Annealed Ising model on random regular graphs

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Dynamics on random graphs and random maps
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Outline

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Ising model on multiple graphs

Given a multiple graph $G_n = (V_n, E_n)$ with n vertices $V_n = \{v_1, \ldots, v_n\}$, we assign to each vertex v_i a spin $\sigma_i \in \{-1, 1\}$.

Ising model on multiple graphs

Given a multiple graph $G_n = (V_n, E_n)$ with n vertices $V_n = \{v_1, \ldots, v_n\}$, we assign to each vertex v_i a spin $\sigma_i \in \{-1, 1\}$. For any configuration $\sigma \in \Omega_n = \{-1, 1\}^n$, the Hamiltonian $H_n(\sigma)$ is given by

$$H_n(\sigma) = -\beta \sum_{i \leq j} k_{i,j} \sigma_i \sigma_j - B \sum_{i \leq n} \sigma_i,$$

where $k_{i,j}$ is the number of edges between v_i and v_j; $\beta > 0$ is the inverse temperature and $B \in \mathbb{R}$ is the uniform external magnetic field.

Ising model on multiple graphs

We remark that

$$\sum_{i \le j} k_{i,j}\sigma_i\sigma_j = e(\sigma_+, \sigma_+) + e(\sigma_-, \sigma_-) - e(\sigma_+, \sigma_-)$$
$$= |E_n| - 2e(\sigma_+, \sigma_-),$$

where

$$\sigma_+ = \{v_i : \sigma_i = 1\} \qquad \text{and} \qquad \sigma_- = \{v_i : \sigma_i = -1\},$$

and

$$e(A, B) = |\{ \text{ edges between } A \text{ and } B\}|.$$

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$$e(A, B) = |\{ \text{ edges between } A \text{ and } B\}|.$$

Therefore,

$$H_n(\sigma) = -\beta\Big(|E_n| - 2e(\sigma_+, \sigma_-)\Big) - B(2|\sigma_+| - n).$$

Configuration model random graphs $CM_n(\mathbf{D})$

- Given a deterministic or random sequence $\mathbf{D} = (D_i)$. Assume that $\sum D_i$ is even, if not we add 1 to a one of D_i. Then we assign D_i half-edges to each vertex v_i.
- Choose pairs of half-edges at random and connect them.

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When

$$D_1 = \ldots = D_n = d,$$

we call $CM_n(\mathbf{D})$ the random d-regular graph and denote it by $G_{n,d}$.

Quenched Boltzmann-Gibbs measure

The *quenched measure* is defined as

$$\mu_n(\sigma) = \frac{e^{-H_n(\sigma)}}{Z_n(\beta, B)} \quad \text{for all} \quad \sigma \in \Omega_n,$$

with

$$Z_n(\beta, B) = \sum_{\sigma \in \Omega_n} e^{-H_n(\sigma)}.$$

Ref. Dembo, Montanari (2010); van der Hofstad et al (2015), den Hollander et al (2017); . . .

Annealed Boltzmann-Gibbs measure

The *annealed measure* is defined as

$$\hat{\mu}_n(\sigma) = \frac{\mathbb{E}(e^{-H_n(\sigma)})}{\mathbb{E}(Z_n(\beta, B))} \quad \text{for all} \quad \sigma \in \Omega_n.$$

Ref. Dommers, Giardinà, Giberti, van der Hofstad, Prioriello (2015, 2016).

AIM $(\Omega_n, \hat{\mu}_n)$: Thermodynamic quantities

(i) The annealed pressure is given by

$$\psi_n(\beta, B) = \frac{1}{n} \log \mathbb{E}(Z_n(\beta, B)).$$

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(i) The annealed pressure is given by
$$\psi_n(\beta, B) = \frac{1}{n} \log \mathbb{E}(Z_n(\beta, B)).$$
(ii) The annealed magnetization is given by
$$\mathcal{M}_n(\beta, B) = \mathbb{E}_{\hat{\mu}_n}\left(\frac{S_n}{n}\right) = \frac{\partial}{\partial B}\psi_n(\beta, B),$$
where $S_n = \sigma_1 + \ldots + \sigma_n$.

