Percolation and isoperimetric inequalities

Elisabetta Candellero

(joint work with Augusto Teixeira, IMPA, Rio de Janeiro)

University of Warwick

Dynamics on random graphs – 2017/10/25

Percolation

Physical phenomenon:

(i) Models how fluid can spread through a medium;

(ii) Models how certain epidemics can spread through a network;

(iii) Many other motivational examples!

Introduced by *Broadbent and Hammersley* in '57 (independent percolation).

Percolation

Ingredients:

(i) A graph $G = (V, E)$ (we consider only $|V| = \infty$);

(ii) A parameter $p \in [0, 1]$.

Percolation

Ingredients:

(i) A graph $G = (V, E)$ (we consider only $|V| = \infty$);

(ii) A parameter $p \in [0, 1]$.

Two types of percolation: *bond* (edges) and *site* (vertices) percolation.

Today we focus on **SITE** percolation.

Example of Percolation

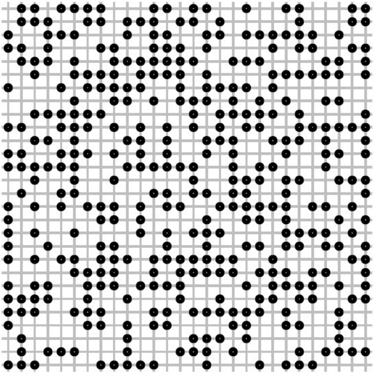


Figure: $\mathbb{Z}^2$ with $p = 0.5$.

Example of Percolation

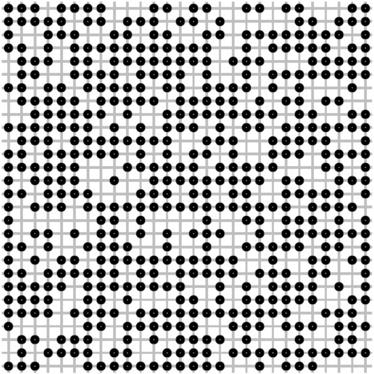


Figure: $\mathbb{Z}^2$ with $p = 0.7$.

Fundamental questions

Study the connectivity properties of the black (random) subgraph.

Fundamental questions

Study the connectivity properties of the black (random) subgraph.

What is the probability that a certain vertex connected to infinity?

Fundamental questions

Study the connectivity properties of the black (random) subgraph.

What is the probability that a certain vertex connected to infinity?

Define

$$\theta(p) := \mathbb{P}_p[\text{vertex } o \text{ is connected to infinity}].$$

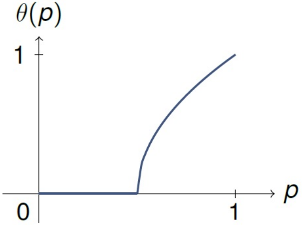


Figure: The function $\theta(p)$.

A critical value

From the previous picture it is then natural to define

$$p_c := \sup\{p \in [0,1] \ : \ \theta(p) = 0\}.$$

A critical value

From the previous picture it is then natural to define

$$p_c := \sup\{p \in [0,1] \ : \ \theta(p) = 0\}.$$

Lots of interesting questions!

(i) Continuity at p_c?

(ii) Behavior of $\theta(p)$ at p_c?

(iii) When is p_c non-trivial?

WARWICK
THE UNIVERSITY OF WARWICK

A critical value

From the previous picture it is then natural to define

$$p_c := \sup\{p \in [0,1] \ : \ \theta(p) = 0\}.$$

Lots of interesting questions!

(i) Continuity at p_c?

(ii) Behavior of $\theta(p)$ at p_c?

(iii) When is p_c non-trivial?

When is $p_c \in (0,1)$?

WARWICK
THE UNIVERSITY OF WARWICK

d-dimensional lattices

On $\mathbb{Z}^d$ we know several things, for example:

- If $d \geq 2$ we know that $p_c \in (0, 1)$;
- If $d = 2$ or $d \geq 11$, then we know that $\theta(p_c) = 0$.

We still don't know what happens in the intermediate range of d's.

WARWICK
THE UNIVERSITY OF WARWICK

Other graphs

In general, for *independent percolation*, it is true that

If the degree of the graph G is at most Δ, then $p_c(G) \geq \frac{1}{\Delta} > 0$.

Other graphs

In general, for *independent percolation*, it is true that

If the degree of the graph G is at most Δ, then $p_c(G) \geq \frac{1}{\Delta} > 0$.

We do not have such an easy way to investigate upper-bounds for p_c.

The first step in a study of percolation on other graphs [...] will be to prove that the critical probability on these graphs is smaller than one.

Benjamini and Schramm

WARWICK
THE UNIVERSITY OF WARWICK

Other graphs

It is known that $p_c(G) < 1$ when

G has **exponential growth** (Lyons, Benjamini and Schramm, Babson and Benjamini...);

as well as

G is the Cayley graph of the Grigorchuck group, an example of a graph with **intermediate growth** (Muchnik and Pak).

