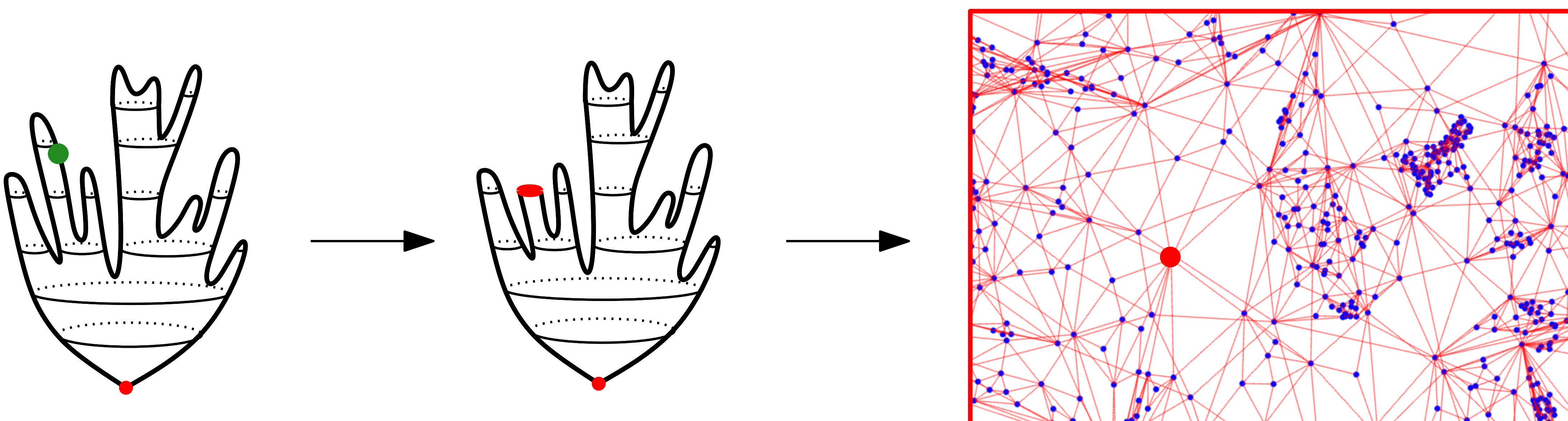
Triangulations with spins : algebraicity and local limit

Marie Albenque (CNRS and LIX)

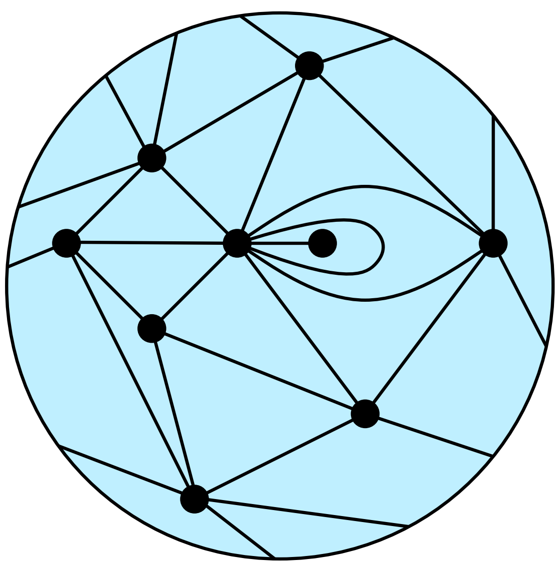
joint work with **Laurent Ménard** (Paris Nanterre)
and **Gilles Schaeffer** (CNRS and LIX)



Dynamics on random graphs, October 2017

Planar Maps as discrete planar metric spaces

Definition:
A **planar map** is a proper embedding of a finite connected graph into the two-dimensional sphere (considered up to orientation-preserving homeomorphisms of the sphere).



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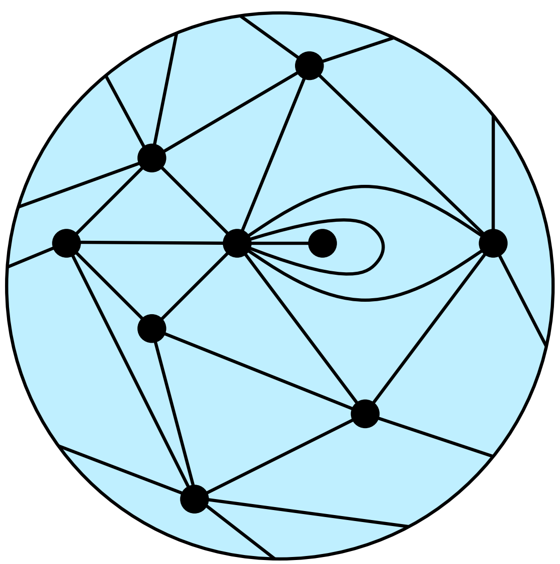

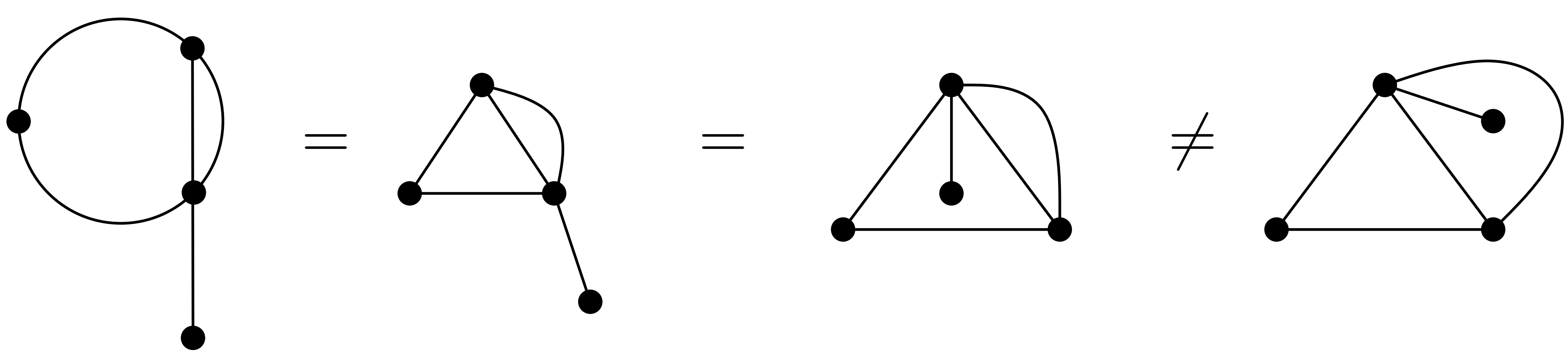

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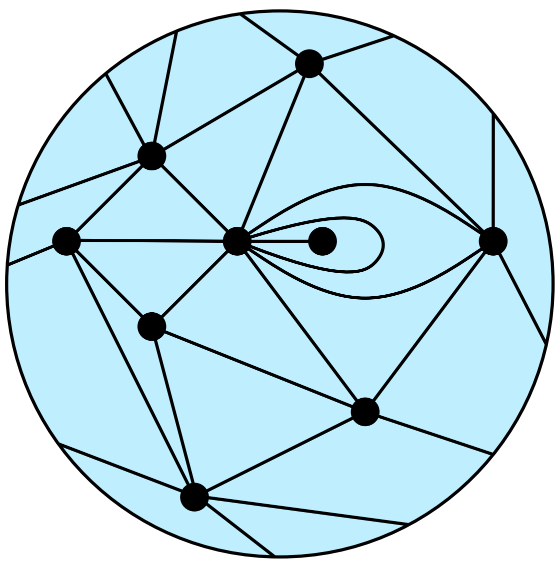
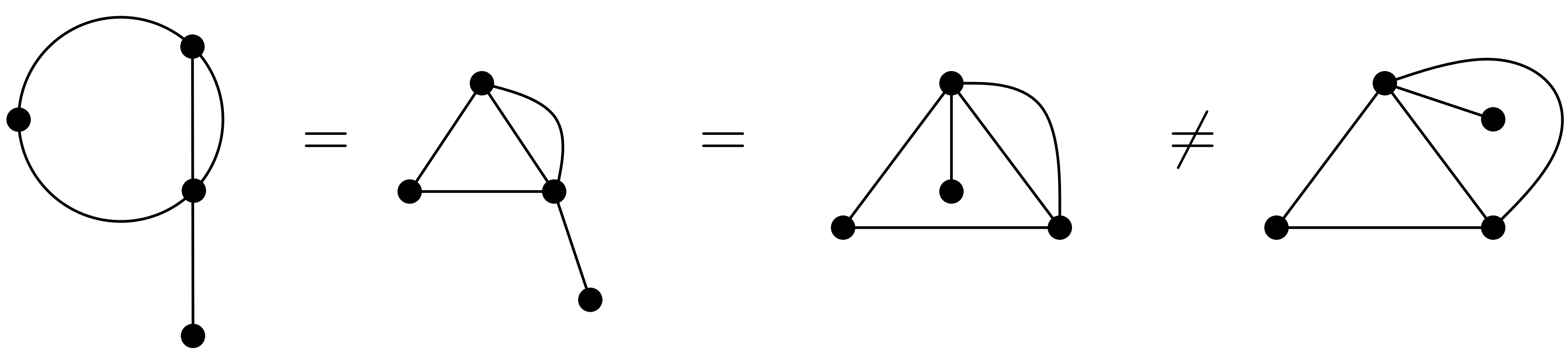


faces: connected components of the complement of edges

p-angulation: each face is bounded by p edges



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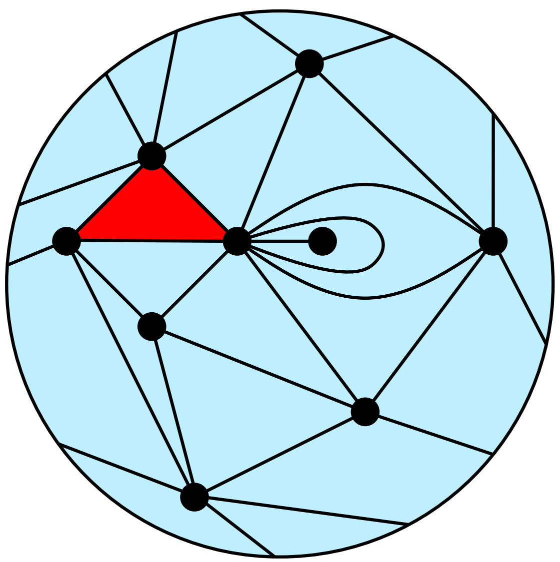

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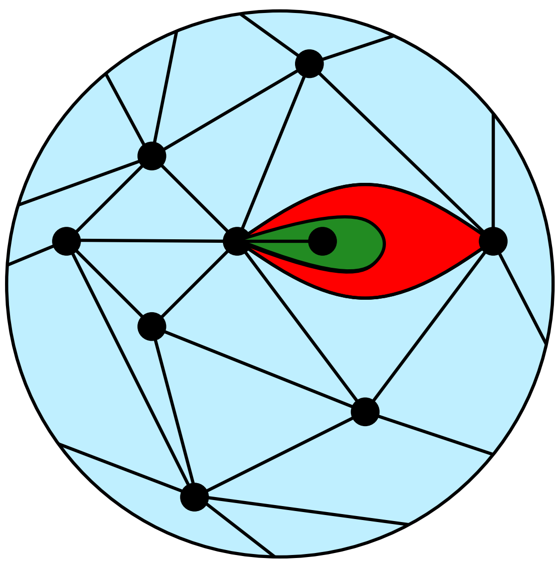

