The lexicographic product of some chordal graphs and of cographs preserves b-continuity

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Joint work with
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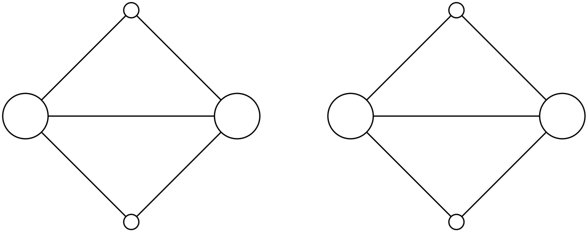
Universidade Federal do Ceará, Brazil

b-coloring algorithm



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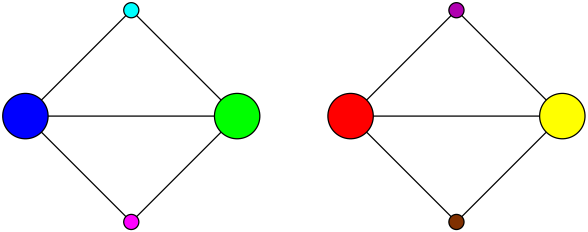

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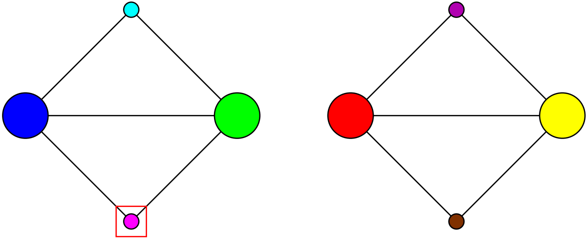

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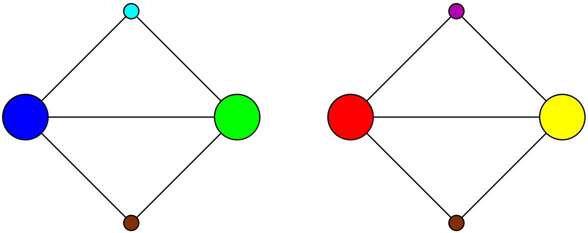

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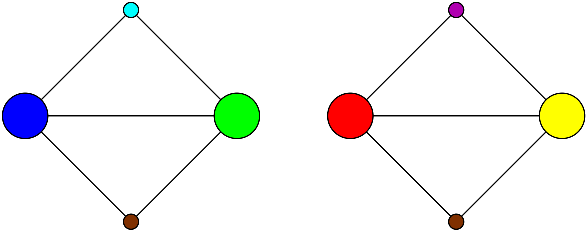

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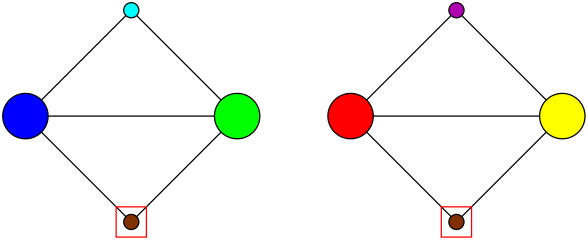

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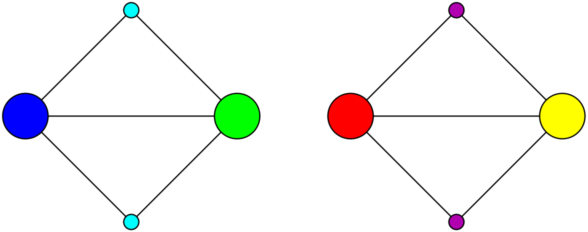

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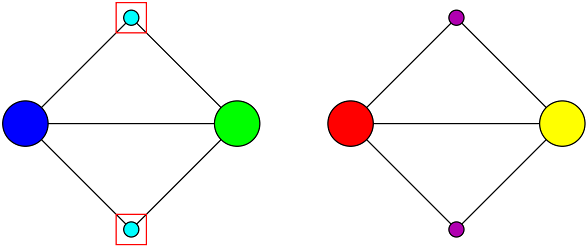

