Advances in Aharoni-Hartman-Hoffman's Conjecture for Split digraphs

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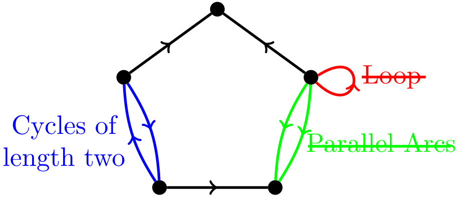
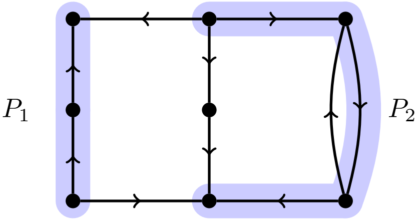
Basic definitions

- Let D be a digraph
  - Vertex set: $V(D)$
  - Arc set: $A(D)$
- Paths are directed



k-pack

A *k-pack* is a collection of at most $k \in \mathbb{Z}_+$ vertex-disjoint paths



$$\mathcal{P} = \{P_1, P_2\}$$

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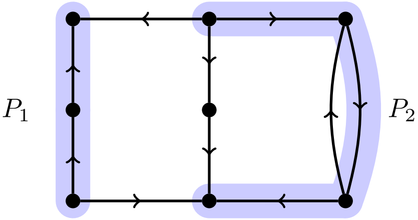
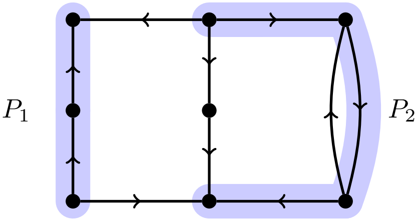


$$\mathcal{P} = \{P_1, P_2\}$$

A vertex v is *covered* by a k-pack $\mathcal{P}$ if $\exists P \in \mathcal{P}$ such that $v \in V(P)$

Weight of a k-pack

The *weight* of a k-pack $\mathcal{P}$, denoted by $||\mathcal{P}||$, is the number of vertices covered by $\mathcal{P}$



$$\mathcal{P} = \{P_1, P_2\}$$

$$||\mathcal{P}|| = 7$$

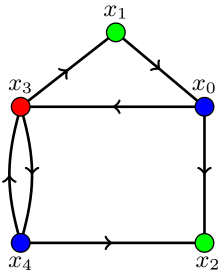
Optimal k-pack

- A k-pack $\mathcal{P}$ of D is *optimal* if its weight is maximum among all k-packs of D

- $\lambda_k(D)$: the weight of an optimal k-pack of D

Coloring

A *coloring* of a digraph D is a partition of its vertices into stable sets *(color classes)*

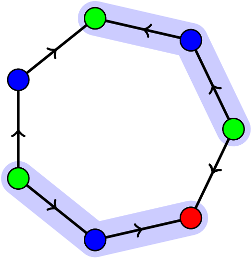


$$\mathcal{S} = \{\{x_0, x_4\}, \{x_1, x_2\}, \{x_3\}\}$$

Orthogonality

A k-pack $\mathcal{P}$ and a coloring $\mathcal{S}$ are *orthogonal* if each color class $C \in \mathcal{S}$ meets $\min\{|C|, k\}$ paths of $\mathcal{P}$

$k = 2$



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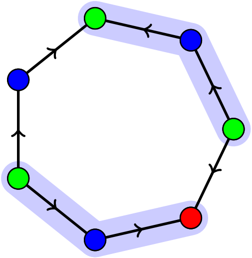

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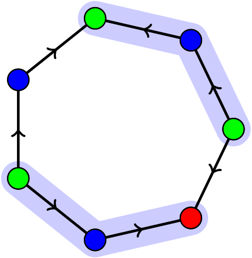

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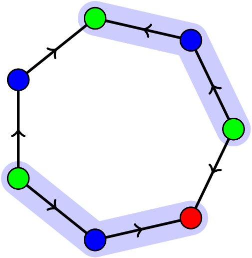


