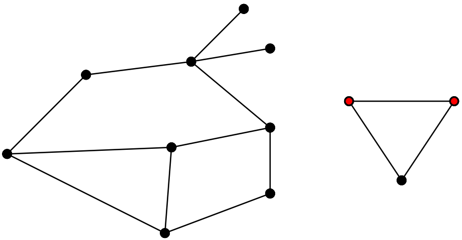
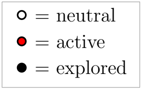
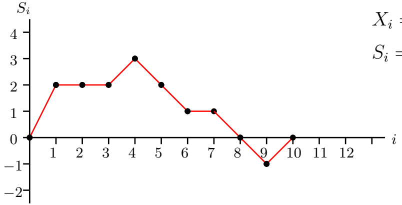
4	2	3
5	0	2
6	0	1
7	1	1
8	0	0
9	0	−1







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i	X_i	S_i
0	0	0
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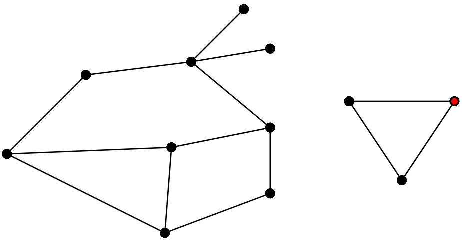
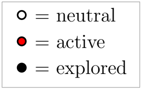
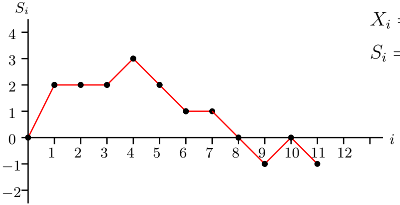
3	1	2
4	2	3
5	0	2
6	0	1
7	1	1
8	0	0
9	0	−1
10	2	0







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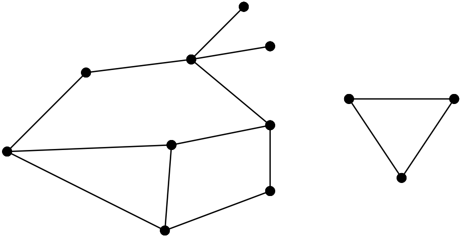

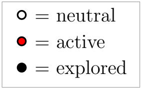

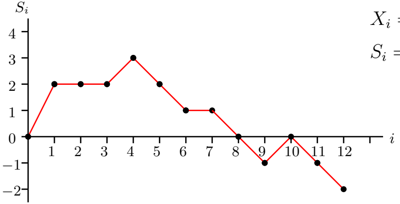

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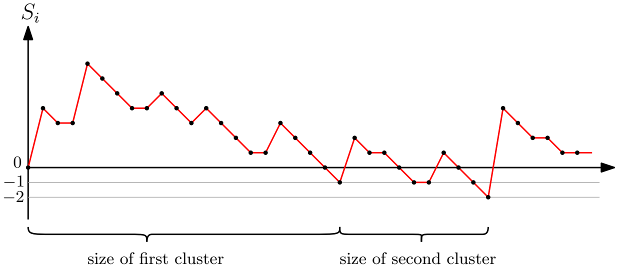
The exploration process and cluster sizes



$\min\{j : S_j = -1\}$ = size of first explored cluster and $\min\{j : S_j = -k\}$ = total size of first k explored clusters.

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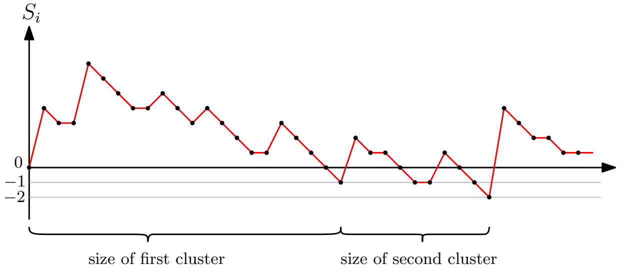

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If $G(n, \frac{1+\theta n^{-1/3}}{n})$ has $\left(n^{-1/3} S_{tn^{2/3}}\right)_{t\geq 0} \xrightarrow{d} (B^{\theta}(t))_{t\geq 0}$, then Aldous' Theorem follows

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The ERRG universality class

It is conjectured that the ERRG phase transition also holds for many other *sparse, high-dimensional* random graph models.