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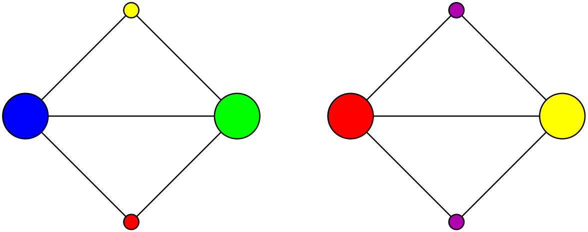

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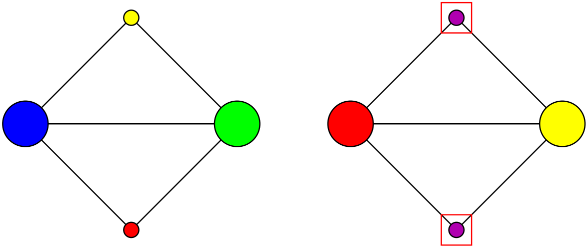

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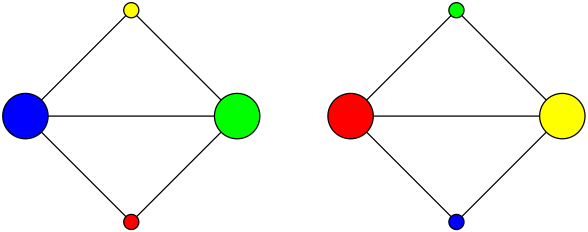

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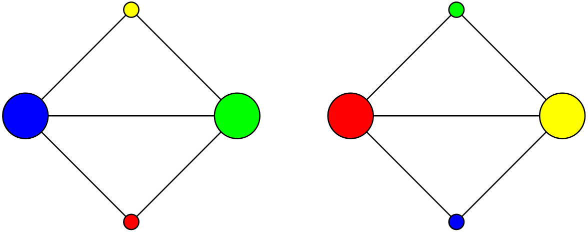
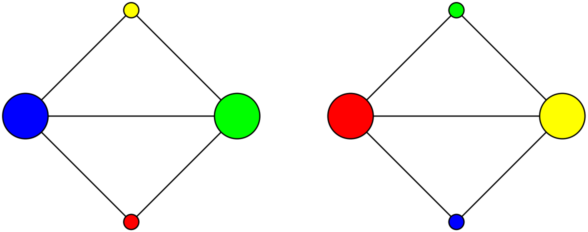


Definition:

- b-vertex: at least one neighbor of each color other than its own.
- b-coloring: each color class has a b-vertex.
- $b(G)$ = max number of colors in a b-coloring of G.

b-coloring



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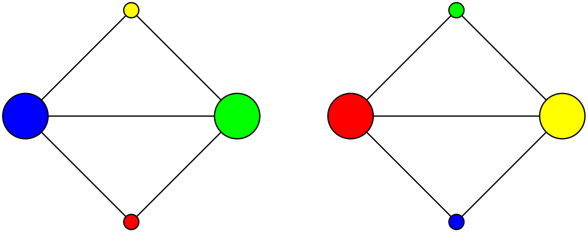

Definition: b-chromatic number

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Complexity

- Finding $b(G)$ is NP-hard, even if G is
  - bipartite;
  - chordal; or
  - line graph.

- And is polynomial if G is
  - Cograph or P_4-sparse graph;
  - Graphs with girth at least 7; etc.

Irving and Manlove.
The b-chromatic number of a graph.
Discrete App. Math. 91 (1999) 127–141.

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Kratochvil, Tuza e Voigt
On the b-chromatic number of graphs.
WG 2002, Lecture Notes in Computer Science 2573.

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Havet, Linhares-Sales and Sampaio.
b-coloring of tight graphs.
Discrete Appl. Math. 160 (18) (2012) 2709–2715.