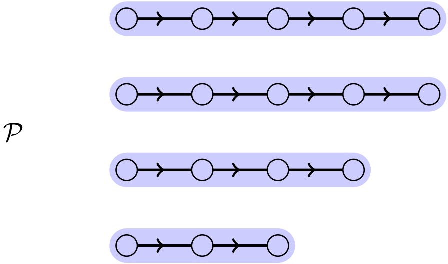
$$\min(|\bullet|, k) = \min(1, 2) = 1$$

Aharoni, Hartman, and Hoffman's Conjecture

Aharoni, Hartman, and Hoffman's Conjecture (AHH), 1985.
If $\mathcal{P}$ is an optimal k-pack, then there exists a coloring orthogonal to $\mathcal{P}$.

Some intuition

Orthogonality: each color class C must meet $\min\{|C|, k\}$ paths



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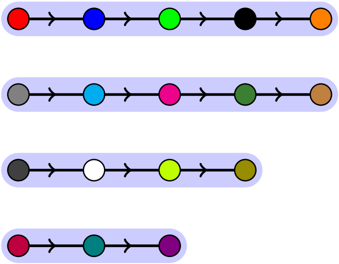
$\mathcal{P}$



$$\mathcal{C} = \{\{v\} \colon v \in V(D)\}$$

$$\min\{|\{v\}|, k\} = 1$$

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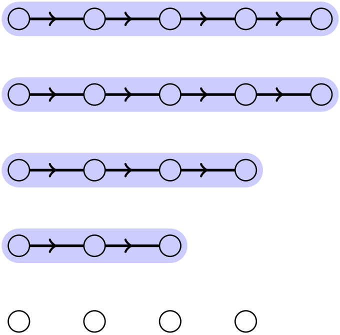

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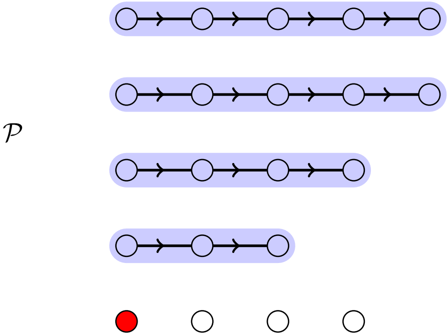

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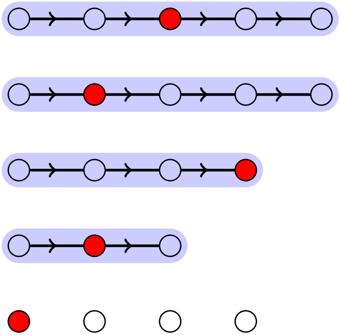

Transversal color: a color which meets every path in the k-pack

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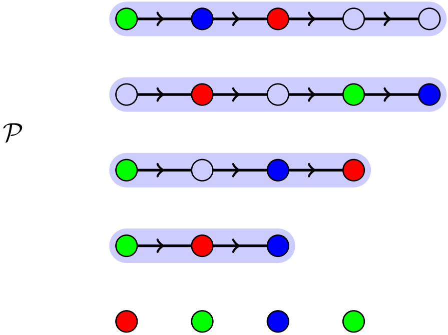

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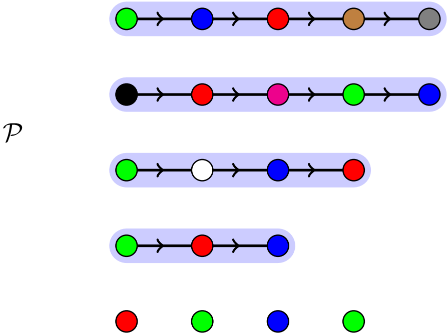

Orthogonal!

