Facets of the polytope of legal sequences

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Definitions

- set V
- function $N\langle \_ \rangle : V \to \mathcal{P}(V)$

$$u \in N\langle v\rangle \implies v \in N\langle u\rangle$$

Dominanting sequence

$S = (v_1, \ldots, v_k)$ sequence of different elements from V

$$S \textit{ dominating} \iff \bigcup_{i=1}^{k} N\langle v_i\rangle = V$$

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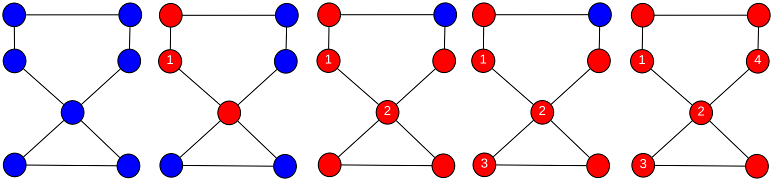

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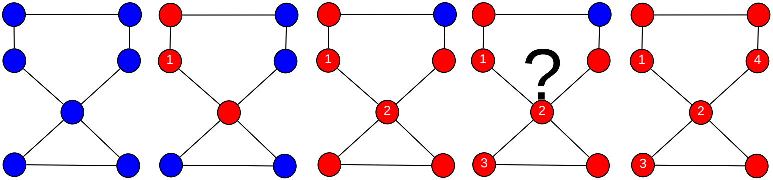

Definitions

Legal sequence

$S = (v_1, \ldots, v_k)$ sequence of different elements from V

$$S \ legal \iff N\langle v_i \rangle \setminus \bigcup_{j=1}^{i-1} N\langle v_j \rangle \neq \emptyset, \ \ \forall \ i = 2, \ldots, k$$

Each v_i dominates at least one vertex from $N\langle v_i \rangle$ not "previously" dominated by v_1, v_2, …, v_{i-1}

We say that v_i *footprints* those vertices from $N\langle v_i \rangle \setminus \bigcup_{j=1}^{i-1} N\langle v_j \rangle$

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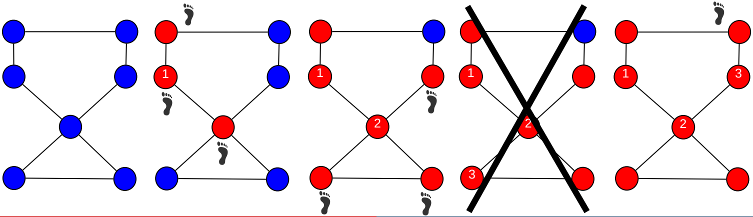

Definitions

¿How "long" can a dominating legal sequence be?

Grundy domination number

For a given graph G, the *Grundy domination number* $\gamma_{\mathrm{gr}}(G)$ computes the size of the longest legal dominating sequence.

In this work, we give integer programming formulations for obtaining $\gamma_{\mathrm{gr}}(G)$ and we study the polytope associated to one of them.

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A bit of history

- “Grundy” concept emerged as a way of studying worst case in greedy coloring heuristics.
  [Christen, Selkow 1979]
  [Bonnet, Foucaud, Kim, Sikora 2015]
- Interest in studying $\gamma_{\rm gr}$ related to “domination game”.
  [Brešar, Klavžar, Rall 2010]
  [Kinnersley, West, Zamani 2013]
  [Košmrlj 2014]
- Finding $\gamma_{\rm gr}(G)$ with “$N[v]$” is NP-Hard even on chordal graphs. On trees, cographs and splits it is linear.
  [Brešar, Gologranc, Milanič, Rall, Rizzi 2014]
- Finding $\gamma_{\rm gr}(G)$ with “$N(v)$” is NP-Hard even on bipartites...
  [Brešar, Henning, Rall 2016]
- ... but is linear on trees, P_4-tidy and distance-hereditary bipartites.
  [Brešar, Kos, Nasini, Torres] (submitted)

Basic properties

- Sequence legal $\wedge$ longest $\implies$ dominating

  ☞ Enough to ask for legality

- G = disjoint union of G_1 and G_2:

$$\gamma_{\rm gr}(G) = \gamma_{\rm gr}(G_1) + \gamma_{\rm gr}(G_2)$$

  ☞ Suppose graphs are connected

- $N\langle u\rangle = N\langle v\rangle$: $\longleftarrow$ u, v "twins"

$$\gamma_{\rm gr}(G) = \gamma_{\rm gr}(G - v)$$

  ☞ Suppose there are no twins

- $\delta^*(G) = \min_{v\in V} |N\langle v\rangle|$ $\longleftarrow$ least "degree"

$$\gamma_{\rm gr}(G) \leq m \doteq n - \delta^*(G) + 1$$

[Brešar, Gologranc, Milanič, Rall, Rizzi 2014]
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Representation of legal sequences