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Campos, Lima, Martins, Sampaio, Santos and Silva.
The b-chromatic index of graphs.
Discrete Math. 338 (11) (2015) 2072–2079.

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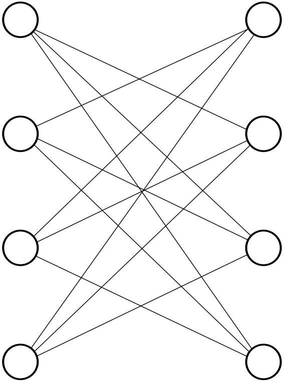
Campos, Lima and Silva.
Graphs with girth at least 7 have high b-chromatic number.
European J. Combinatorics 48 (2015) 154–164.

An interesting aspect

G may not have a b-coloring with k colors, for some $k \in \{\chi(G), \cdots, b(G)\}$.



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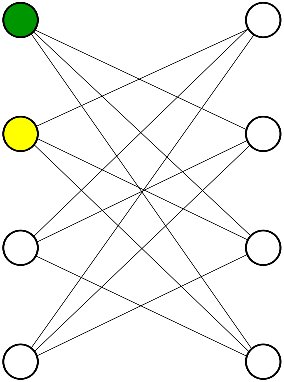

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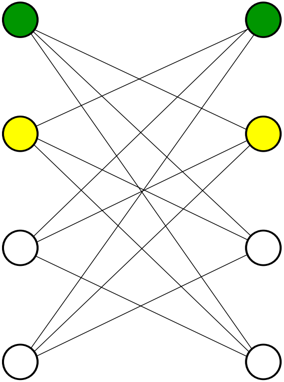

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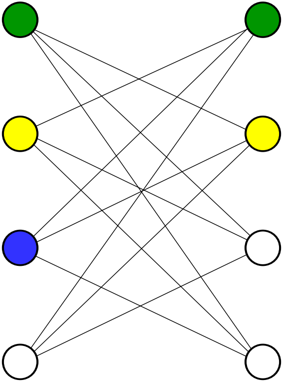

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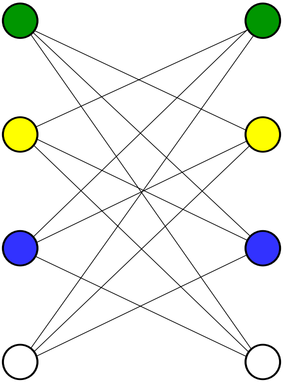

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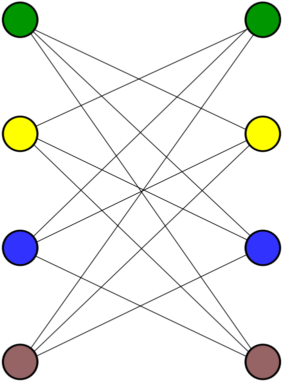

b-Continuity

Definition: b-spectrum

- $S_b(G)$: set of integers k such that G has a b-coloring with k colors;
- If $S_b(G) = \{\chi(G), \cdots, b(G)\}$, we say that G is b-continuous.

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Some results on b-continuity

- For every $S \subseteq \mathbb{N} - \{1\}$, $\exists G$ s.t. $S_b(G) = S$; and
- Decide whether G is b-continuous is NP-complete.

b-continuous graph classes

- Chordal graphs;
- Kneser graphs $K(n, 2)$, $n \geq 17$;
- P_4-sparse graphs;
- P_4-tidy graphs;
- Regular graphs with girth at least 6 and no cycles of length 7;
- Graphs with girth 10.

D. Barth, J. Cohen and T. Faik.
On the b-continuity property of graphs.
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T. Faik.
About the b-continuity of graphs.
CTW 2004. Electron. Notes in Discrete Math. 17 (2004) 151–156.

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Kara, Kratochvil and Voigt.
b-Continuity.
Preprint no. M14/04, Technical University Ilmenau, 2004.