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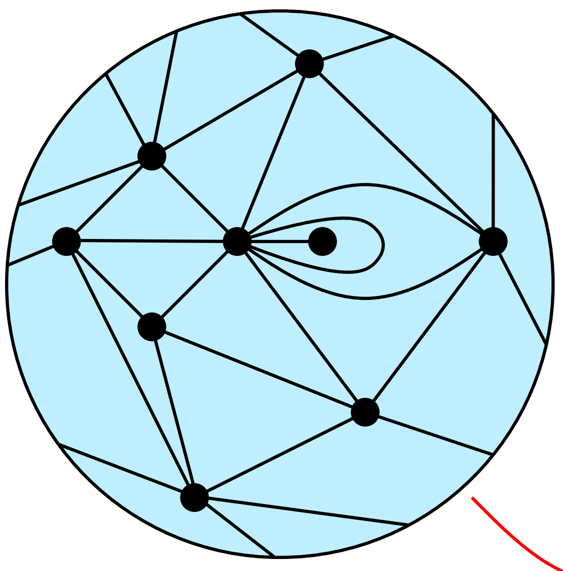

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This is a triangulation

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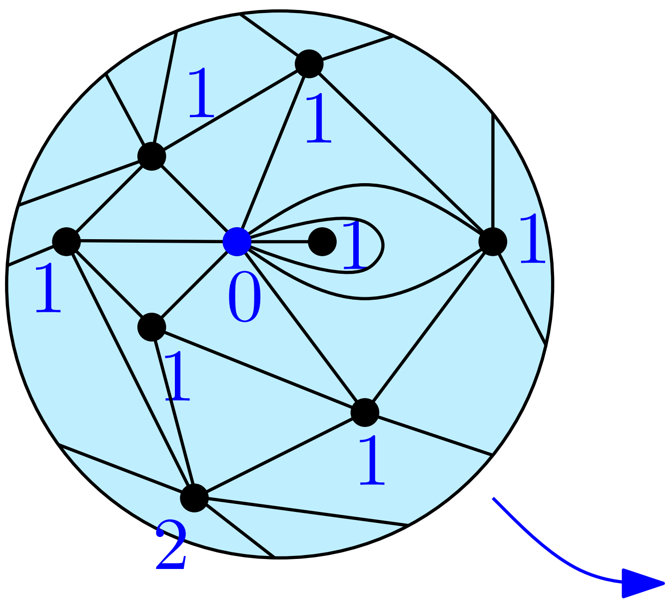

In blue, distances from •

M Planar Map:

- $V(M) :=$ set of vertices of M
- $d_{gr} :=$ graph distance on $V(M)$
- $(V(M), d_{gr})$ is a (finite) metric space

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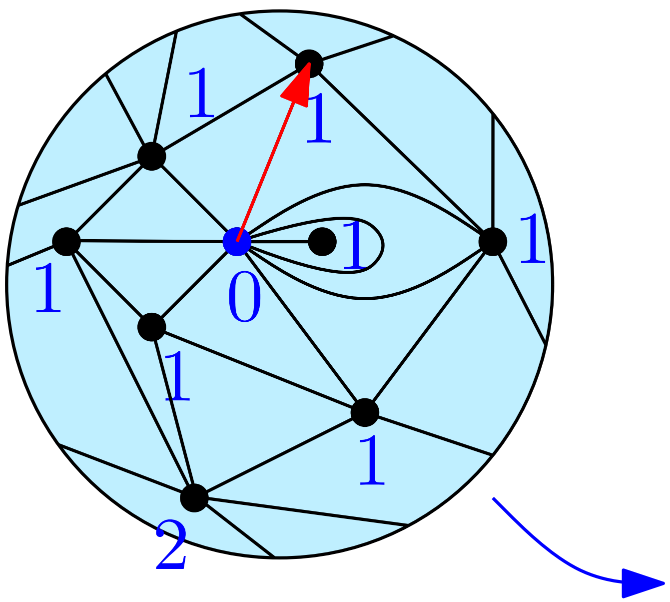

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Rooted map: mark an oriented edge of the map →

"Classical" large random triangulations

Take a triangulation with n edges uniformly at random. What does it look like if n is large ?

Two points of view : global/local, continuous/discrete

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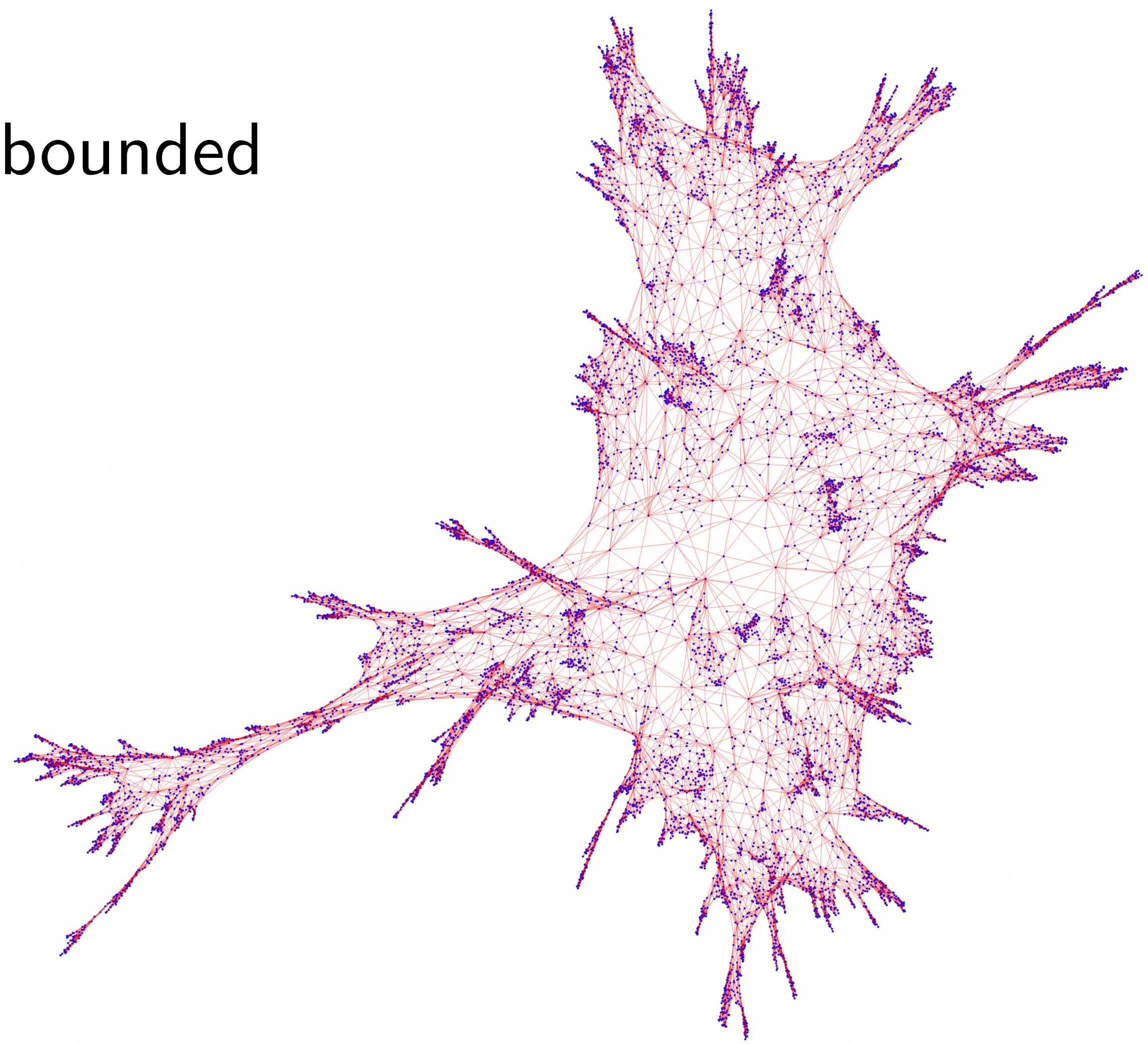
Global :
Rescale distances to keep diameter bounded

[Le Gall 13, Miermont 13] :
converges to the **Brownian map**



- Gromov-Hausdorff topology
- Continuous metric space
- Homeomorphic to the sphere
- Hausdorff dimension 4
- **Universality**

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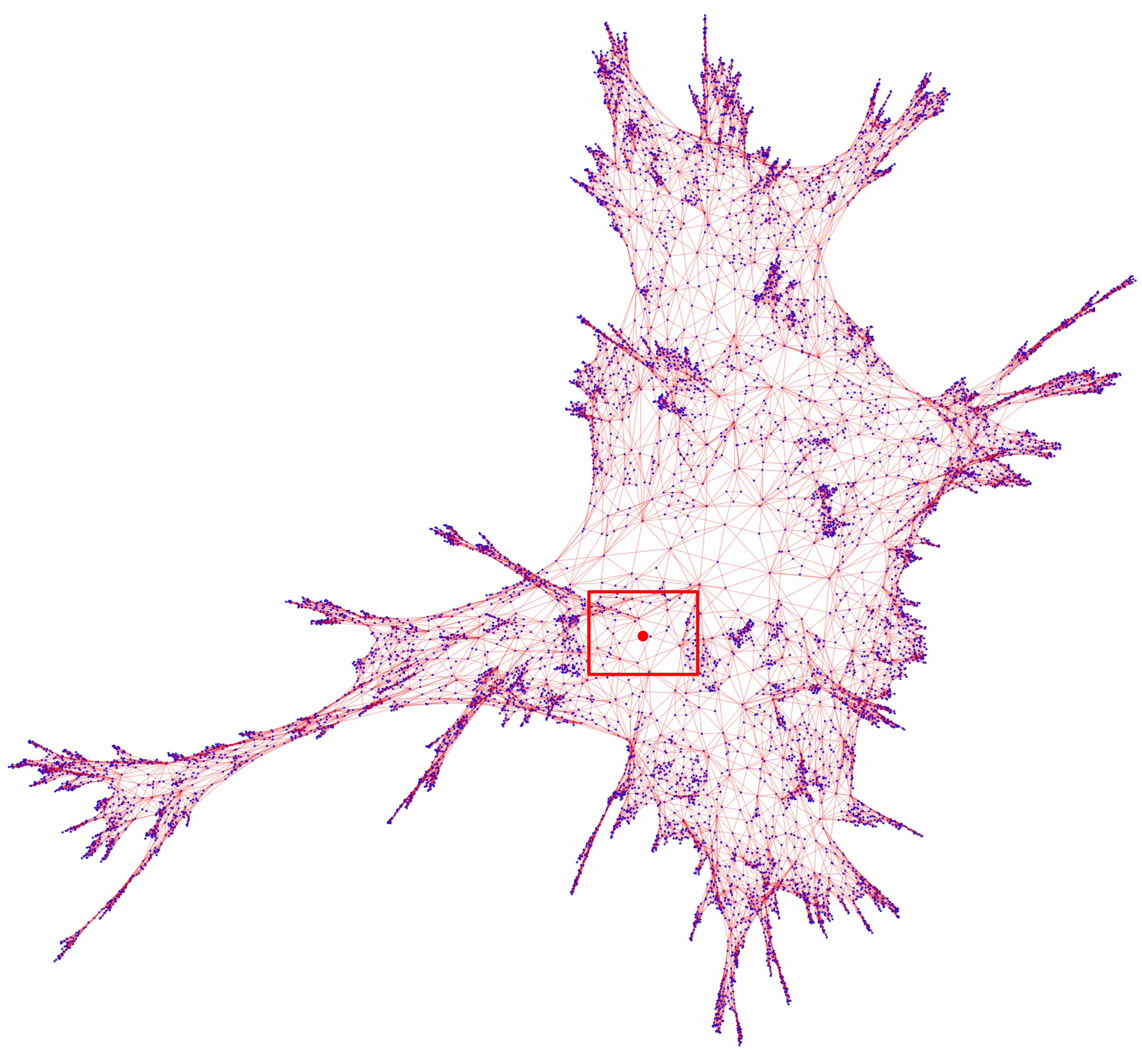
Local :
Don't rescale distances and look at neighborhoods of the root