Known results

- $k = 1$ (Gallai, 1968, and Roy, 1967)
- Bipartite digraphs (Hartman *et al.*, 1994)
- Acyclic digraphs (Aharoni *et al.*, 1985)
- The optimal k-pack has at least one trivial path (Hartman *et al.*, 1994)

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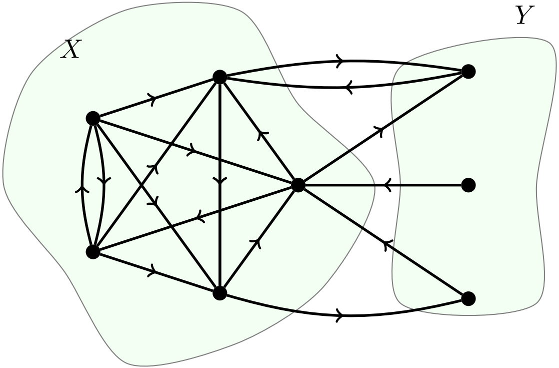
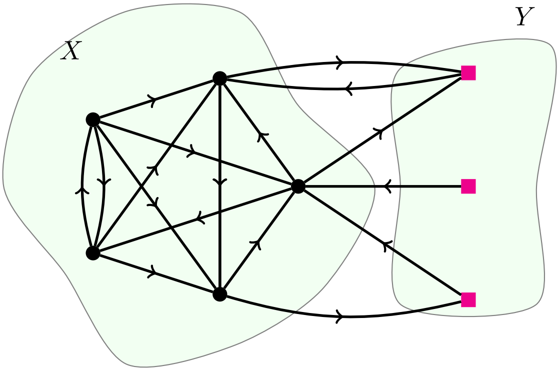
AHH's Conjecture	$\Rightarrow$	Linial's Conjecture (1981)
✔		✔
???		Split digraphs

Split digraph



X is a clique and Y a stable set

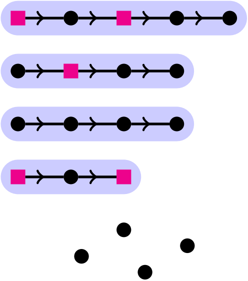
Split digraph

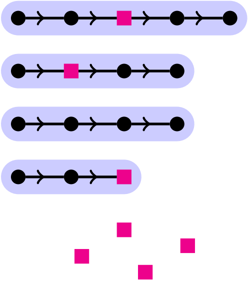


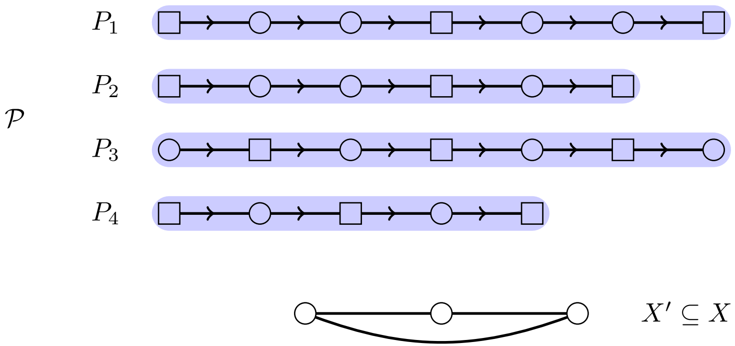
X is a clique and Y a stable set

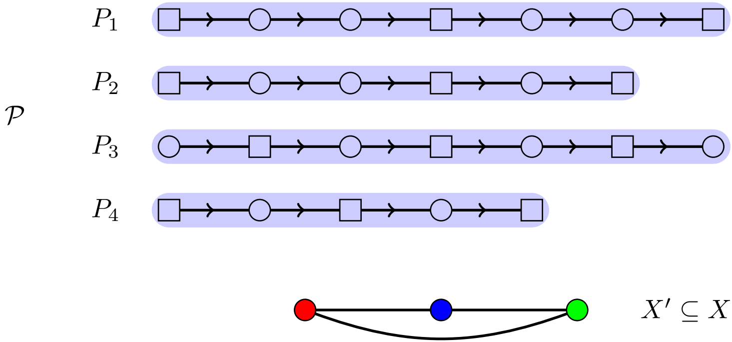
Our results

Given a split digraph D, we verified Aharoni, Hartman, and Hoffman's Conjecture when the optimal k-pack $\mathcal{P}$ satisfies one of the following conditions:






$\mathcal{P}$
P_1
P_2
P_3
P_4
$X' \subseteq X$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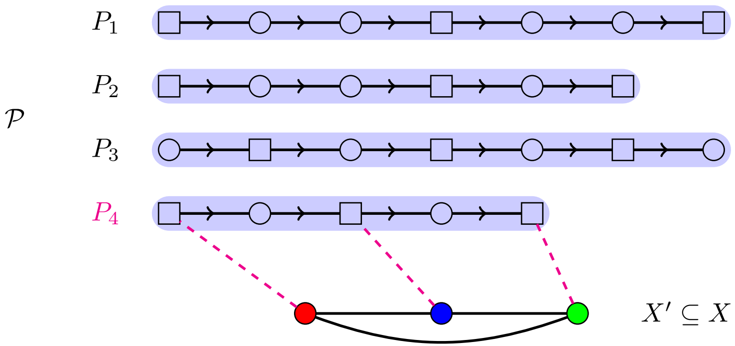
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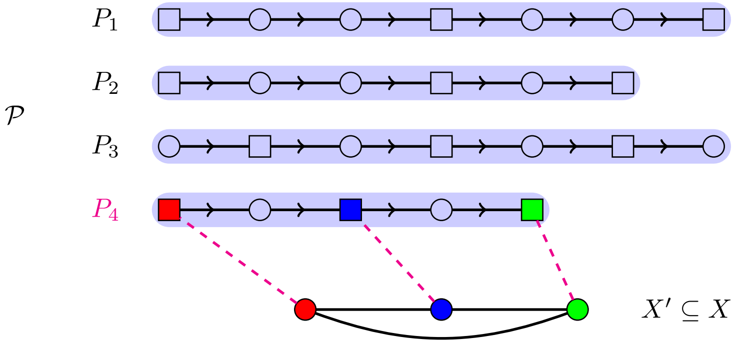
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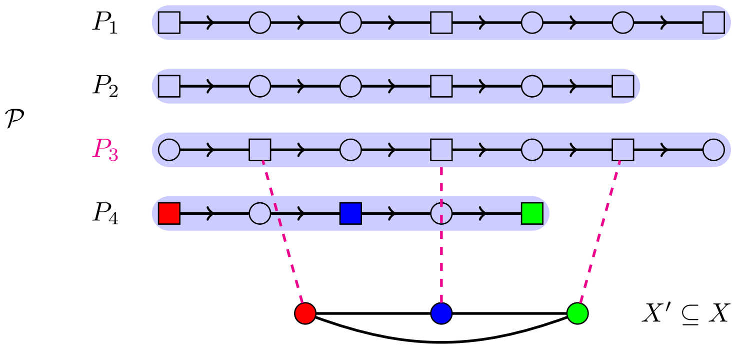

