Maximum Cuts in Edge Colored Graphs

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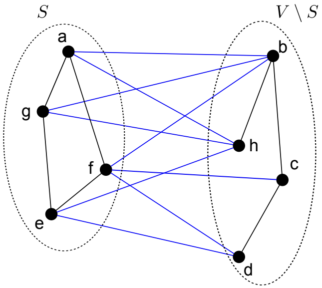

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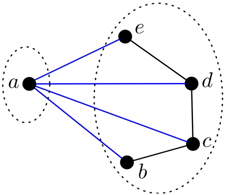
- Let $S \subset V$ be a proper subset of V. The *edge cut ∂S* is the set of edges with one endpoint in S and the other in $V \setminus S$.
- (SIMPLE MAXCUT): Given a graph $G = (V, E)$, find $S \subset V$ such that $|\partial S| \geq |\partial T|$ for all $T \subset V$.



Colored Edge Cuts

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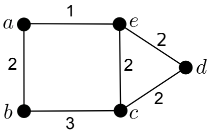

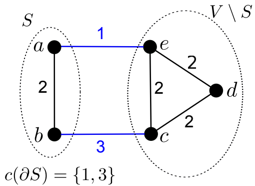

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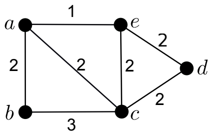
- Optimization Problem (MAX-COLORED-CUT): Find $S \subset V$ such that $|c(\partial S)| \geq |c(\partial T)|$ for all $T \subset V$.



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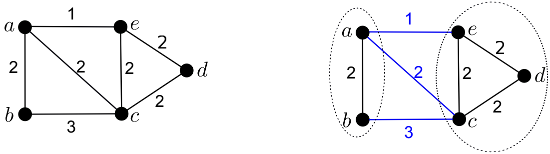

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- If $c : E \to \mathbb{N}$ is injective, then MAX-COLORED-CUT is exactly SIMPLE MAXCUT, that is, MAX-COLORED-CUT is NP-hard.
- Our goal is to analyze the complexity of MAX-COLORED-CUT on graph classes where SIMPLE MAXCUT is solvable in polynomial time.

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Although SIMPLE MAXCUT is easily solvable on complete graphs, not necessarily the same thing happens with MAX-COLORED-CUT.

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Theorem

MAX-COLORED-CUT *is* NP*-complete on complete graphs.*

Proof.

- Take an instance $G = (V, E)$ of SIMPLE MAXCUT

□

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- Take an instance $G = (V, E)$ of SIMPLE MAXCUT
- Consider $G' = (V \cup \{v_1\}, E)$, such that v_1 is a new isolated vertex and create an injective function $c : E \rightarrow \{1, 2, \cdots, |E|\}$.

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- Construct the complete graph G'' connecting all pairwise nonadjacent vertices of G' and give to all these new edges the color $|E| + 1$.

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- Construct the complete graph G'' connecting all pairwise nonadjacent vertices of G' and give to all these new edges the color $|E| + 1$.
- G'' has a colored cut of size $k + 1$ if and only if G has a cut of size k (the color $|E| + 1$ is in all edge cut of G'').

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SIMPLE MAXCUT can be solved in polynomial time on planar graphs [Hadlock'75]

What can we say about the complexity of COLORFUL CUT and MAX-COLORED-CUT on planar graphs?

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Reduction from 3SAT

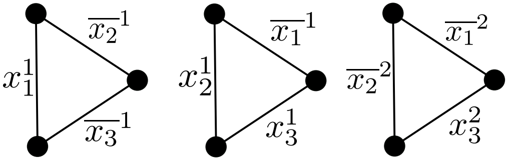

Figure: Graph G_F corresponding to the instance $F = (x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3)$.

- Associate to each pair $\{{x_i}^j, {\overline{x_i}}^k\}$ the color ${S_i}^{j,k}$.

COLORFUL CUT

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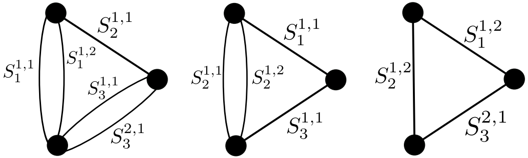

Figure: Multigraph G corresponding to the graph G_F with colored edges.
$F = (x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3)$.

- In each clause, choose a true literal x_i and put the corresponding edge in the same part of the partition.

