Locally self-avoiding Eulerian graphs

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The (Chinese) postman problem

I want to visit every street exactly once and return to the beginning.

- He wants: an Eulerian tour on the map.
- Satisfiable $\iff$ the map is an Eulerian graph (connected + even degree).

A hard-to-please postman

Moreover, I don't want to visit the same place twice in 4 hours.

- He has to cross a 6-way intersection 3 times.
  Example: (6:00, 12:00, 17:00): OK,
  (6:00, 12:00, 15:00): nope.

- He wants: an Eulerian tour on the map with some additional property.
- **Question**: what is the additional property exactly?

What does he want?

Given an Eulerian tour $\mathcal{T}$ of a (simple) graph G.

- Segment: a subwalk of $\mathcal{T}$.
- **Observation:** A segment either intersects itself or is a path.
- The postman wants: every segment of length 4 of $\mathcal{T}$ is a path.
- $\mathcal{T}$ is ℓ-Eulerian: if every segment of length ℓ of $\mathcal{T}$ is a path.
- **Question:** Which graphs admit an ℓ-Eulerian tour?
- **Question:** A sufficient condition?

Conjecture

Conjecture (Häggkvist 1989, Kriesell 2011)

For every $\ell \geq 1$, there is d_ℓ such that

$$\begin{cases} \textit{(simple) Eulerian,} \\ \textit{mindeg} \geq d_\ell \end{cases} \implies \exists \textit{ an } \ell\textit{-Eulerian tour.}$$

- **Remark:** Clearly false for multigraphs.

Results

$$\textit{Conjecture}: \begin{cases} \text{Eulerian}, \\ \textit{mindeg} \geq d_\ell \end{cases} \implies \exists \text{ an } \ell\text{-Eulerian tour.}$$

- $\ell = 1, 2$: trivial.
- $\ell = 3$ (i.e. triangle-free Eulerian tour):
  Oksimets 1997: True with $d_3 = 6$ (sharp).
- Bensmail, Harutyunyan, L., Thomassé 2014:

$$\begin{cases} \text{Eulerian}, \\ \textit{mindeg} \geq d_\ell, \\ \text{4-edge connected} \end{cases} \implies \exists \text{ an } \ell\text{-Eulerian tour.}$$

- L. 2016+: The conjecture is true.

A Corollary

- P_ℓ: path with ℓ edges.
- G is P_ℓ-decomposable: if G can be decomposed into copies P_ℓ, and an additional shorter path when $\ell \not| |E|$.

Conjecture (Barát-Thomassen 2006, path case)

$$\textit{edge connectivity} \geq c_\ell \implies P_\ell\textit{-decomposable.}$$

[Proved in 2014 independently by Botler-Mota-Oshiro-Wakabayashi and our team.]

- L. 2016+:

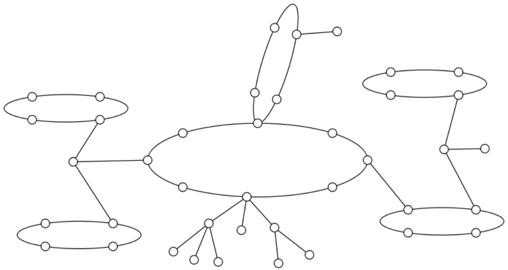
$$\begin{cases} \text{Eulerian,} \\ \textit{mindeg} \geq d_\ell \end{cases} \implies P_\ell\text{-decomposable.}$$

Sketch of proof

Cactus graph

Given a loopless multigraph G:

- G is a cactus: every edge belongs to at most one cycle.
- If a cactus is Eulerian: every edge belongs to exactly one cycle.
- Key **property** of an Eulerian cactus: From x visit y and come back to x, then you will never visit y again.



First observations

- Given G: Eulerian + high mindeg.
- If G is 4-edge-connected, then apply BHLT'14.
- Else, G has a cut of size 2 $\Longrightarrow$ cut them $\Longrightarrow$ obtain two partitions of G.
- "Heal" missing degrees by one dummy edge (or loop) in each partition $\Longrightarrow$ get two Eulerian "induced" sub-multigraphs of G.
- What do we get if we exhaustively repeat this process?

Partitioning the big graph

Lemma

Every Eulerian multigraph G can be partitioned into "induced" sub-multigraphs $G_1, \ldots, G_k$ such that:

- *Each G_i is Eulerian + 4-edge-connected, and*
- *$G_1, \ldots, G_k$ are globally linked by a giant cactus.*

Proved by induction.

Upgrade to multigraphs

Upgrade BHLT'14 to multigraphs:

$$G_i : \begin{cases} \text{multi Eulerian}, \\ \textit{mindeg} \geq d_\ell, \\ \text{4-edge connected} \end{cases} \implies \exists \text{ an } \ell\text{-Eulerian tour.}$$

- Obviously false.
- Avoidable by relaxing the definition of ℓ-Eulerian tour, which is sufficient for the proof:

Weaker "ℓ-Eulerian property: G_i has an Eulerian tour $\mathcal{T}_i$ s.t. every segment with no dummy-edge of length $\leq \ell$ is a path.

Final step

- Every G_i has a "weak ℓ-Eulerian" tour $\mathcal{T}_i$.
- Carefully rewiring $\mathcal{T}_i$ by the giant cactus to get an ℓ-Eulerian tour of G, using the key property of cactus graphs.

Open questions

- The proof gives a tower bound for d_ℓ.
- **Question 1:** Can we obtain a sharp (or good) bound for d_ℓ?

- The proof uses some probabilistic methods.
- **Question 2:** Can we have an efficient algorithm to find an ℓ-Eulerian tour?

- The theorem gives a sufficient condition.
- **Question 3:** Can we characterize graphs admitting an ℓ-Eulerian tour?

Thank you.