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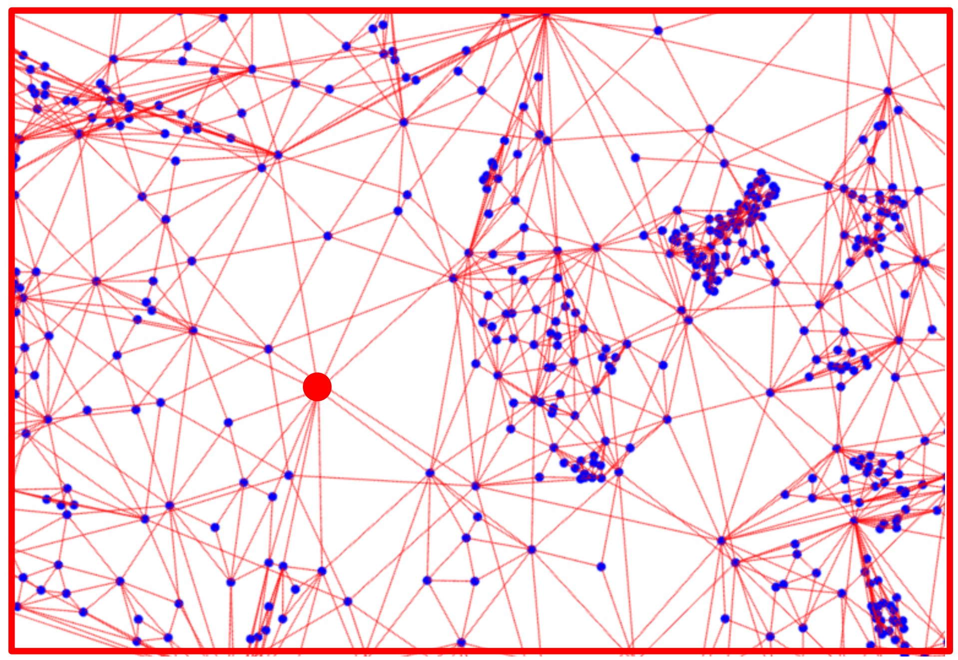
[Angel – Schramm 03, Krikun 05] :
Converges to the **Uniform Infinite Planar Triangulation**



- Local topology
- Metric balls of radius R grow like R^4
- **"Universality"** of the exponent 4.

Adding matter: Ising model on triangulations

How does Ising model influence the underlying map?

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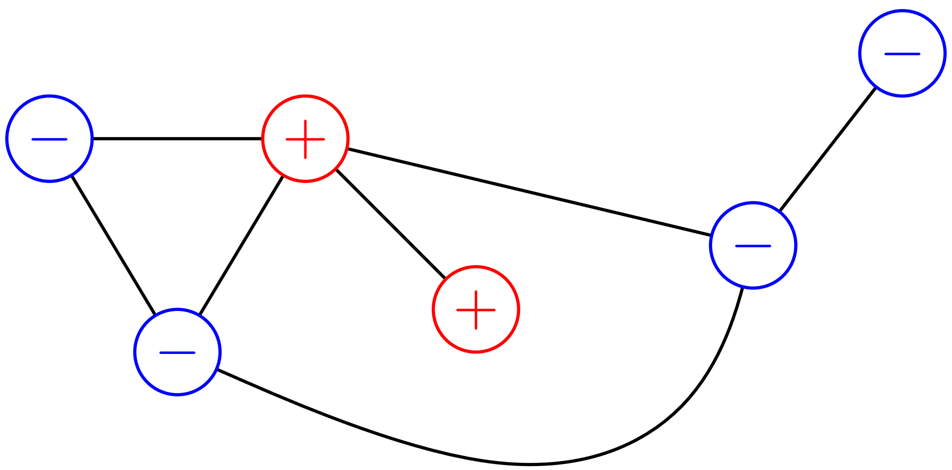
$G = (V, E)$ finite graph



Spin configuration on G:

$$\sigma : V \to \{-1, +1\}.$$

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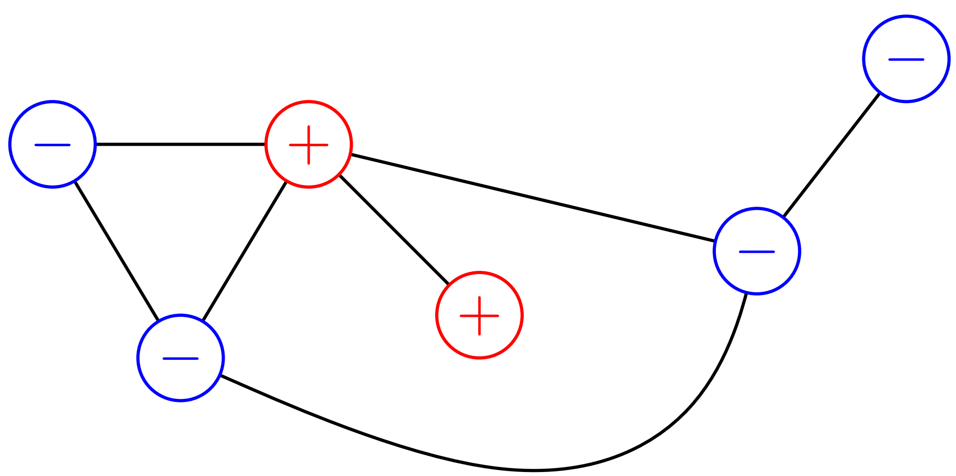

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Ising model on G: take a random spin configuration with probability

$$P(\sigma) \propto e^{-\frac{\beta}{2} \sum_{v \sim v'} \mathbf{1}_{\{\sigma(v) \neq \sigma(v')\}}}$$

$\beta > 0$: inverse temperature.

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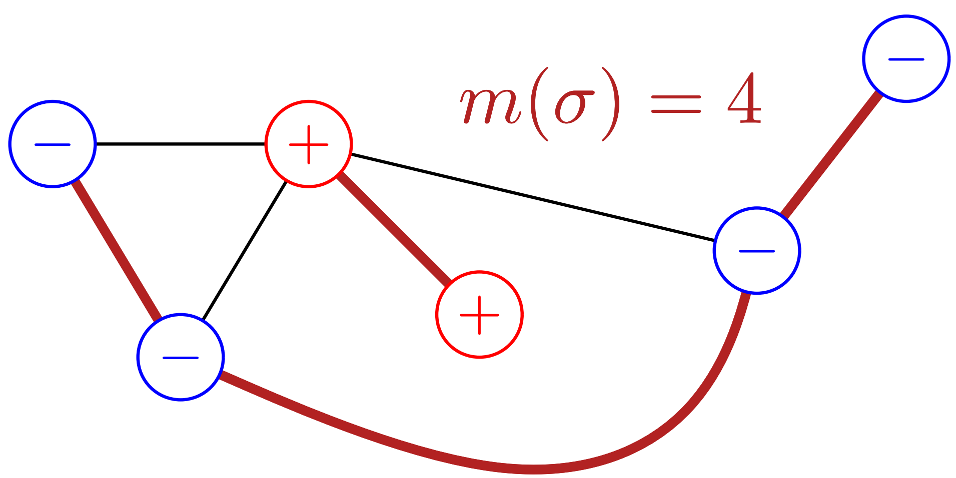

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Combinatorial formulation: $P(\sigma) \propto \nu^{m(\sigma)}$
with $m(\sigma) =$ number of monochromatic edges and $\nu = e^{\beta}$.

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$\mathcal{T}_n = \{$rooted planar triangulations with $3n$ edges$\}$.

Random triangulation with spins in $\mathcal{T}_n$ with probability $\propto \nu^{m(T,\sigma)}$?

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$$\mathbb{P}_n^\nu\Big(\{(T,\sigma)\}\Big) = \frac{\nu^{m(T,\sigma)}}{[t^{3n}]Q(\nu,t)}.$$

where $Q(\nu,t) =$ generating series of **Ising-weighted triangulations**:

$$Q(\nu,t) = \sum_{T \in \mathcal{T}_f} \sum_{\sigma : V(T) \to \{-1,+1\}} \nu^{m(T,\sigma)} t^{e(T)}.$$

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Theorem [A. – Ménard – Schaeffer]
As $n\to\infty$, the sequence $\mathbb{P}_n^\nu$ converges **weakly** to a probability measure $\mathbb{P}^\nu$ for the **local topology**.
The measure $\mathbb{P}^\nu$ is supported on infinite triangulations with **one end**.

Adding matter: New asymptotic behavior

Counting exponent for undecorated maps:

coeff $[t^n]$ of generating series of (undecorated) maps
(e.g.: triangulations, quadrangulations, general maps, simple maps,...)
$\sim \kappa\rho^{-n} n^{-\mathbf{5/2}}$

Note : κ and ρ depend on the combinatorics of the model.

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Note : κ and ρ depend on the combinatorics of the model.

Theorem [Bernardi – Bousquet-Mélou 11]
For every ν the series $Q(\nu, t)$ is algebraic, has $\rho_\nu > 0$ as unique dominant singularity and satisfies

$$[t^{3n}]Q(\nu,t) \underset{n\to\infty}{\sim} \begin{cases} \kappa\, \rho_{\nu_c}^{-n}\, n^{-7/3} & \text{if } \nu = \nu_c = 1 + \frac{1}{\sqrt{7}}, \\ \kappa\, \rho_{\nu}^{-n}\, n^{-5/2} & \text{if } \nu \neq \nu_c. \end{cases}$$

This suggests an unusual behavior of the underlying maps for $\nu = \nu_c$.
See also [Boulatov – Kazakov 1987], [Bousquet-Melou – Schaeffer 03] and [Bouttier – Di Francesco – Guitter 04].

Adding matter: Watabiki's (controversial?) predictions

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coeff $[t^n]$ of generating series of (decorated) maps $\sim \kappa \rho^{-n} n^{-\alpha}$

Central charge :

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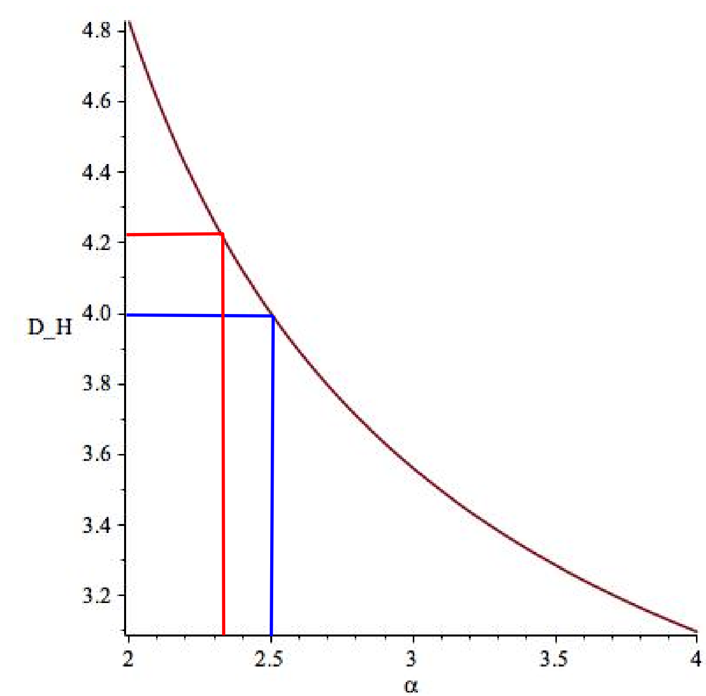
Hausdorff dimension : [Watabiki 93]

$$D_H = 2\frac{\sqrt{25-c}+\sqrt{49-c}}{\sqrt{25-c}+\sqrt{1-c}}$$

- $\alpha = 5/2$ gives $D_H = 4$
- $\alpha = 7/3$ gives $D_H = \frac{7+\sqrt{97}}{4} \approx 4.21$



Local convergence of triangulations with spins

Probability measure on triangulations of $\mathcal{T}_n$ with a spin configuration:

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Local topology

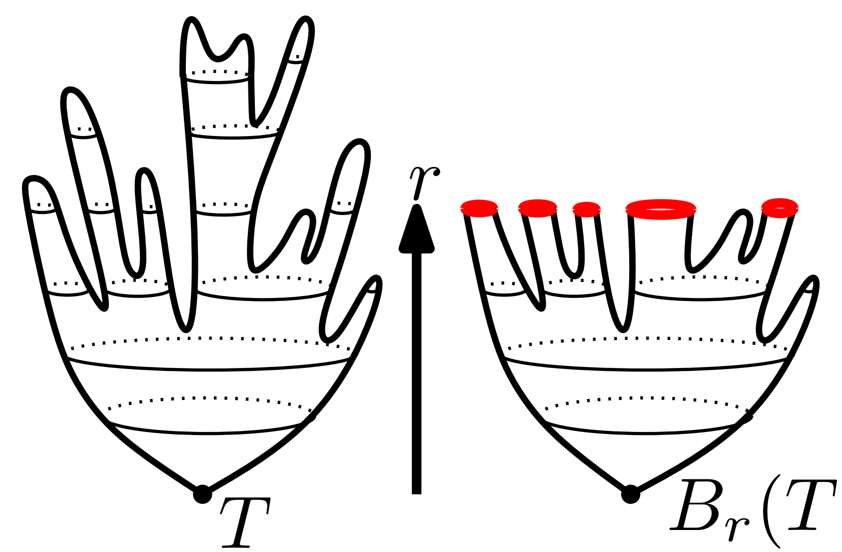
$$\mathcal{T}_f := \{\text{finite rooted planar triangulations with spins}\}.$$

Definition:

The **local topology** on $\mathcal{T}_f$ is induced by the distance:

$$d_{loc}(T, T') := (1 + \max\{r \geq 0 : B_r(T)$$

where $B_r(T)$ is the submap (with spins) of T composed by the faces of T with a vertex at distance $< r$ from the root.