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(iii) The annealed susceptibility is given by

$$\chi_n(\beta, B) = \mathrm{Var}_{\hat{\mu}_n}\left(\frac{S_n}{\sqrt{n}}\right) = \frac{\partial^2}{\partial B^2}\psi_n(\beta, B).$$

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Let $\mathcal{M}(\beta, B) = \lim_{n\to\infty} \mathcal{M}_n(\beta, B)$. The critical inverse temperature is

$$\beta_c = \inf\{\beta > 0 : \mathcal{M}(\beta, 0^+) > 0\}.$$

Finally, the region of the existence of the limit magnetization is

$$\mathcal{U} = \{(\beta, B) : \beta > 0, B \neq 0 \text{ or } 0 < \beta < \beta_c, B = 0\}.$$

Problems and known results

1. The limits of thermodynamic quantities.
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Problems and known results

1. The limits of thermodynamic quantities.
2. The value of the critical inverse temperature β_c.
3. The law of large number for S_n on the region $\mathcal{U}$.
4. The central limit theorem for S_n on the region $\mathcal{U}$.
5. Critical behavior of $\mathcal{M}(\beta, B)$ and $\chi(\beta, B)$ when $\beta \to \beta_c$ and $B \to 0$.
6. Critical behavior of S_n when $\beta = \beta_c$ and $B = 0$.

R. van der Hofstad and coauthors consider all $1 - 6$ for inhomogeneous random graphs, configuration model with degrees 1 and 2, random 2-regular graphs.

Contents

Thermodynamic limits

Theorem (1)

Let us consider the Ising model on the random d-regular graph with $d \geq 2$. Then the following assertions hold.

(i) *For all $\beta \geq 0$ and $B \in \mathbb{R}$, the annealed pressure converges*

$$\lim_{n\to\infty} \psi_n(\beta, B) = \psi(\beta, B)$$
$$= \frac{\beta d}{2} - B + \max_{0\leq t\leq 1}\left[t \log\left(\frac{1}{t}\right) + (1-t)\log\left(\frac{1}{1-t}\right) + 2Bt + dF(t)\right],$$

where

$$F(t) = \int_0^{u(t)} \log f(s) ds,$$

with $u(t) = t \wedge (1-t)$ and

$$f(s) = \frac{e^{-2\beta}(1-2s) + \sqrt{1 + (e^{-4\beta} - 1)(1-2s)^2}}{2(1-s)}.$$

Thermodynamic limits

Theorem (1)

(ii) *For all* $(\beta, B) \in \mathcal{U}$*, the magnetization converges*

$$\lim_{n\to\infty} \mathcal{M}_n(\beta, B) = \mathcal{M}(\beta, B) = \frac{\partial}{\partial B}\psi(\beta, B).$$

Moreover, the critical inverse temperature is

$$\beta_c = atanh(1/(d-1)) = \begin{cases} \frac{1}{2}\log\left(\frac{d}{d-2}\right) & \text{if} \quad d \geq 3 \\ \infty & \text{if} \quad d = 2. \end{cases}$$

(iii) *For all* $(\beta, B) \in \mathcal{U}$*, the annealed susceptibility converges*

$$\lim_{n\to\infty} \chi_n(\beta, B) = \chi(\beta, B) = \frac{\partial^2}{\partial B^2}\psi(\beta, B).$$

Thermodynamic limits

Dembo and Montanari have been proved that the quenched pressure converges almost surely to a non random function $\varphi(\beta, B)$. We can show that

$$\psi(\beta, B) = \varphi(\beta, B).$$

Annealed Law of large number

Theorem (2)

Suppose that $(\beta, B) \in \mathcal{U}$. *Then for any* $\varepsilon > 0$, *there exists a positive constant* $L = L(\varepsilon)$, *such that for all sufficiently large* n

$$\hat{\mu}_n\left(\left|\frac{S_n}{n} - \mathcal{M}(\beta, B)\right| > \varepsilon\right) \leq \exp(-nL),$$

where $\mathcal{M}(\beta, B)$ *is defined in Theorem (1).*

Annealed Central limit theorem

Theorem (3)