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R. Javadi and B. Omoomi.
On b-coloring of the Kneser graphs.
Discrete Math. 309 (2009) 4399–4408.

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C.I. Betancur Velasquez, F. Bonomo, and I. Koch.
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Discrete Appl. Math. 159 (2011) 67–76.

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R. Balakrishnan and T. Kavaskar.
b-coloring of Kneser graphs.
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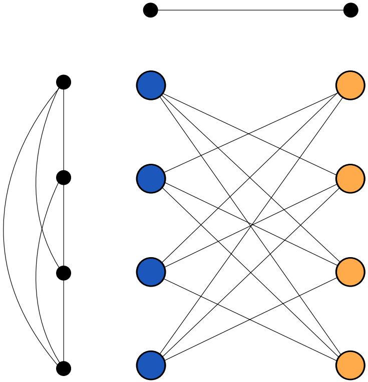
C. Linhares Sales and A. Silva.
The b-continuity of graphs with large girth.
Graphs and Combinatorics. To appear.

Do graph products preserve b-continuity?

Direct product and cartesian product do not preserve b-continuity.



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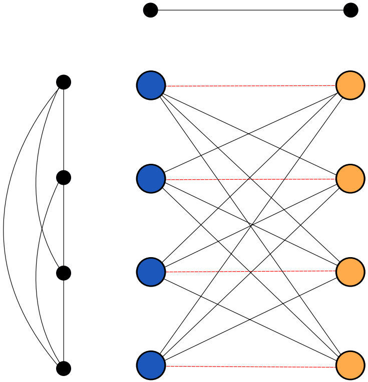

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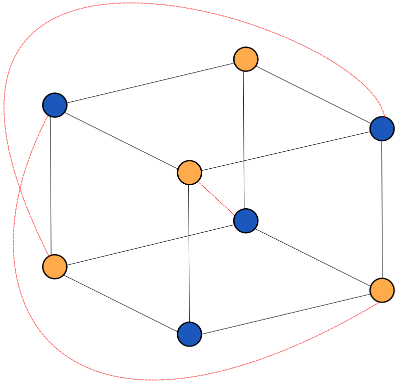

What about the lexicographic product?

Theorem

If G is chordal and H is b-continuous, then

$$[\chi(G)\chi(H), b(G)b(H)] \cap \mathbb{N} \subseteq S_b(G[H])$$

C. Linhares Sales, L. Sampaio and A. Silva.
On the b-continuity of the lexicographic product of graphs.
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b-continuity and the lexicographic product of graphs.
LAGOS'15. Electron. Notes in Discrete Math. 50 (2015) 134–144.

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This talk:

Theorem

If G is an interval graph or a block graph, and H is a b-continuous graph, then G[H] is b-continuous.

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If G is an interval graph or a block graph, and H is a b-continuous graph, then $G[H]$ is b-continuous.

Theorem

If G is a P_4-sparse and H is a b-continuous graph, then $G[H]$ is b-continuous.

Outline of the Proof

- G be a chordal graph and H be b-continuous;
- ψ be a b-coloring of $G[H]$ with $k > b(G)b(H)$ colors;
- Decrease the number of b-vertices in ψ;
- Eventually end up with a b-coloring with $k - 1$ colors.

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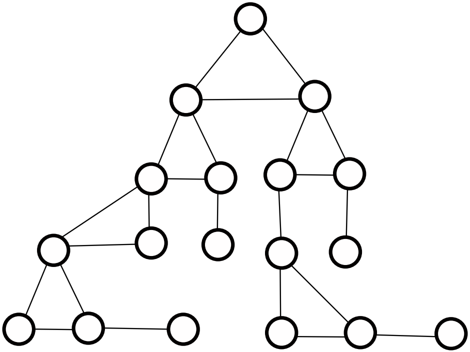