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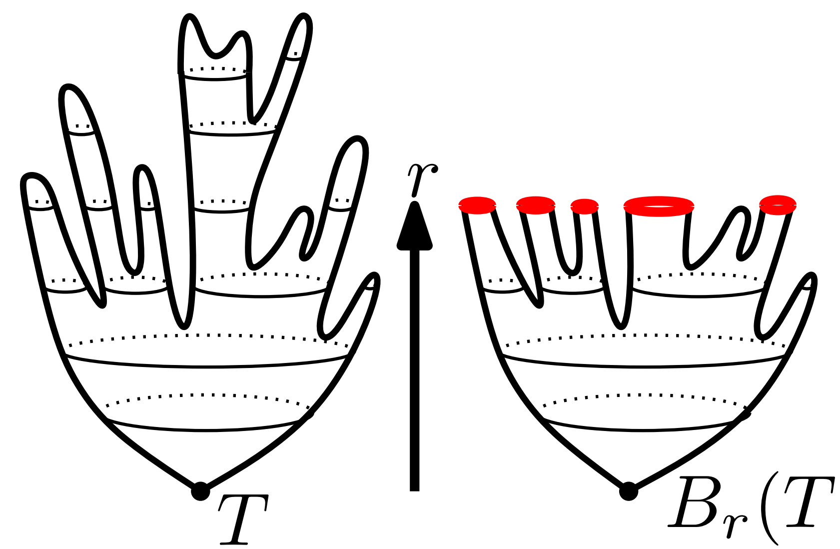

- $(\mathcal{T}, d_{loc})$: closure of $(\mathcal{T}_f, d_{loc})$. It is a **Polish** space.
- $\mathcal{T}_\infty := \mathcal{T} \setminus \mathcal{T}_f$ set of **infinite** planar triangulations with spins.

Weak convergence for the local topology

Portemanteau theorem + Levy – Prokhorov metric:

To show that $\mathbb{P}_n^\nu$ converges weakly to $\mathbb{P}^\nu$, prove

1. For every $r > 0$ and every possible ball Δ, show:

$$\mathbb{P}_n^\nu\left(\{(T,v) \in \mathcal{T}_n \,:\, B_r(T,v) = \Delta\} \right) \underset{n\to\infty}{\longrightarrow} \mathbb{P}^\nu\left(\{T \in \mathcal{T}_\infty \,:\, B_r(T) = \Delta\} \right).$$

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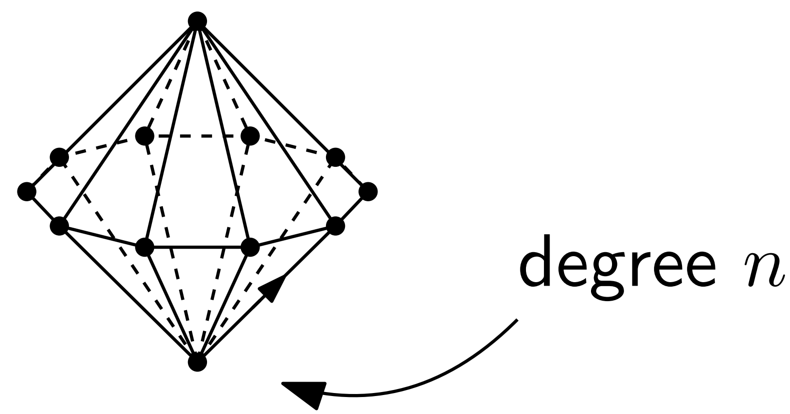

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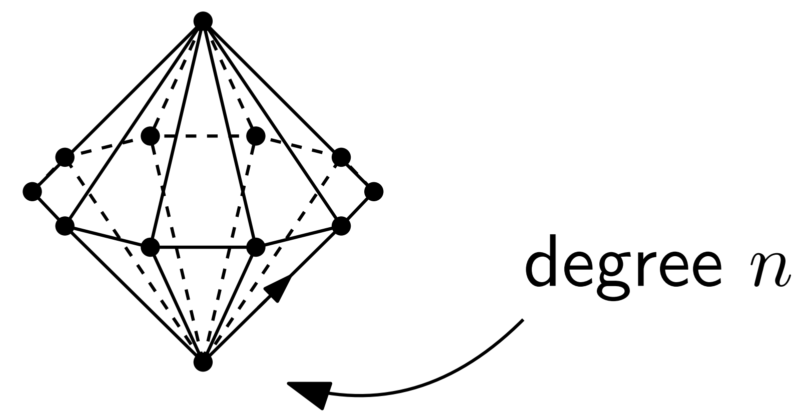

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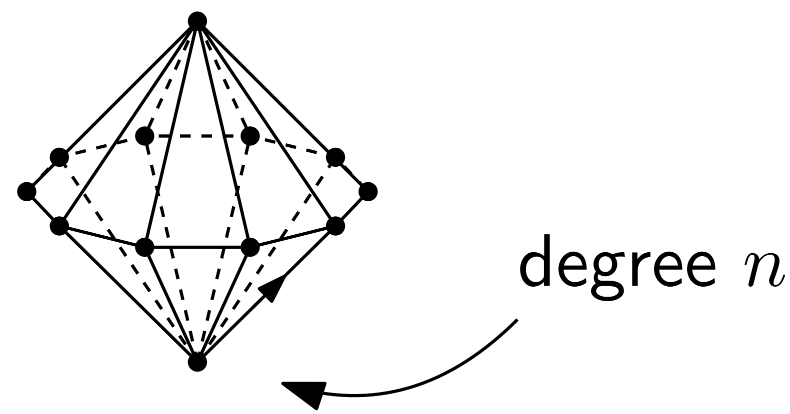

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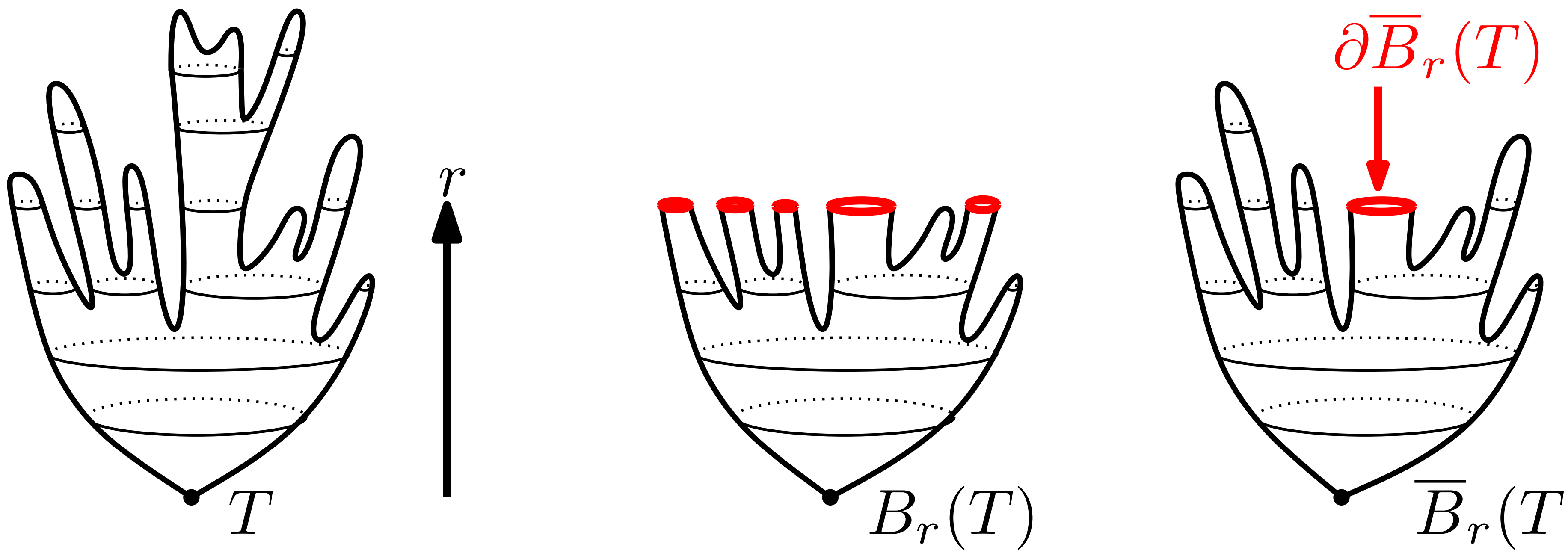
$$\forall r \geq 0, \qquad \sum_{r-\text{balls } \Delta} \mathbb{P}^\nu\Big(\{T \in \mathcal{T}_\infty \,:\, B_r(T) = \Delta\} \Big) = 1.$$

Local topology: Hulls

Balls $B_r(T)$ not practical (multiple holes). Take **hulls** instead:

$$\overline{B}_r(T) := \begin{array}{l}\text{everything not in the largest connected} \\ \text{component of } T \setminus B_r(T)\end{array}$$



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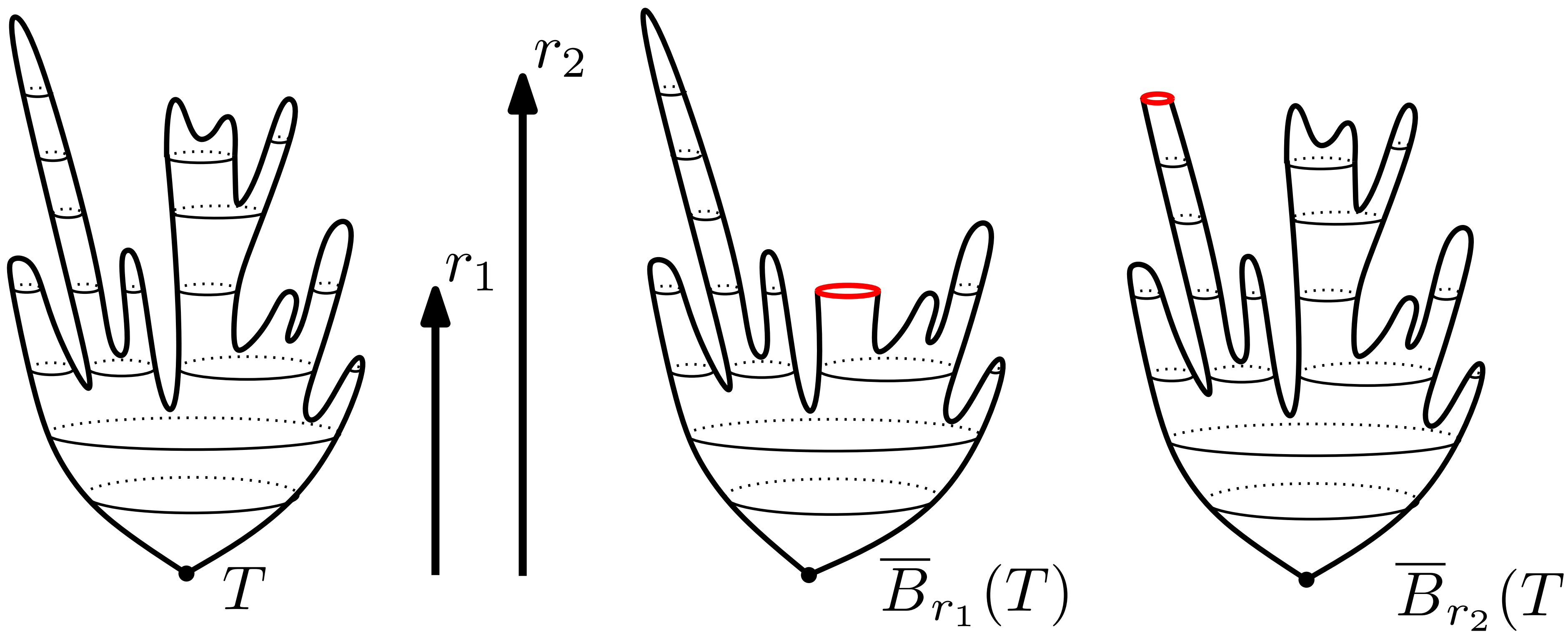

Problem: Hulls are not **nested** !