Other graphs

What we **didn't know** before starting this project was that $p_c(G) < 1$ holds also for

Vertex-transitive graphs with polynomial growth.

Other graphs

What we **didn't know** before starting this project was that $p_c(G) < 1$ holds also for

Vertex-transitive graphs with polynomial growth.

The proof of this fact involves Gromov's theorem (a very difficult and powerful result from group theory) and combinatorial techniques developed by Babson and Benjamini, and later on simplified by Timar.

Isoperimetric inequalities

It is known that $p_c(\mathbb{Z}) = 1$, but on the other hand, when $d \geq 2$ then $p_c(\mathbb{Z}^d) < 1$. Hence, the natural question here is

Isoperimetric inequalities

It is known that $p_c(\mathbb{Z}) = 1$, but on the other hand, when $d \geq 2$ then $p_c(\mathbb{Z}^d) < 1$. Hence, the natural question here is

Does the dimension play a role for $p_c(G) < 1$?
How important?

Isoperimetric inequalities

For every finite set $A \subset V(G)$, define the (internal) vertex-boundary as

$$\partial A := \left\{ x \in A \ : \ \exists\, y \in V(G) \setminus A \ : \ \{x, y\} \in E(G) \right\}.$$

Isoperimetric inequalities (dimension)

Define the (isoperimetric) **dimension** of G as follows: we say that $\dim(G) = d > 1$

if and only if d is the largest value for which

there is a constant $c > 0$ such that

$$\inf_{A \subset V(G),\ A \text{ finite}} \frac{|\partial A|}{|A|^{(d-1)/d}} \geq c.$$

Isoperimetric inequalities (remarks)

Remark: for every $d \geq 2$, $\mathbb{Z}^d$ has isoperimetric dimension d.

Remark: If G has isoperimetric dimension $d > 1$, then we can say that it satisfies IS_d (d-isoperimetric inequality).

Question (Benjamini and Schramm '96)

Is it true that dim$(G) > 1$ implies that $p_c(G) < 1$?

WARWICK
THE UNIVERSITY OF WARWICK

Some results

- If G is *planar*, has *polynomial growth* and *no accumulation points* then $\dim(G) > 1 \;\Rightarrow\; p_c(G) < 1.$ [Kozma]

Some results

- If G is *planar*, has *polynomial growth* and *no accumulation points*

  then $\dim(G) > 1 \;\Rightarrow\; p_c(G) < 1.$ [Kozma]

- If G satisfies a *stronger* condition than the isoperimetric inequality (called *local isoperimetric inequality*), and has *polynomial growth*

  then $\dim_\ell(G) > 1 \;\Rightarrow\; p_c(G) < 1.$ [Teixeira]

Our results

Definition: A measure $\mathbb{P}$ satisfies the *decoupling inequality* $\mathcal{D}(\alpha, c_\alpha)$ (where $\alpha > 0$ is a fix parameter) if for all $r \geq 1$ and any two decreasing events $\mathcal{G}$ and $\mathcal{G}'$ such that

$$\mathcal{G} \in \sigma\big(Y_z, z \in B(o,r)\big) \qquad \text{and} \qquad \mathcal{G}' \in \sigma\big(Y_w, w \notin B(o,2r)\big),$$

we have

$$\mathbb{P}(\mathcal{G} \cap \mathcal{G}') \leq \big(\mathbb{P}(\mathcal{G}) + c_\alpha r^{-\alpha}\big)\mathbb{P}(\mathcal{G}').$$

In other words: we admit dependencies, as long as they decay fast enough in the distance.

Our results

With a completely **probabilistic approach** we showed:

Theorem [C. and Teixeira]: If G is transitive, with polynomial growth, $\dim(G) > 1$, and $\mathbb{P}$ satisfies $\mathcal{D}(\alpha, c_\alpha)$ with α "large enough", then

(i) There exists a $p_* < 1$, such that if $\inf_{x \in V} \mathbb{P}[Y_x = 1] > p_*$, then the graph contains almost surely a unique infinite open cluster.

(ii) Moreover, fixed any value $\theta > 0$, we have

$$\lim_{v \to \infty} v^\theta \mathbb{P}[v < |\mathcal{C}_o| < \infty] = 0,$$

where $\mathcal{C}_o =$ open connected component containing the origin.

Our results

Moreover, in the dependent case we also need to show that:

Theorem [C. and Teixeira]: If G is transitive, with polynomial growth, $\dim(G) > 1$, and $\mathbb{P}$ satisfies $\mathcal{D}(\alpha, c_\alpha)$ with α "large enough", then

(i) There exists a $p_{**} > 0$, such that if $\sup_{x\in V} \mathbb{P}[Y_x = 1] < p_{**}$, then the graph contains almost surely **NO** infinite open cluster.