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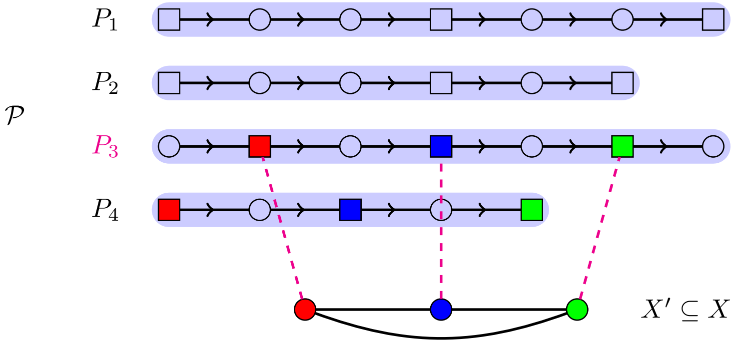
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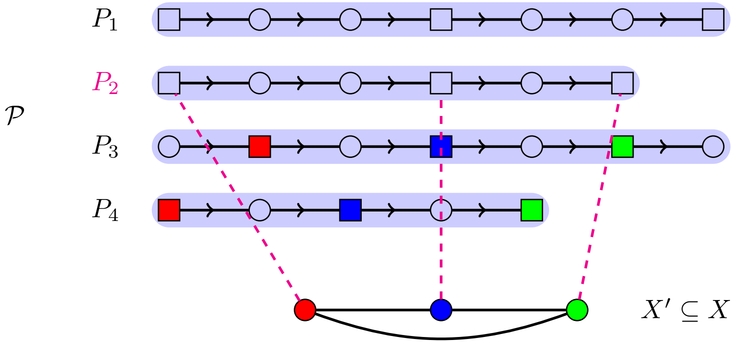

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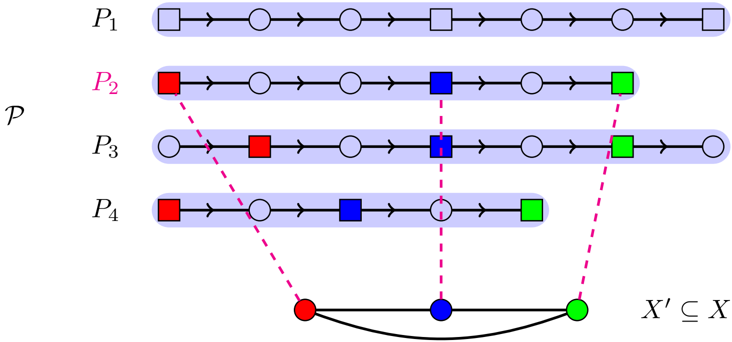