Local topology: Pointed hulls

For $(T,v) \in \mathcal{T}_f^\bullet :=$ { finite rooted triangulations with pointed vertex }:

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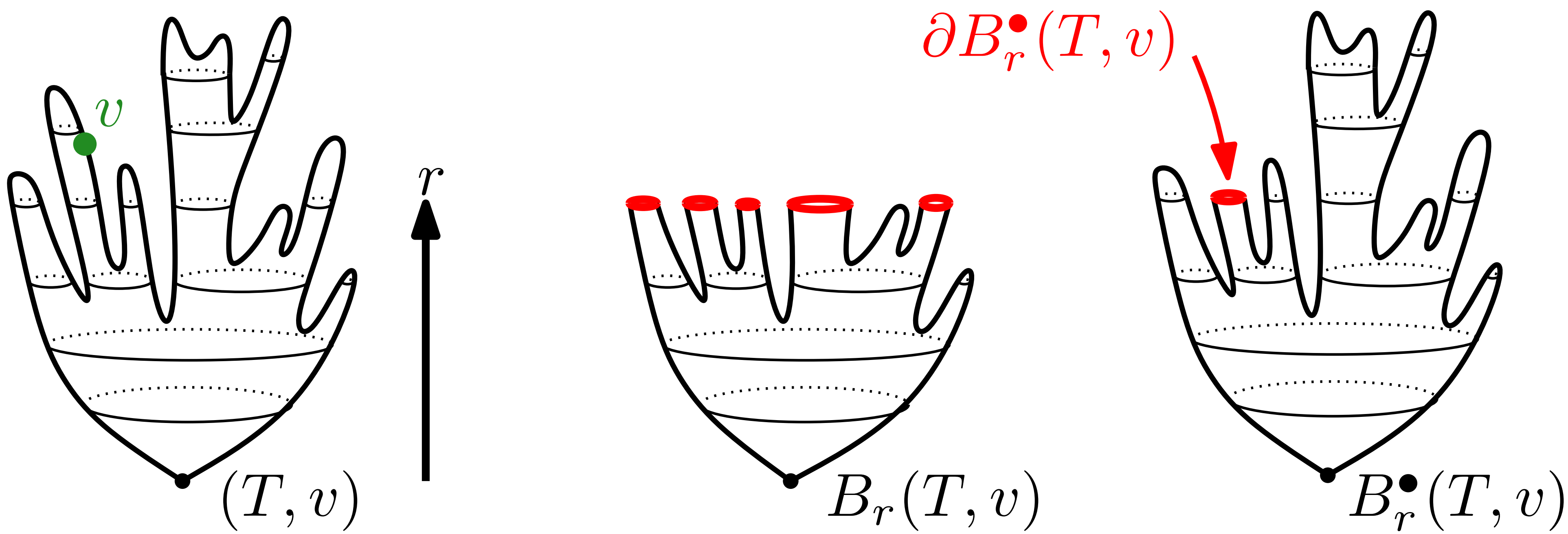

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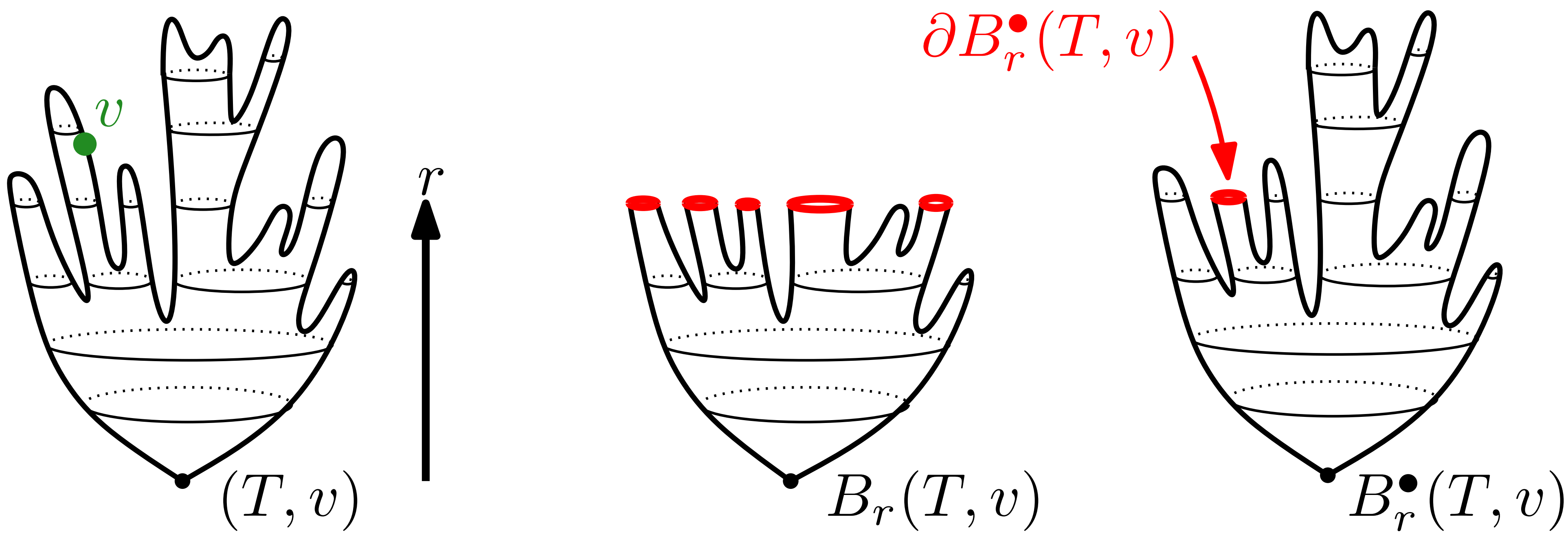

Convergence for $d_{loc}^\bullet \Rightarrow$ convergence for d_{loc} with the same limit.

Local convergence and generating series

Need to evaluate, for every possible hull Δ

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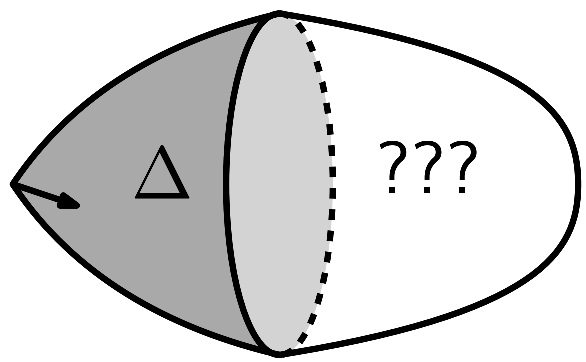

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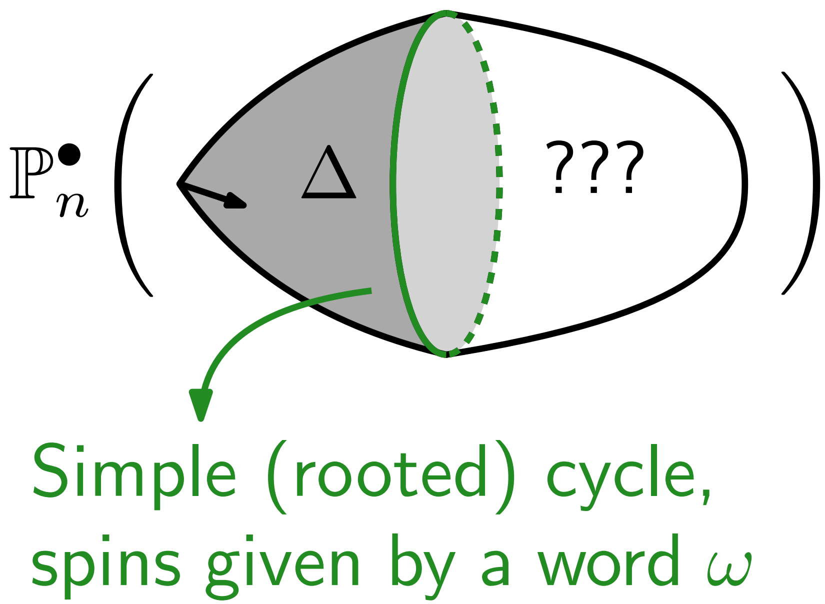

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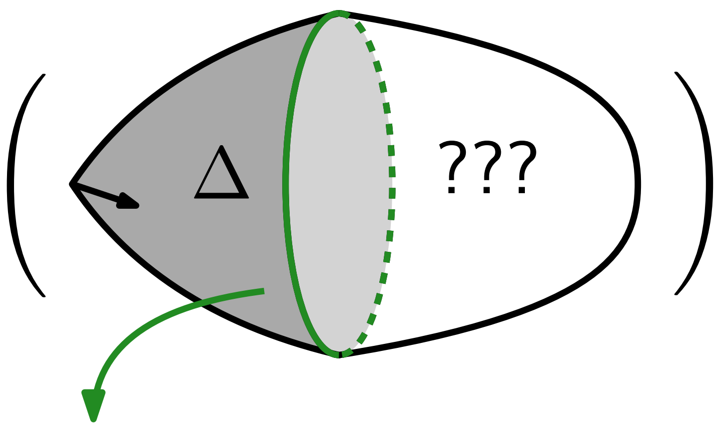

$$\mathbb{P}_n^\bullet\left(\dots \right) = \frac{\nu^{m(\Delta)-m(\omega)}\,[t^{3n-e(\Delta)+|\omega|}]\mathbf{Z}^\bullet_\omega(\nu,t)}{[t^{3n}]Q^\bullet(\nu,t)}$$

Simple (rooted) cycle, spins given by a word ω

Generating series of triangulations with simple boundary ω

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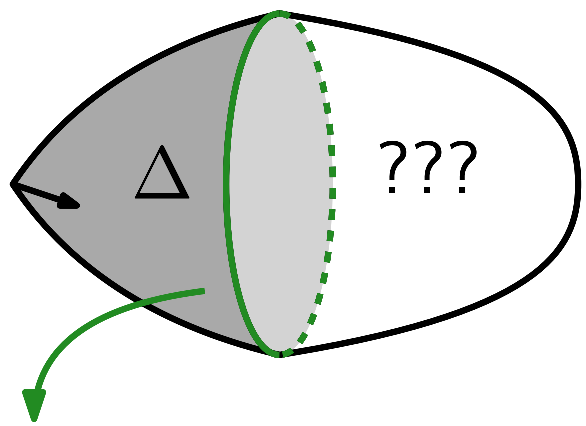