For all $(\beta, B) \in \mathcal{U}$, the total spin under the annealed measure satisfies a central limit theorem:

$$\frac{S_n - \mathbb{E}_{\hat{\mu}_n}(S_n)}{\sqrt{n}} \xrightarrow{(\mathcal{D})} \mathcal{N}(0, \chi(\beta, B)) \qquad w.r.t.\ \hat{\mu}_n,$$

where $\chi(\beta, B)$ is defined in Theorem (1) and $\mathcal{N}(0, \chi)$ denotes a centered Gaussian random variable with variance χ.

Annealed Law of large number

Theorem (4)

Suppose that $\beta > \beta_c$ *and* $B = 0$. *Then as* $n \to \infty$,

$$\hat{\mu}_n\left(\left|\frac{S_n}{n} - \nu\right| \le n^{-1/6}\right) \longrightarrow 1/2 \quad \text{and} \quad \hat{\mu}_n\left(\left|\frac{S_n}{n} + \nu\right| \le n^{-1/6}\right) \longrightarrow 1/2,$$

with $\nu = \mathcal{M}(\beta, 0^+) > 0$.

Critical exponents

The annealed critical exponents $\boldsymbol{\beta}, \boldsymbol{\delta}, \boldsymbol{\gamma}, \boldsymbol{\gamma'}$ are defined by:

$$\begin{aligned}
\mathcal{M}(\beta, 0^+) &\asymp (\beta - \beta_c)^{\boldsymbol{\beta}} && \text{for } \beta \searrow \beta_c, \\
\mathcal{M}(\beta_c, B) &\asymp B^{1/\boldsymbol{\delta}} && \text{for } B \searrow 0, \\
\chi(\beta, 0^+) &\asymp (\beta_c - \beta)^{-\boldsymbol{\gamma}} && \text{for } \beta \nearrow \beta_c, \\
\chi(\beta, 0^+) &\asymp (\beta - \beta_c)^{-\boldsymbol{\gamma'}} && \text{for } \beta \searrow \beta_c,
\end{aligned}$$

where we write $f(x) \asymp g(x)$ if the ratio $f(x)/g(x)$ is bounded from 0 and infinity for the specified limit.

Annealed critical exponents

Theorem (5)

Let us consider the annealed Ising model on random d-regular graph with $d \geq 3$. Then the critical exponents are given by

$$\begin{aligned}\beta &= 1/2\\ \delta &= 3\\ \gamma = \gamma' &= 1.\end{aligned}$$

Annealed critical exponents

Theorem (5)

Let us consider the annealed Ising model on random d-regular graph with $d \geq 3$. Then the critical exponents are given by

$$\begin{aligned} \beta &= 1/2 \\ \delta &= 3 \\ \gamma = \gamma' &= 1. \end{aligned}$$

Note that these critical exponents are equal to the ones of Curie-Weiss model.

Non standard limit theorem at criticality

Theorem (6)

Consider the annealed Ising model on random d-regular graphs with $d \geq 3$. Suppose that $\beta = \beta_c$ and $B = 0$. Then

$$\frac{S_n}{n^{3/4}} \xrightarrow{(\mathcal{D})} X \qquad \text{w.r.t. } \hat{\mu}_n,$$

where X is a random variable with density proportional to

$$\exp\left(\frac{-(d-1)(d-2)x^4}{12d^2}\right).$$

Contents

Annealed pressure

Recall that

$$H_n(\sigma) = -\beta\Big(\frac{dn}{2} - 2e(\sigma_+, \sigma_-)\Big) - B(2|\sigma_+| - n).$$

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Observe that if $|\sigma_+| = |\sigma'_+|$ then $e(\sigma_+, \sigma_-) =^{(\mathcal{D})} e(\sigma'_+, \sigma'_-)$.