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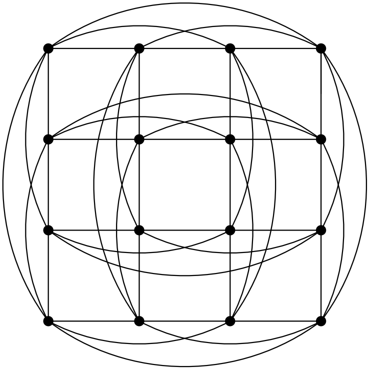
The main difficulty in going from the ERRG to geometric graphs is that K_n is highly *symmetric* and *self-similar,* which makes everything easier. For instance, if we remove a component of size k from $G(n, p)$, the (conditional) law of what remains is $G(n-k, p)$. This is obviously not true for percolation on any other graph.

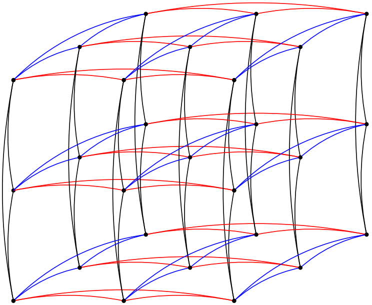
The Hamming graph

$H(d, n)$ is defined as the $(d-1)$-fold Cartesian product of K_n,

$$H(d, n) \simeq K_n \times K_n \times \cdots \times K_n$$

$H(d, n)$ has degree $m := d(n-1)$ and $V := n^d$ vertices.





The critical window

Theorem [FHHH '17]

For percolation on $H(d,n)$ with degree $m = d(n-1)$ and $d = 2, 3, \ldots, 6$,

$$p_c^{H(d,n)} = \frac{1}{m} + \frac{2d^2-1}{2(d-1)^2}\frac{1}{m^2}$$

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REMARK: The width of the critical window is $O(m^{-1} V^{-1/3}) = O(n^{-d/3-1})$ [Borgs *et al.* '05], so $1/m$ is not in the critical window when $d \geq 4$.

Critical percolation on the Hamming graph

Theorem [FHHH '17+]

For percolation on $H(d,n)$ with $d = 2, 3, 4$, fix $\theta \in \mathbb{R}$ and $p = p_c^{H(d,n)}(1 + \theta V^{-1/3})$. Then,

$$\left(\frac{|\mathcal{C}_i|}{V^{2/3}} \right)_{i \geq 1} \xrightarrow{\text{d}} (\gamma_i(\theta))_{i \geq 1}$$

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[Exactly the same as the ERRG]

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The proof uses an exploration process, just like Aldous. But non-trivial geometry gives rise to two problems:

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The proof uses an exploration process, just like Aldous. But non-trivial geometry gives rise to two problems:

- PROBLEM 1: consecutive steps in the exploration are highly dependent
- PROBLEM 2: current cluster is dependent on all explored clusters

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Percolation and branching random walks

We describe percolation configurations as a projection of randomly embedded $\mathsf{Bin}(m, p)$-Galton-Watson trees into $H(d, n)$, where particles are killed when