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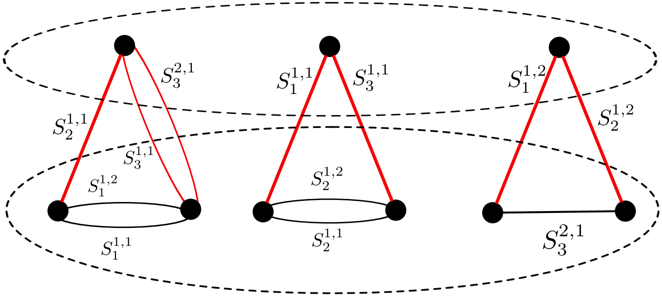

Figure: Colorful cut of the multigraph G.

COLORFUL CUT

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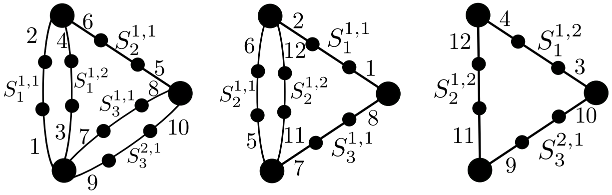

Figure: Simple graph H corresponding to the multigraph G.

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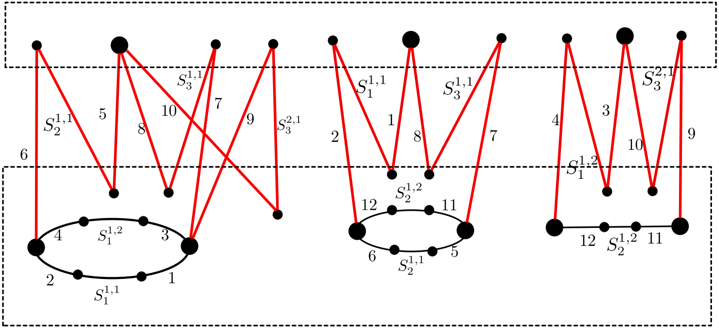

Figure: Colorful cut of H.

COLORFUL CUT

Corollary

COLORFUL CUT *is* NP*-complete, even on planar graphs with odd cycle transversal number at most 1.*

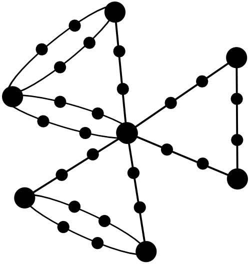

Figure: Graph obtained from H identifying three vertices into a single vertex.

Theorem

COLORFUL CUT *is* NP*-complete when each color class induces a clique of size at most three.*

Reduction from NAE3SAT

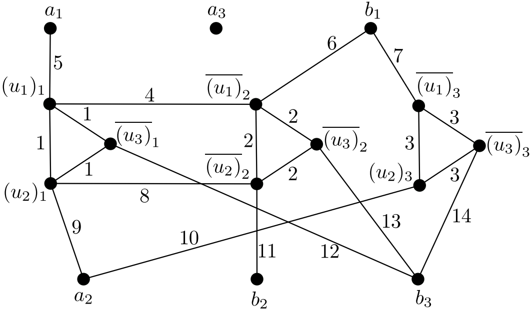

Figure: Graph corresponding to the instance $F = (u_1 \vee u_2 \vee \overline{u_3}) \wedge (\overline{u_1} \vee \overline{u_2} \vee \overline{u_3}) \wedge (\overline{u_1} \vee u_2 \vee \overline{u_3})$.

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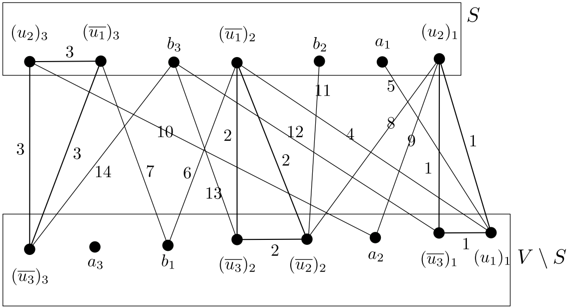

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- For c colors we obtain a kernel of size $O(c^3)$

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Corollary

MAX-COLORED-CUT *admits a cubic kernel when parameterized by the cost of the solution.*

- Consider a simple graph $G = (V, E)$ with an edge coloring $c : E \to \{1, 2, \cdots, p\}$

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Corollary

MAX-COLORED-CUT *can be solved in polynomial time on graphs G colored with a constant number of colors.*

Thank You!