Annealed pressure

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Observe that if $|\sigma_+| = |\sigma'_+|$ then $e(\sigma_+,\sigma_-) =^{(\mathcal{D})} e(\sigma'_+,\sigma'_-)$. Hence

$$\begin{aligned}\mathbb{E}(Z_n(\beta,B)) = \sum_{\sigma\in\Omega_n} \mathbb{E}\left(e^{-H_n(\sigma)}\right) \quad &= \quad \sum_{j=0}^{n} \sum_{\sigma:|\sigma_+|=j} \mathbb{E}\left(e^{-H_n(\sigma)}\right) \\ &= \quad e^{\left(\frac{\beta d}{2}-B\right)n} \sum_{j=0}^{n} \binom{n}{j} e^{2Bj} \mathbb{E}\left(e^{-2\beta e(U_j,U_j^c)}\right),\end{aligned}$$

where

$$U_j = \{v_1,\ldots,v_j\}.$$

Annealed pressure

We have

$$e(U_j, U_j^c) \overset{(\mathcal{D})}{=} X(dj, dn),$$

where for $k \leq m$,

$$X(k, m) = \#\{ \text{ edges in } G_{m,1} \text{ between } \{w_1, \ldots, w_k\} \text{ and } \{w_{k+1}, \ldots, w_m\}\}.$$

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In conclusion,

$$\mathbb{E}(Z_n(\beta, B)) = e^{\left(\frac{\beta d}{2} - B\right)n} \times \sum_{j=0}^{n} \binom{n}{j} e^{2Bj} g(\beta, dj, dn),$$

where

$$g(\beta, k, m) = \mathbb{E}\Big(\exp\Big(-2\beta X(k,m)\Big)\Big).$$

Annealed pressure

$$\mathbb{E}(Z_n(\beta, B)) = e^{\left(\frac{\beta d}{2} - B\right)n} \times \sum_{j=0}^{n} \binom{n}{j} e^{2Bj} g(\beta, dj, dn).$$

Lemma

There exists a positive constant C*, such that*

$$\max_{0 \le k \le m} \left| \frac{\log g(\beta, k, m)}{m} - F(\beta, k/m) \right| \le \frac{C}{m},$$

with $F(\beta, k/m)$ *as in Theorem (1).*

Contents

Annealed Ising model on configuration model random graphs

Let G_n be the configuration model $CM_n(\mathbf{D})$, where $\mathbf{D} = (D_i)$ is an i.i.d sequence of random variables.

1. Thermodynamic limits: annealed pressure is done; but magnetization, susceptibility?
2. Limit theorems ?
3. Critical behaviors?

Proposition (7)

Consider the annealed Ising model on $CM_n(\boldsymbol{D})$ *with* $\boldsymbol{D} = (D_i)$ *an i.i.d sequence of random variables satisfying* $\mathbb{E}(e^{sD_1}) < \infty \; \forall \, s \in \mathbb{R}$.

Proposition (7)

Consider the annealed Ising model on $CM_n(\boldsymbol{D})$ *with* $\boldsymbol{D} = (D_i)$ *an i.i.d sequence of random variables satisfying* $\mathbb{E}(e^{sD_1}) < \infty \; \forall \, s \in \mathbb{R}$.
For all $\beta > 0$ *and* $B \in \mathbb{R}$, *the annealed pressure* $\psi_n(\beta, B)$ *converges to a limit given by*

$$\psi(\beta, B) = -B + \max_{0 \le t \le 1} \left[t \log\left(\frac{1}{t}\right) + (1-t)\log\left(\frac{1}{1-t}\right) + 2Bt + G(\beta, t) \right],$$

where

$$G(\beta, t) = \sup_{a,b} \left\{ b(\beta/2 + F(a/b)) - t\Lambda^* \left(\frac{a}{t}\right) - (1-t)\Lambda^* \left(\frac{b-a}{1-t}\right) \right\},$$

with $F(t)$ *as in Theorem (1) and*

$$\Lambda^*(x) = \sup_{s \in \mathbb{R}} \{ xs - \log \mathbb{E}(e^{sD_1}) \}.$$

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