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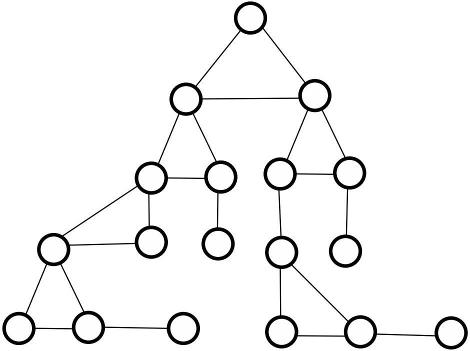
ψ be a b-coloring of $G[H]$ with $k > b(G)b(H)$ colors

Choose a block C containing b-vertices closer to the leaves;



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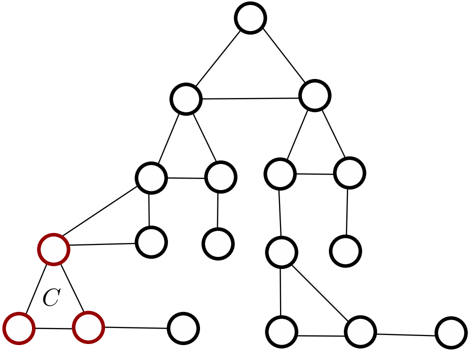

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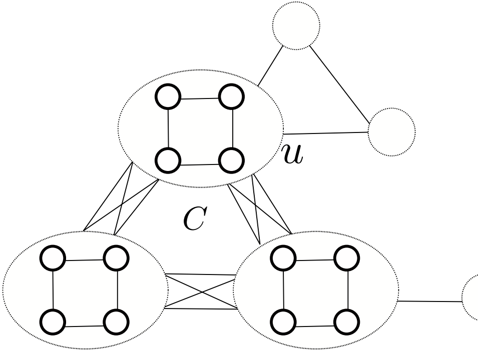

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If color d has no b-vertices in $C - u$,

- choose d' a color with b-vertices to switch with d;
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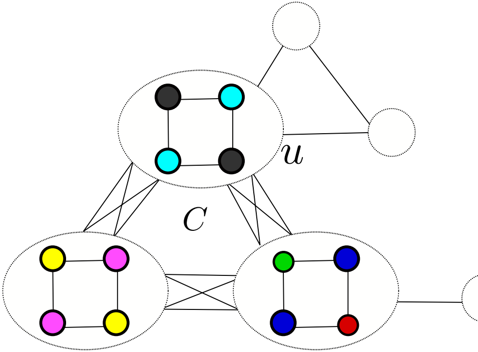

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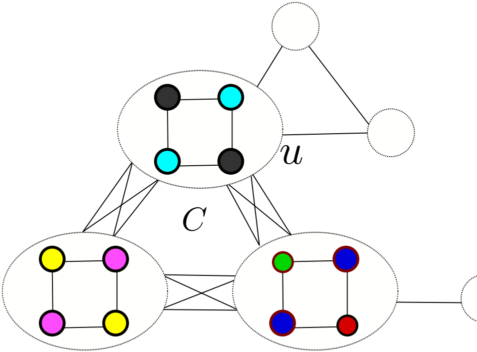

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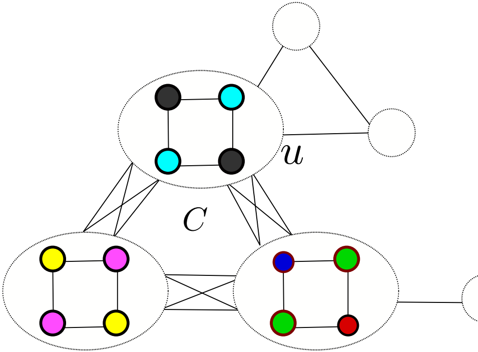