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Theorem [A. – Ménard – Schaeffer]
For every ω, the series $t^{|\omega|}Z_\omega(\nu,t)$ is algebraic, has ρ_ν as unique dominant singularity and satisfies

$$[t^{3n}]t^{|\omega|}Z_\omega(\nu,t) \underset{n\to\infty}{\sim} \begin{cases} \kappa_\omega(\nu_c)\,\rho_{\nu_c}^{-n}\,n^{-7/3} & \text{if } \nu=\nu_c=1+\frac{1}{\sqrt{7}}, \\ \kappa_\omega(\nu)\,\rho_\nu^{-n}\,n^{-5/2} & \text{if } \nu\neq\nu_c. \end{cases}$$

Triangulations with simple boundary

Fix a word ω, with injections from and into triangulations of the sphere:

$$[t^{3n}]t^{|\omega|}Z_\omega = \Theta\left(\rho_\nu^{-n} n^{-\alpha}\right), \text{ with } \alpha = 5/2 \text{ of } 7/3 \text{ depending on } \nu.$$

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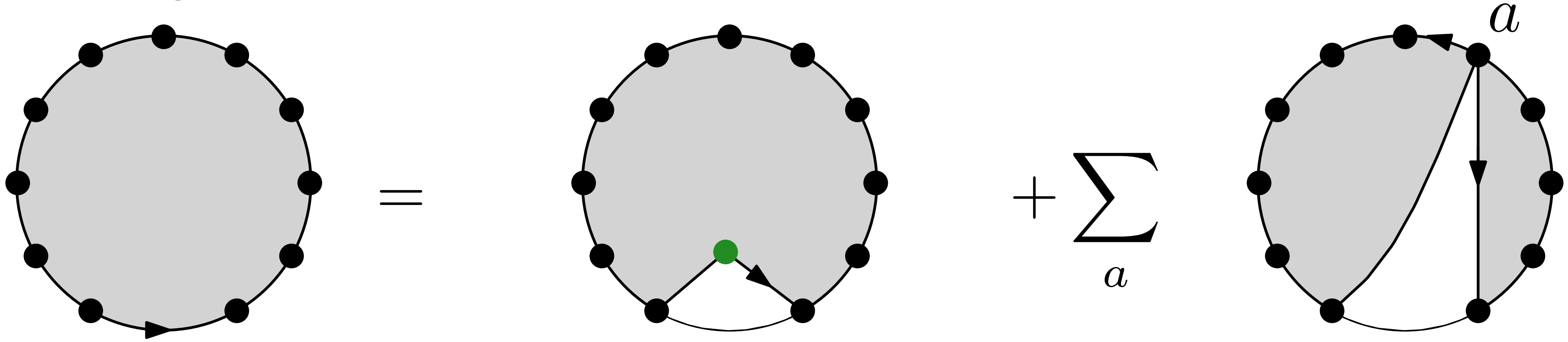
Peeling equation :



$$|\omega| \leq 3, \qquad Z_\omega = \left(Z_{\oplus\omega} + Z_{\ominus\omega} + \sum_{\omega=\omega_1 a \omega_2} Z_{a\omega_1} \cdot Z_{a\omega_2} \right) \times \nu^{\mathbf{1}_{\overleftarrow{\omega}=\overrightarrow{\omega}}}\, t$$

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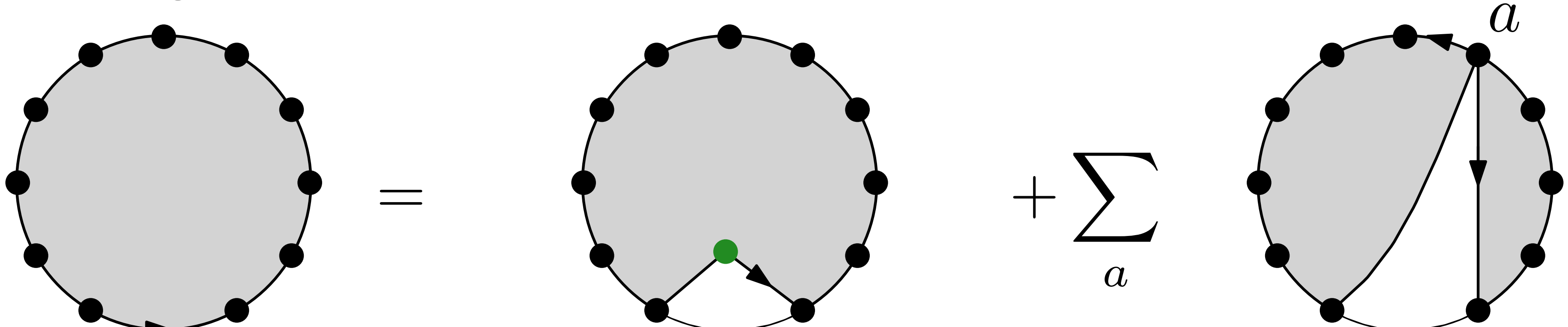

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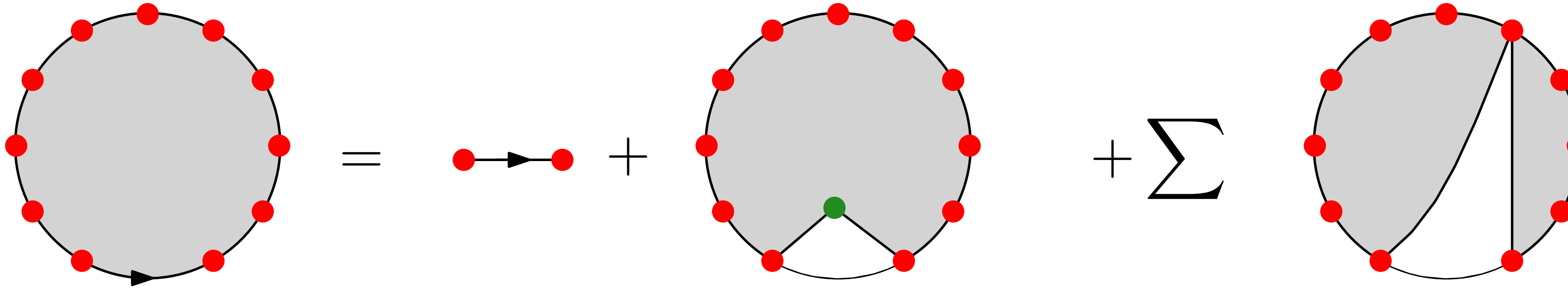
Double recursion on $|\omega|$ and number of $\ominus$'s :
enough to prove 1. and 2. for the $t^p Z_{\oplus^p}$'s

Positive boundary conditions : two catalytic variables



$$A(x) := \sum_{p\geq 1} Z_{\oplus^p} x^p = \nu t x^2 + \qquad\qquad + \frac{\nu t}{x}(A(x))^2$$

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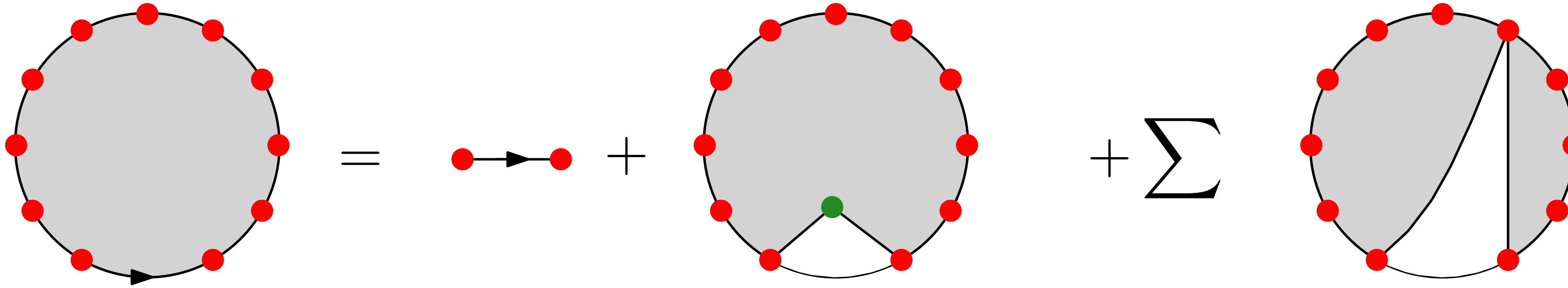

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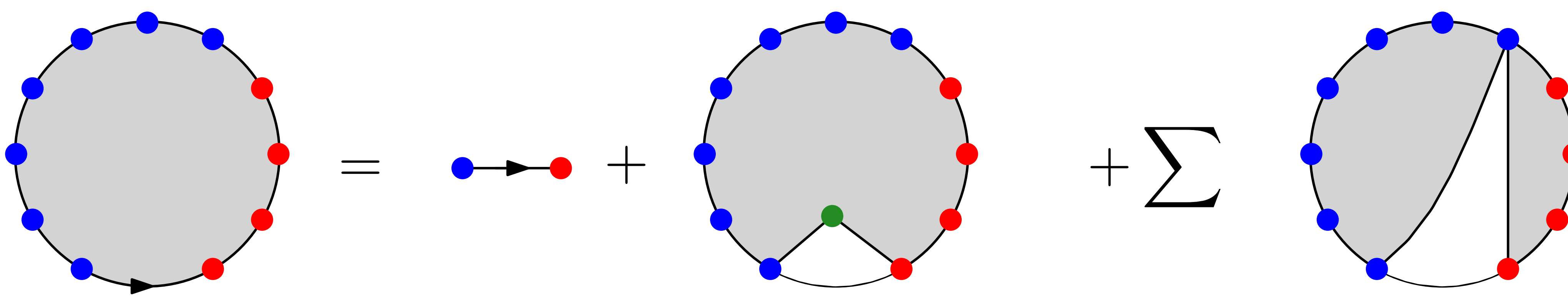

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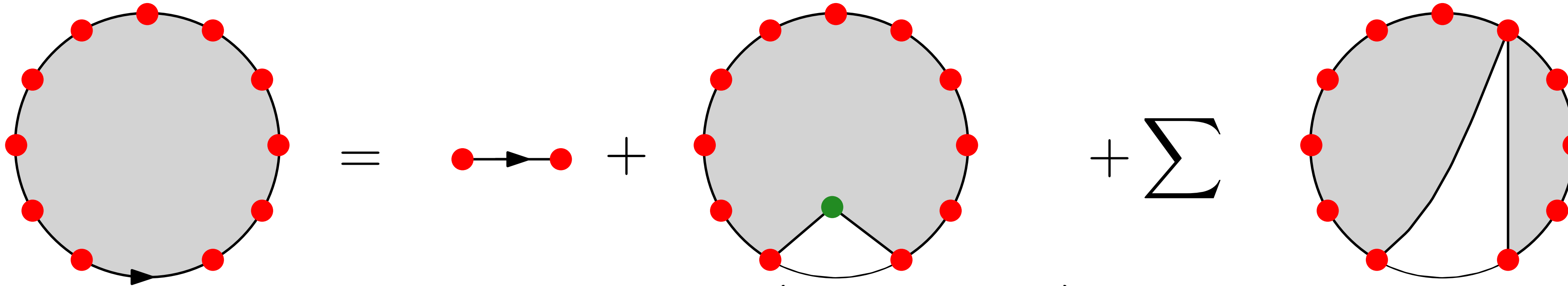