$\forall\ v \in V,\ \ i = 1, \ldots, m$:

$$y_{vi} = \begin{cases} 1, & v \text{ is chosen in step } i \\ 0, & \text{otherwise} \end{cases}$$

$\forall\ u \in V,\ \ i = 1, \ldots, m$:

$$x_{ui} = \begin{cases} 1, & u \text{ is not footprinted in steps } 1, \ldots, i \\ 0, & \text{otherwise} \end{cases}$$

Example: $u \in N\langle v \rangle$

$$y_v = (0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0)$$
$$\downarrow$$
$$x_u = (1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0)$$

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Integer linear formulation

F_1:

$$\max \sum_{i=1}^{m} \sum_{v \in V} y_{vi}$$

$s.a$

$$
\begin{array}{llr}
asig.y & \sum_{v \in V} y_{vi} \leq 1, & \forall \ i = 1, \ldots, m \quad (1) \\
asig.y & \sum_{i=1}^{m} y_{vi} \leq 1, & \forall \ v \in V \quad (2) \\
legal. & y_{vi+1} \leq \sum_{u \in N\langle v \rangle} (x_{ui} - x_{ui+1}), & \forall \ v \in V, \ i = 1, \ldots, m-1 \quad (3) \\
def.x & x_{ui} + \sum_{v \in N\langle u \rangle} y_{vi} \leq 1, & \forall \ u \in V, \ i = 1, \ldots, m \quad (4) \\
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 & x, y \in \{0, 1\}^{nm}, &
\end{array}
$$

- x_{ui} can switch to 0 even if nobody footprints u
- There are steps where nobody is chosen

☞ Symmetric solutions

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☞ Symmetric solutions

Breaking symmetries

F_2: x_{ui} set to 0 only when u is footprinted

$$\max \sum_{i=1}^{m} \sum_{v \in V} y_{vi}$$

$$\begin{aligned} s.a \quad & \cdots & & \cdots \\ & x_{u1} + \sum_{v \in N\langle u \rangle} y_{v1} \geq 1, & & \forall \ u \in V \quad (6) \\ & x_{ui+1} + \sum_{v \in N\langle u \rangle} y_{vi+1} \geq x_{ui}, & & \forall \ u \in V, \ i = 1, \ldots, m-1 \quad (7) \end{aligned}$$

$$x, y \in \{0,1\}^{nm},$$

Breaking symmetries

F_3: Vertices are chosen in first steps

$$\max \sum_{i=1}^{m} \sum_{v \in V} y_{vi}$$

$$s.a \quad \cdots \qquad \qquad \cdots$$

$$\sum_{v \in V} y_{v1} = 1, \tag{8}$$

$$\sum_{v \in V} y_{vi+1} \leq \sum_{v \in V} y_{vi}, \qquad \forall \ i = 1, \ldots, m-1 \tag{9}$$

$$x, y \in \{0,1\}^{nm},$$

Breaking symmetries

F_4: 1-to-1 corresp.: legal seq. $\Longleftrightarrow$ integer sol.

$$\max \sum_{i=1}^{m} \sum_{v \in V} y_{vi}$$

$$s.a \quad \cdots \qquad \cdots$$

$$x_{u1} + \sum_{v \in N\langle u \rangle} y_{v1} \geq 1, \qquad \forall \; u \in V \qquad (6)$$

$$x_{ui+1} + \sum_{v \in N\langle u \rangle} y_{vi+1} \geq x_{ui}, \quad \forall \; u \in V, \; i = 1, \ldots, m-1 \qquad (7)$$

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Breaking symmetries

F_5: sequences are dominating

$$\max \sum_{i=1}^{m} \sum_{v \in V} y_{vi}$$

$$s.a \quad \cdots \qquad \cdots$$

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$$x, y \in \{0,1\}^{nm},$$

Breaking symmetries

F_6:

$$\max \sum_{i=1}^{m} \sum_{v \in V} y_{vi}$$

$$\begin{aligned}
s.a \quad & \cdots & & \cdots & \\
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& \sum_{i=1}^{m} \sum_{v \in N\langle u \rangle} y_{vi} \geq 1, & & \forall \ \ u \in V & (10) \\
& x, y \in \{0, 1\}^{nm}, & & &
\end{aligned}$$

Breaking symmetries

F_7:

$$\max \sum_{i=1}^{m} \sum_{v \in V} y_{vi}$$

$$s.a \quad \cdots \qquad\qquad \cdots$$

$$\sum_{v \in V} y_{v1} = 1, \tag{8}$$

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Breaking symmetries

F_8: 1-to-1 cor.: legal dom. seq. $\Longleftrightarrow$ int. sol.