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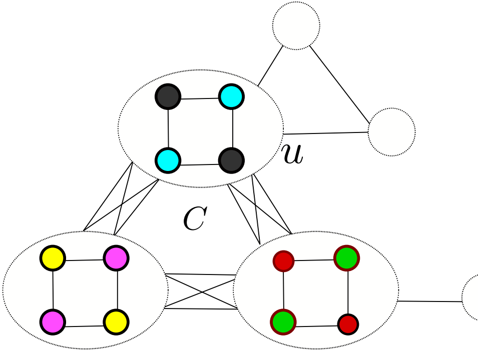

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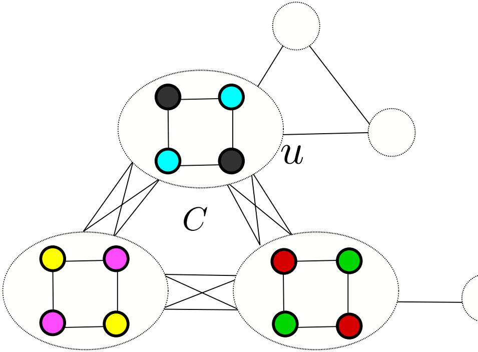

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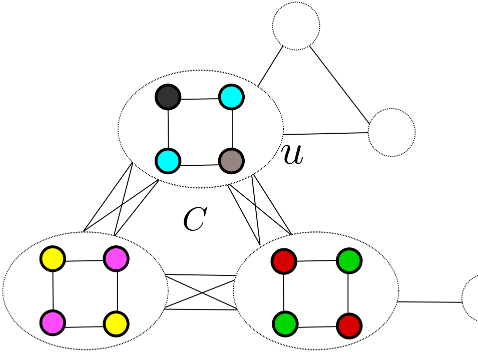

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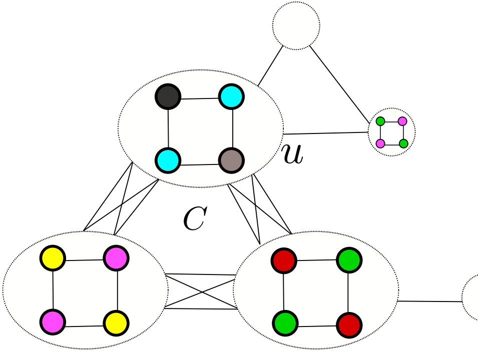

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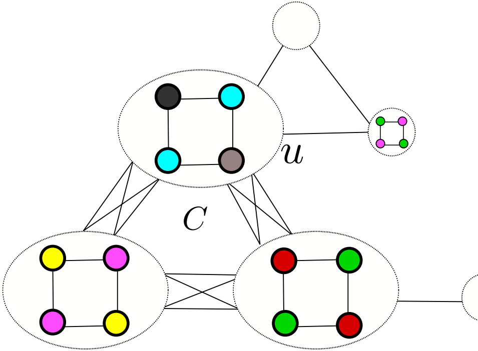

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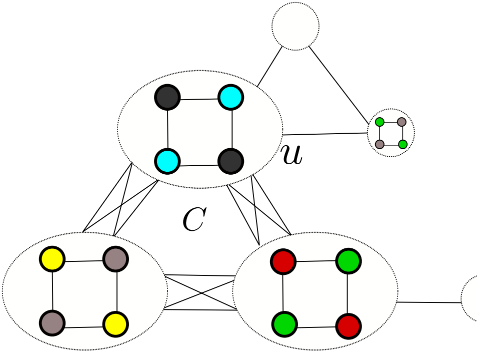

Open Problems

- Is $G[H]$ b-continuous, when G is chordal and H is b-continuous?
- Is $G[H]$ b-continuous, whenever G, H are?
- Is $G[K_\ell]$ b-continuous whenever G is b-continuous?
  This would imply $[\chi(G)\chi(H), b(G)b(H)] \cap \mathbb{N} \subseteq S_b(G[H])$, when G and H are b-continuous;
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C. Linhares Sales, L. Sampaio and A. Silva.
On the b-continuity of the lexicographic product of graphs.
Graphs and Combinatorics. To appear.

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Thank you!