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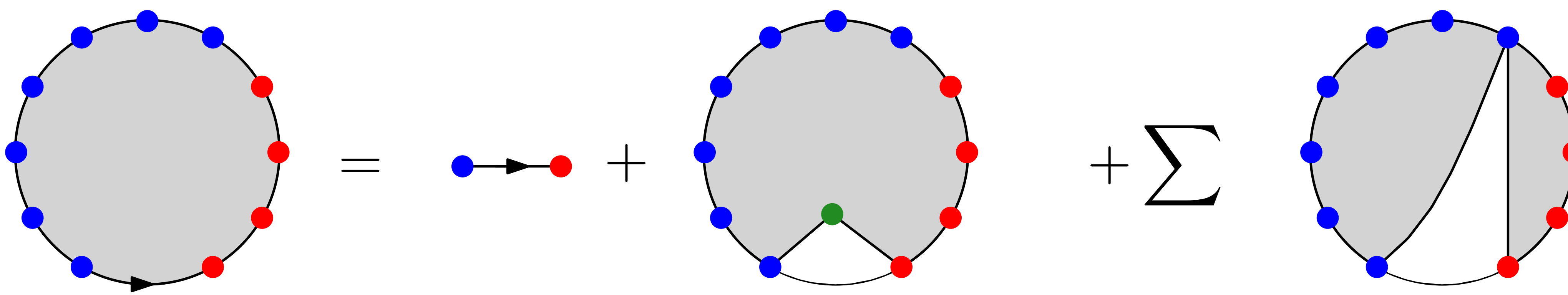

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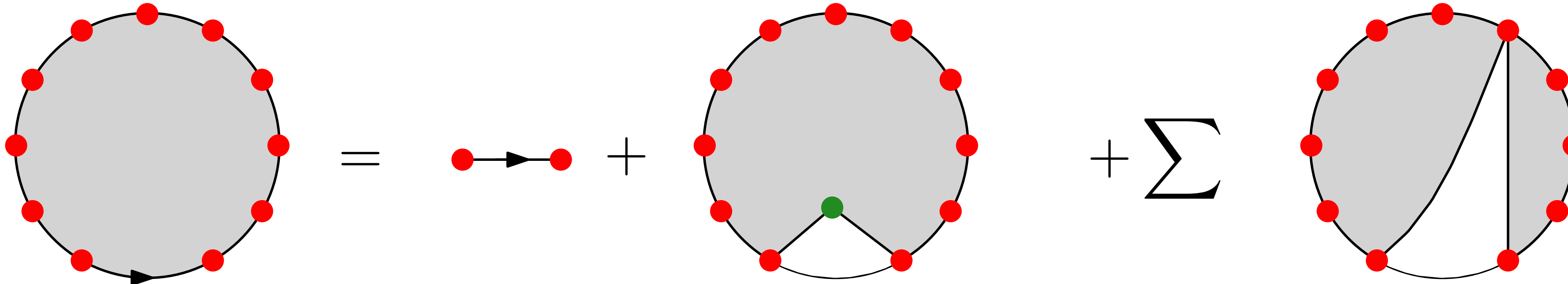

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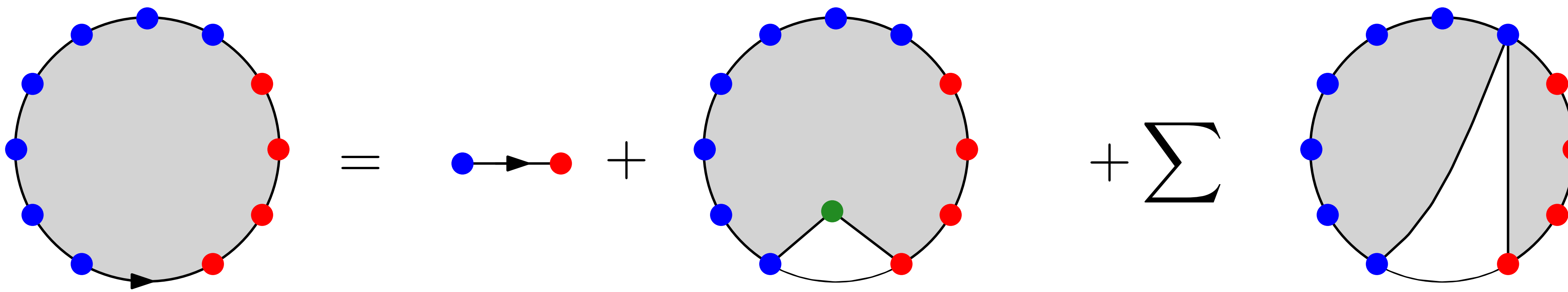

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$$= txy + \frac{t}{x}\Big(S(x,y) - x[x]S(x,y)\Big) + \frac{t}{y}\Big(S(x,y) - y[y]S(x,y)\Big) + \frac{t}{x}S(x,y)A(x) + \frac{t}{y}S(x,y)A(y)$$

From two catalytic variables to one: Tutte's invariants

Kernel method: equation for S reads

$$K(x,y)\cdot S(x,y)=R(x,y)$$

where $$K(x,y)=1-\frac{t}{x}-\frac{t}{y}-\frac{t}{x}A(x)-\frac{t}{y}A(y).$$

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3. Prove that $J(y) = C_0(t) + C_1(t)I(y) + C_2(t)I^2(y)$ with C_i's explicit polynomials in $t, Z_{\oplus}(t)$ and $Z_{\oplus^2}(t)$.

Equation with one catalytic variable for $A(y)$ with $Z_{\oplus}$ and $Z_{\oplus^2}$!

Explicit solution for positive boundary conditions

Equation with one catalytic variable reads:

$$2t^2\nu(1-\nu)\left(\frac{A(y)}{y} - Z_{\oplus}\right) = y \cdot \mathrm{Polynom}\left(\nu, \frac{A(y)}{y}, Z_{\oplus}, Z_{\oplus^2}, t, y\right)$$

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Maple: **rational parametrization !**

$$t^3 = U\frac{P_1(\mu, U)}{4(1-2U)^2(1+\mu)^3}$$

$$ty = V\frac{P_2(\mu, U, V)}{(1-2U)(1+\mu)^2(1-V)^2}$$

$$t^3A(t, ty) = \frac{V P_3(\mu, U, V)}{4(1-2U)^2(1+\mu)^3(1-V)^3}$$

with $\nu = \frac{1+\mu}{1-\mu}$ and P_i's explicit polynomials.

Going back to local convergence

Fix $r \geq 0$ and take Δ a r-hull with boundary spins $\partial\Delta$:

$$\mathbb{P}_n^\bullet\left(B_r^\bullet(T,v)=\Delta\right)=\frac{\nu^{m(\Delta)-m(\partial\Delta)}\,[t^{3n-e(\Delta)+|\partial\Delta|}]Z^\bullet_{\partial\Delta}(\nu,t)}{[t^{3n}]Q^\bullet(\nu,t)}$$

$$\underset{n\to\infty}{\longrightarrow}\ \frac{\kappa_{\partial\Delta}}{\kappa}\rho^{|\Delta|-|\partial\Delta|}\nu^{m(\Delta)-m(\partial\Delta)}.$$

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$$\sum_{r-\text{hulls }\Delta} \frac{\kappa_{\partial\Delta}}{\kappa}\rho^{|\Delta|-|\partial\Delta|}\nu^{m(\Delta)-m(\partial\Delta)} = 1$$

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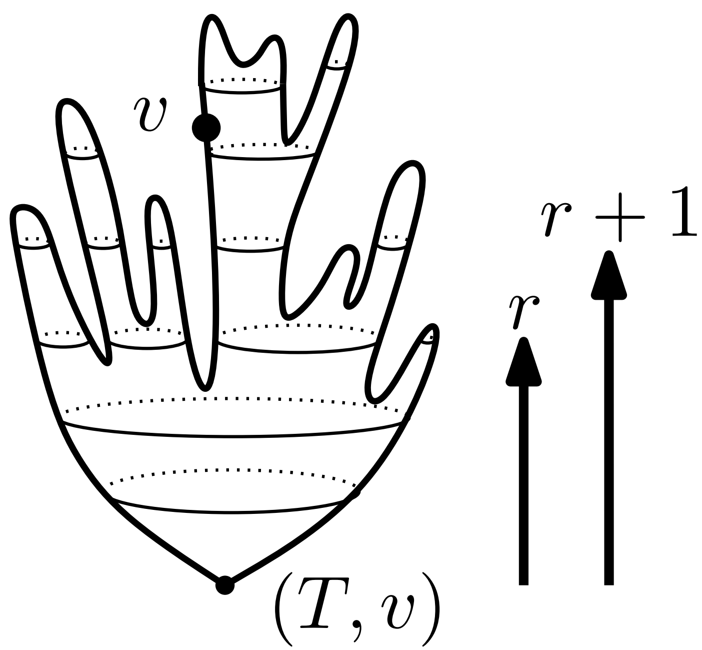
Easy for $r = 0$ and nested hulls : by induction !

Between hulls: Krikun's layers



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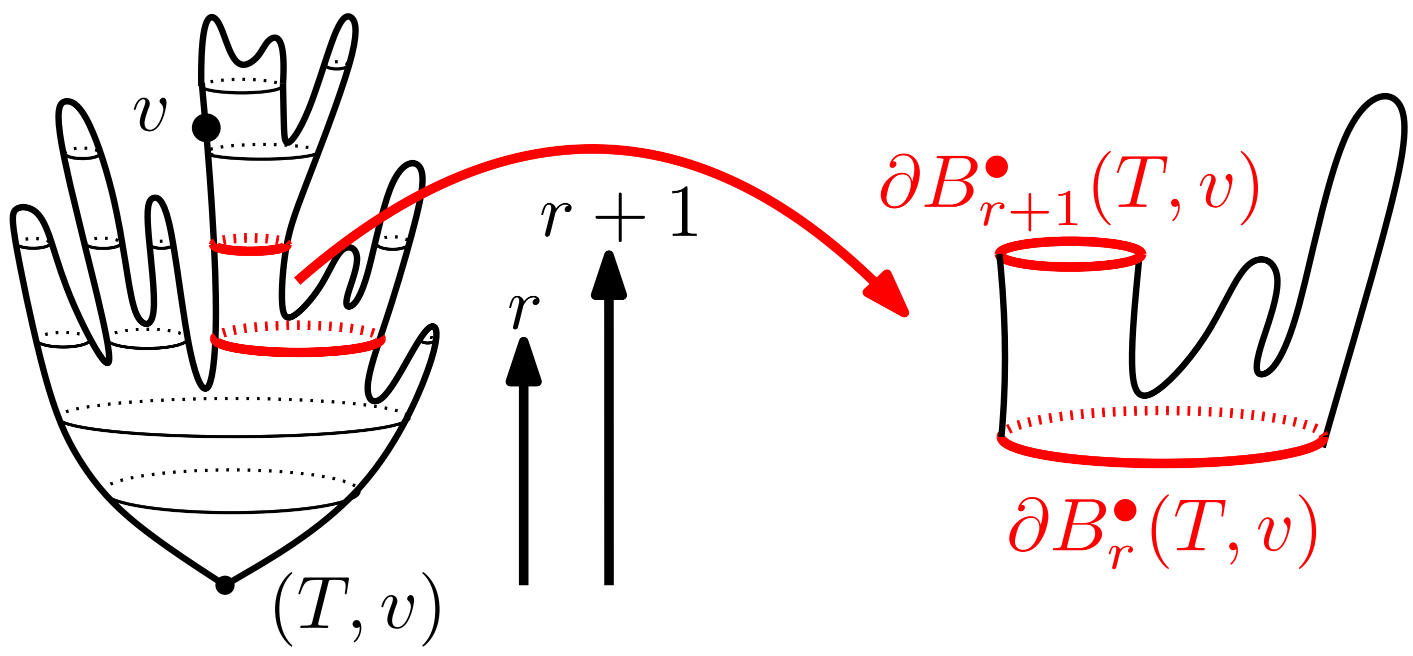