$$\max \sum_{i=1}^{m} \sum_{v \in V} y_{vi}$$

$$
\begin{aligned}
s.a \quad & \cdots & & \cdots \\
& x_{u1} + \sum_{v \in N\langle u \rangle} y_{v1} \geq 1, & \forall \ \ u \in V & \quad (6) \\
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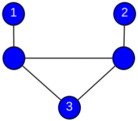
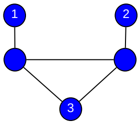
Comparing formulations



$m = 4$ $\qquad \gamma_{\mathrm{gr}}(G) = 3$

Form.	(6)-(7)	(8)-(9)	(10)	Solutions
F_1				16253
F_2	✓			205
F_3		✓		463
F_4	✓	✓		43
F_5			✓	1668
F_6	✓		✓	124
F_7		✓	✓	68
F_8	✓	✓	✓	28

Comparing formulations



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F_7		✓	✓	68
F_8	✓	✓	✓	28

From computational experiments, we determined that the best form. is F_4 $\longleftarrow$ "dominating" ineq. hinder the optimization

Our results: Dimension

- P_i = Polytope associated to F_i

Proposition

P_1 is full dimensional.

Proposition

$$dim(P_3) = nm - m|V_0| - (m-1)|V_1| + \sum_{v \in V} i(G; C, v) - 1$$

where:

$V_0 = \{v \in V : N\langle v\rangle = V\}$, $V_1 = \{v \in V : N\langle v\rangle = V \setminus \{v\}\}$,

$i(G; C, v)$ = largest index where v can be chosen

Minimal system:

1) $y_{vi} = 0 \quad \forall\ v \in V,\ \ i = i(G; C, v) + 1, \ldots, m,$

2) $x_{vi} = 0 \quad \forall\ v \in V_0,\ \ i = 1, 2, \ldots, m,$

3) $x_{vi} = 0 \quad \forall\ v \in V_1,\ \ i = 2, \ldots, m,$

4) $\sum_{v \in V} y_{v1} = 1.$

- $P_2, P_4, \ldots, P_8$ are harder to study.
- $P_i \subset P_1 \quad \longrightarrow \quad$ valid on $P_1 \Rightarrow$ valid on P_i.

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where:

$V_0 = \{v \in V : N\langle v \rangle = V\}$, $V_1 = \{v \in V : N\langle v \rangle = V \setminus \{v\}\}$,
$i(G; C, v)$ = largest index where v can be chosen

Minimal system:

1) $y_{vi} = 0 \quad \forall\ v \in V, \quad i = i(G; C, v) + 1, \ldots, m,$
2) $x_{vi} = 0 \quad \forall\ v \in V_0, \quad i = 1, 2, \ldots, m,$
3) $x_{vi} = 0 \quad \forall\ v \in V_1, \quad i = 2, \ldots, m,$
4) $\sum_{v \in V} y_{v1} = 1.$

- $P_2, P_4, \ldots, P_8$ are harder to study.
- $P_i \subset P_1 \quad \longrightarrow \quad$ valid on $P_1 \Rightarrow$ valid on P_i.

Our results: Facet-defining inequalities

1. Ineq. that generalize constraints (3):
$$\sum_{w \in W} y_{wi+1} \leq \sum_{u \in N\langle w_1 \rangle} (x_{ui} - x_{ui+1})$$
   - "⇔" condition for facet-definition on P_1.
2. Ineq. that generalize constr. (4) and dominates constr. (2):
$$x_{ui} + \sum_{v \in N} y_{vi} + \sum_{r=1}^{t} \sum_{j=j_r}^{j_{r+1}} y_{w_r j} \leq 1$$
   - "⇔" condition for facet-definition on P_1.
   - Leads to new valid inequalities:
$$x_{ui} + \sum_{j=1}^{i} y_{wj} \leq 1 \quad \longleftarrow \quad \textit{Family 1 (sep. polytime)}$$
3. New valid inequalities:
$$x_{u_1 i} + x_{u_2 i} + \sum_{j=1}^{i} y_{wj} + \sum_{v \in N\langle u_1 \rangle \cup N\langle u_2 \rangle} y_{vk} \leq 2 \quad \longleftarrow \quad \textit{Family 2 (sep.)}$$
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Results on clutters

Clutter

$\mathcal{H} = (V, \mathcal{E})$ where $\mathcal{E} = \{N\langle v\rangle : v \in V\}$

$$\mathcal{H} \text{ clutter} \iff N\langle u\rangle \setminus N\langle v\rangle \neq \emptyset \text{ for all } u \neq v$$

Proposition

If $\mathcal{H}$ is a clutter, then:

- Constraint (3), (4) and (5) define facets of P_1
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Computational experiments

- CPLEX 12.7, 1 thread, 1 hour (limit)
- Form. F_4
- 24 instances
- $B\&C_1 = B\&B + \text{Family 1 as cuts}$
- $B\&C_2 = B\&B + \text{Family 1 and 2 as cuts}$

	All instances		High density ($p = 0.8$)	
Algorithm	Nodes	Time	Nodes	Time
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Merci!