(ii) Moreover, fixed any value $\theta > 0$, we have

$$\lim_{v\to\infty} v^\theta \mathbb{P}[v < |\mathcal{C}_o|] = 0,$$

where $\mathcal{C}_o$ = open connected component containing the origin.

WARWICK
THE UNIVERSITY OF WARWICK

Our results (Remark)

We always assume α to be *large enough*.

Although we don't have sharp bounds on its critical value, if α is too small, there are *counterexamples*!

[One counterexample in paper by Tykesson and Windisch]

Our results

Definition: Two metric spaces (X_1, d_1) and (X_2, d_2) are *roughly isometric* (sometimes called "quasi-isometric") if there is a map $\varphi : X_1 \to X_2$ s.t.:

(i) There are $A \geq 1$, $B \geq 0$ such that for all $x, y \in X_1$

$$A^{-1} d_1(x, y) - B \leq d_2\big(\varphi(x), \varphi(y)\big) \leq A d_1(x, y) + B.$$

(ii) There is $C \geq 0$ such that for all $z \in X_2$ there is $x \in X_1$ s.t.

$$d_2(z, \varphi(x)) \leq C.$$

Our results (Remarks)

Definition: A graph G is *roughly transitive* if there is a rough isometry between any two vertices of G.

Our results (Remarks)

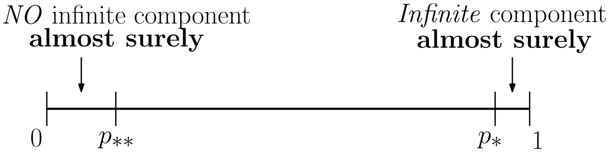
Definition: A graph G is *roughly transitive* if there is a rough isometry between any two vertices of G.

ROUGHLY TRANSITIVE $\neq$ ROUGHLY ISOMETRIC TO TRANSITIVE!
[One counterexample in paper by Elek and Tardos]

Our results

Our proof also works when G is a *roughly transitive graph*:

Theorem [C. and Teixeira]: If G is *roughly-transitive graph*, with polynomial growth, $\dim(G) > 1$, and $\mathbb{P}$ satisfies $\mathcal{D}(\alpha, c_\alpha)$ with α "large enough", then



and, for every $\theta > 0$,

$$\begin{cases} \lim_{v\to\infty} v^\theta \mathbb{P}[v < |\mathcal{C}_\infty|] = 0 & \text{if } p < p_{**} \\ \lim_{v\to\infty} v^\theta \mathbb{P}_p[v < |\mathcal{C}_\infty| < \infty] = 0 & \text{if } p > p_*. \end{cases}$$

Idea of the proof: renormalization (multiscale argument)

On the blackboard.

Main hypothesis I: *polynomial growth*

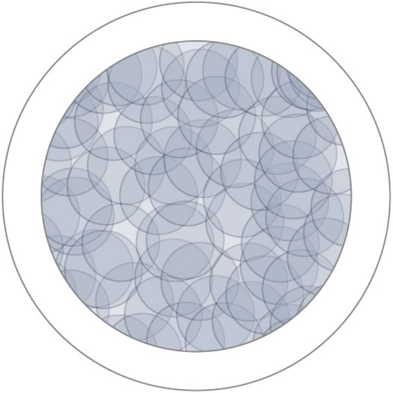


Figure: Polynomial growth allows us to split the graph into cells.

Main hypothesis II: *Isoperimetric inequality*

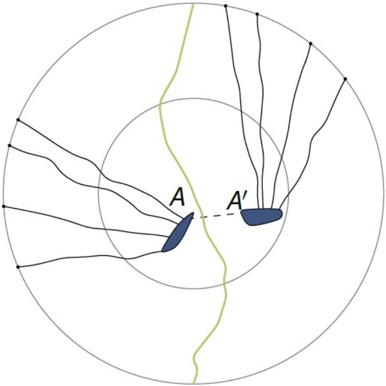


Figure: Isoperimetric inequality implies that there are lots of paths between large connected sets and infinity.

Main hypothesis III: *transitivity* (or rough-transitivity)

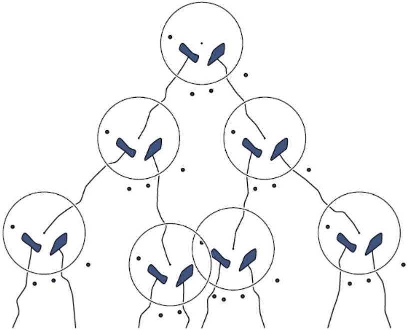


Figure: Transitivity allows us to repeat the same reasoning in different areas of the graph...

Proof

If G satisfies conditions I, II, and III (i.e., polynomial growth, isoperimetric dimension > 1, rough transitivity), then

$$\text{assuming } p_c(G) = 1$$
$$\Downarrow$$
$$\text{it is possible to construct a binary tree inside } G.$$

CONTRADICTION with polynomial growth of G! □

WARWICK
THE UNIVERSITY OF WARWICK

Thank you for your attention!