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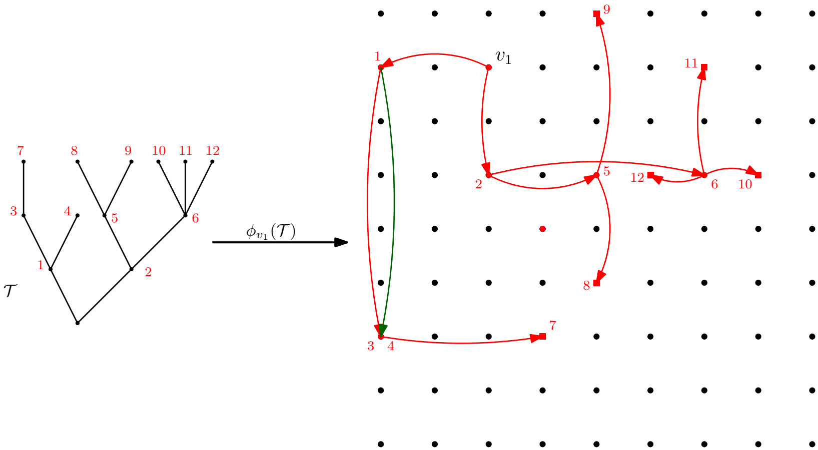
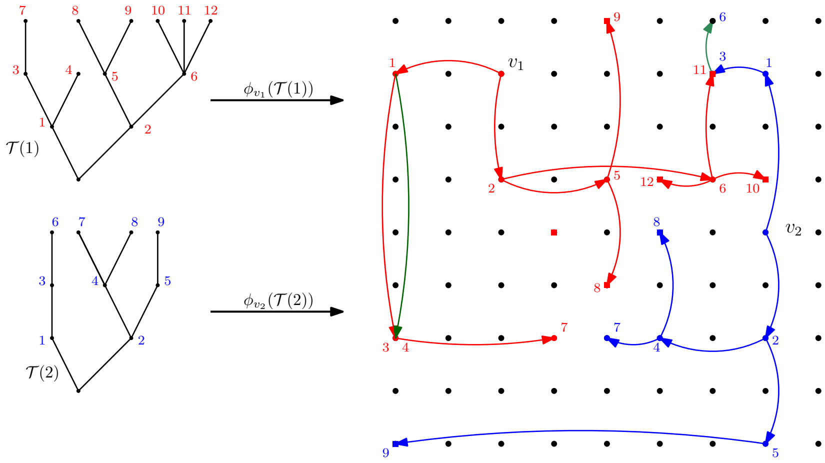


7
8
9
10
11
12
3
4
5
6
1
2
$\mathcal{T}$
$\phi_{v_1}(\mathcal{T})$
9
1
v_1
11
2
5
12
6
10
8
7
3
4



7 8 9 10 11 12
3 4 5 6
1 2
$\mathcal{T}(1)$
6 7 8 9
3 4 5
1 2
$\mathcal{T}(2)$
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v_1
v_2

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Disadvantage:

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A two-scale exploration

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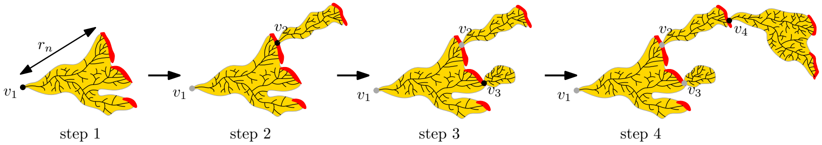

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Disadvantage:

- The number of explored vertices is now random. But for r_n small enough the number concentrates.

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Reducing dependence between current cluster and explored clusters

A sticky coupling

When exploring the ERRG the geometry of the already explored clusters does not matter (removing a cluster of size k from $G(n,p)$ gives $G(n-k,p)$). On the Hamming graph it does matter.

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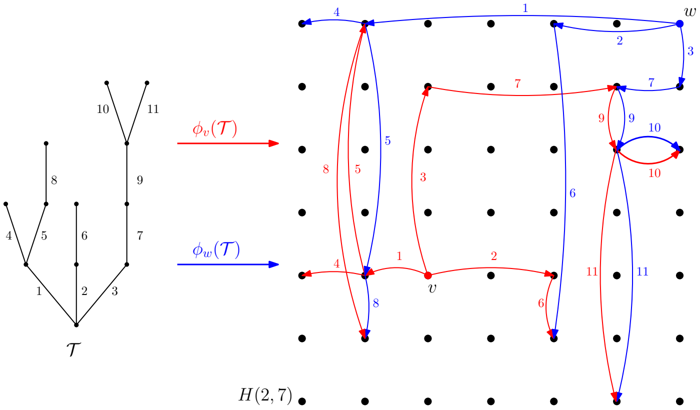
We use a *sticky coupling* between the actual BRW exploration and a BRW started from a uniformly random vertex to exploit this fact.



$\mathcal{T}$
$\phi_v(\mathcal{T})$
$\phi_w(\mathcal{T})$
$H(2,7)$
v
w

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Disadvantage:

- Many different processes and couplings going on at the same time

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What about $d > 4$, or other graphs?

Improving our result to $d \leq 9$ is feasible but hard work. Improving beyond that, or to other graphs (e.g. hypercubes) requires some new ideas. The main problem is that our method requires explicit knowledge of p_c.

Thank you