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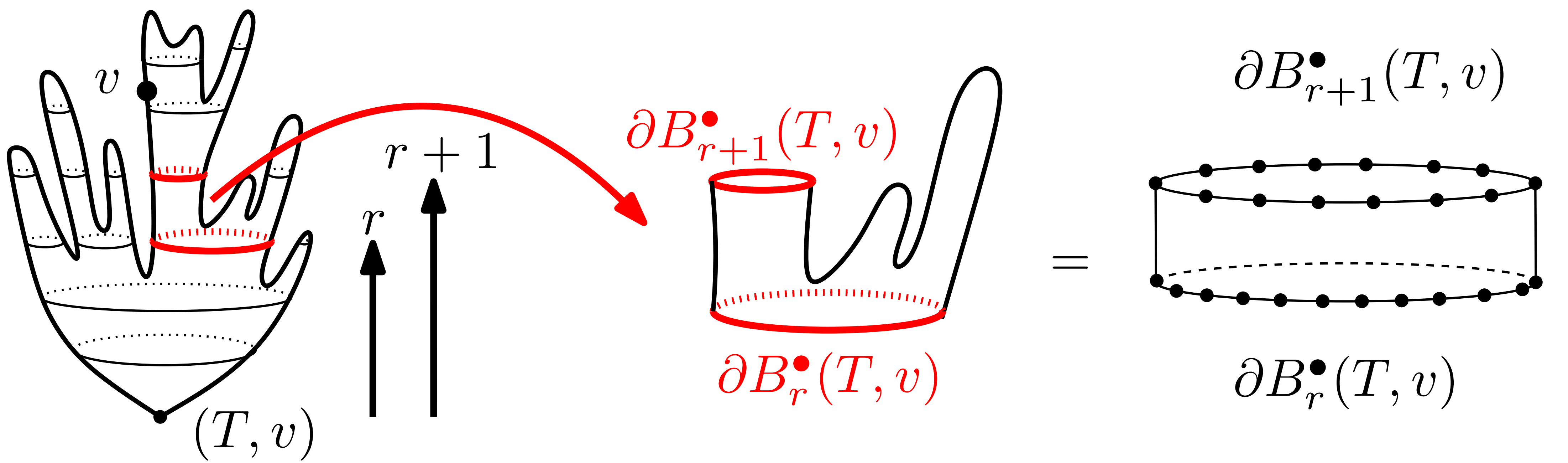

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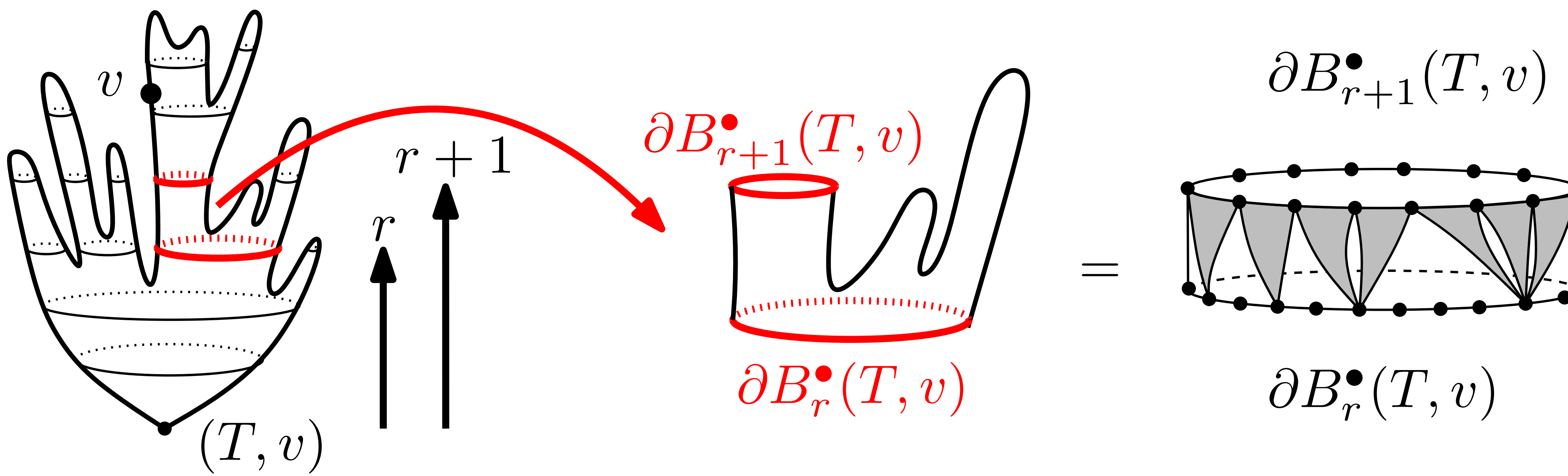

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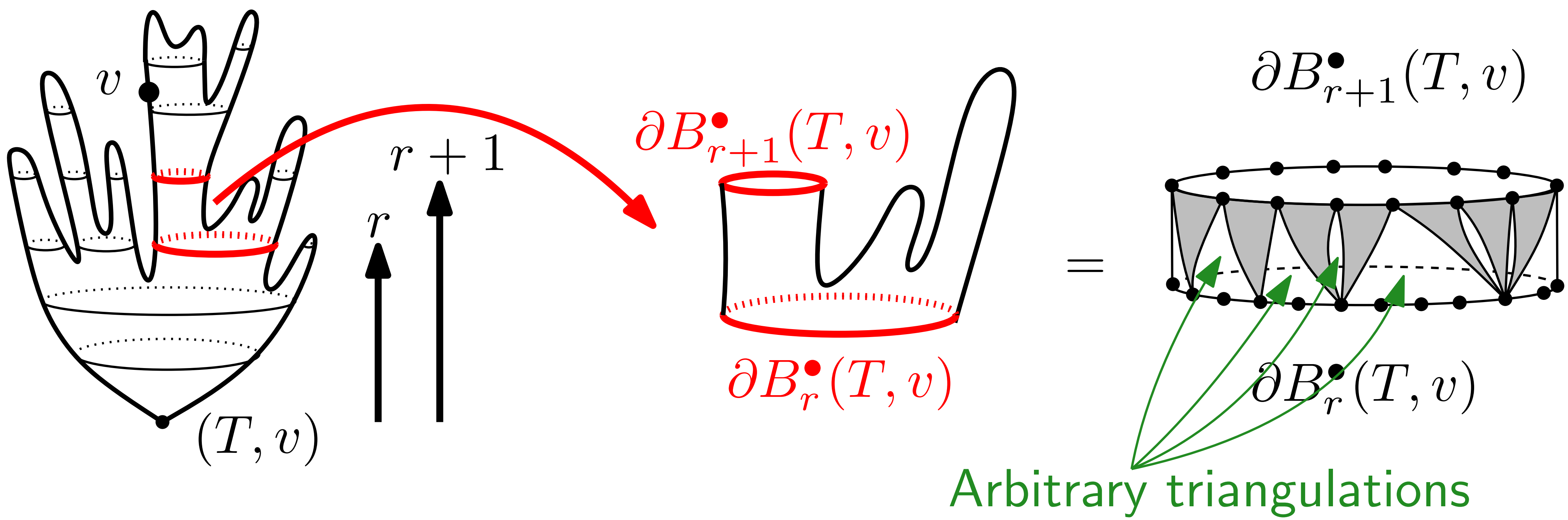

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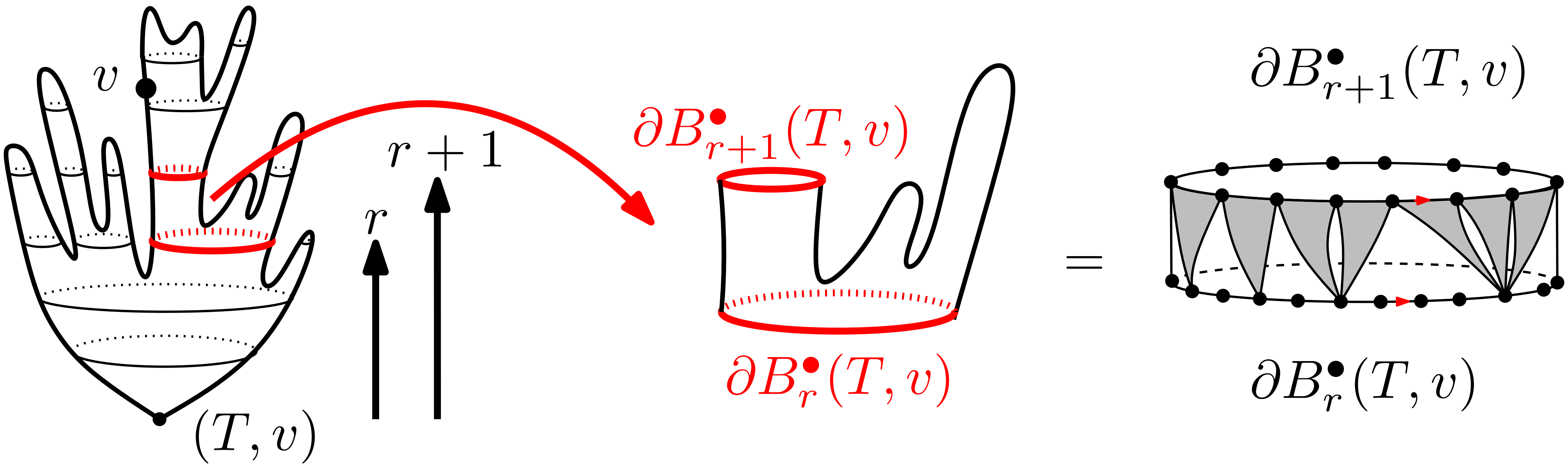

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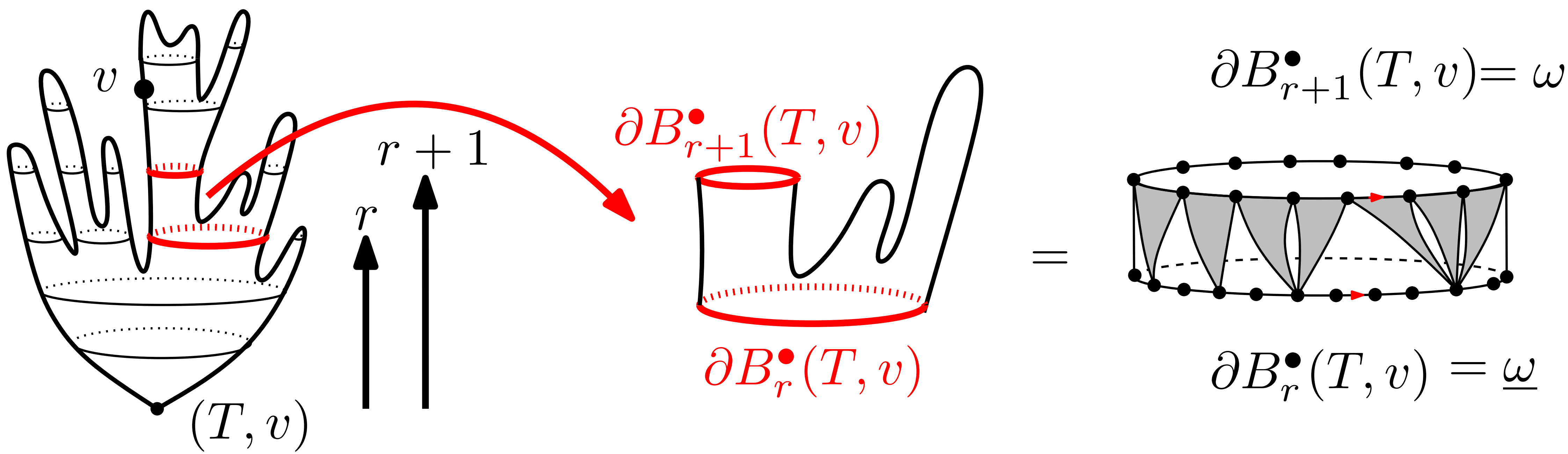

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$$= \sum_{\underline{\omega}} \frac{\kappa_\omega}{\kappa} Z_{\mathcal{L}_{\underline{\omega},\omega}}(\rho_c,\nu) \rho_c^{-|\omega|} \nu^{-m(\omega)} \sum_{\substack{r-\text{hulls } \underline{\Delta} \\ \text{s.t. } \partial\underline{\Delta}=\underline{\omega}}} \rho_c^{|\underline{\Delta}|-|\underline{\omega}|} \nu^{m(\underline{\Delta})-m(\underline{\omega})}$$

where $Z_{\mathcal{L}_{\underline{\omega},\omega}}$ is the generating series of layers with boundary conditions given by ω and $\underline{\omega}$.

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Thank you for your attention!