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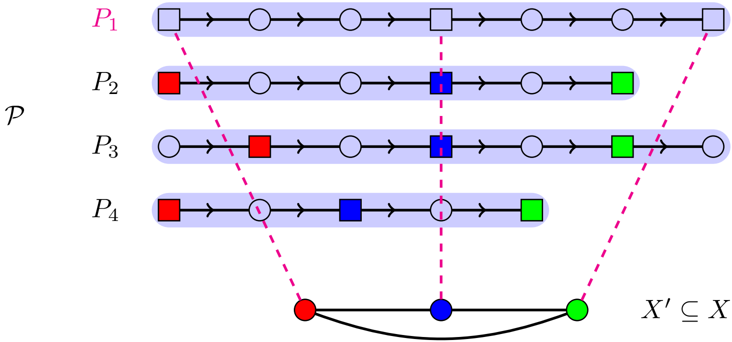
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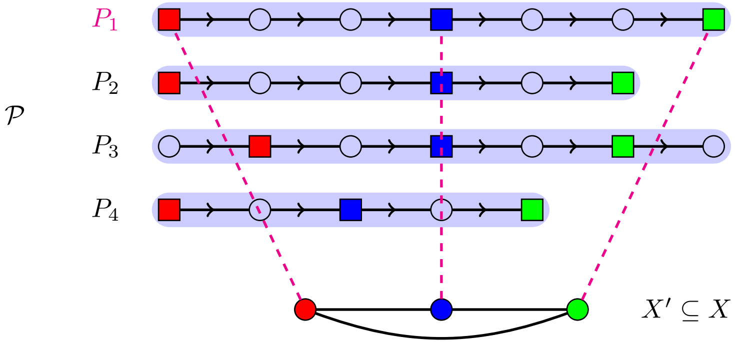
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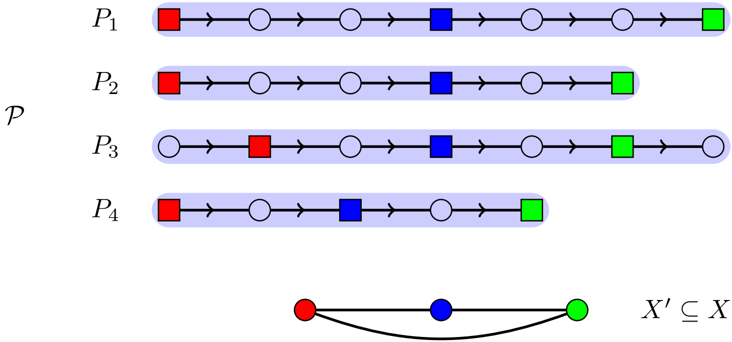
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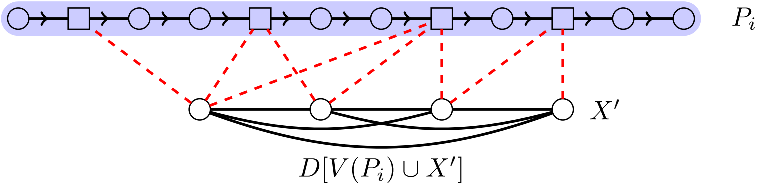
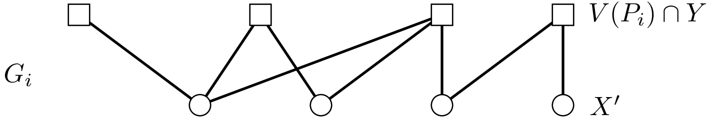
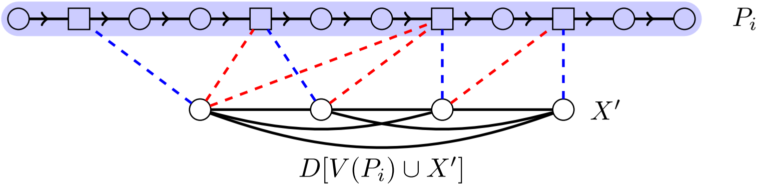
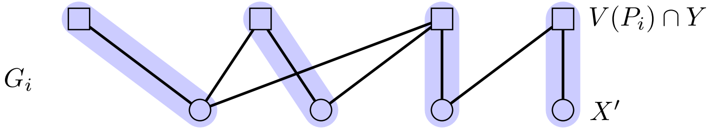




$$E(G_i) = \{uv \colon u \in V(P_i) \cap Y \text{ and } v \in X' \text{ are nonadjacent in } D\}$$





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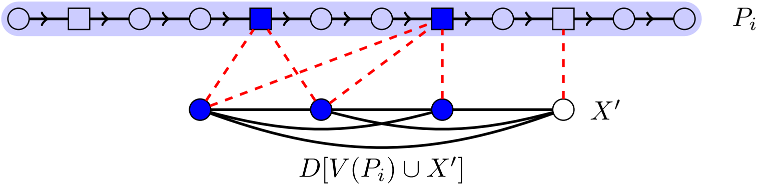

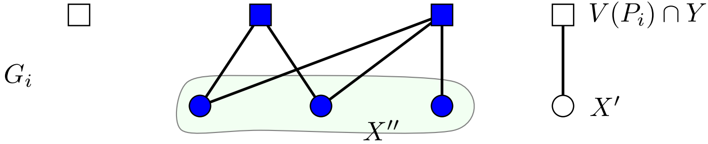

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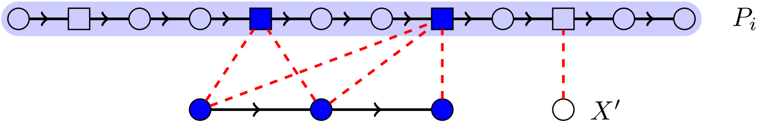

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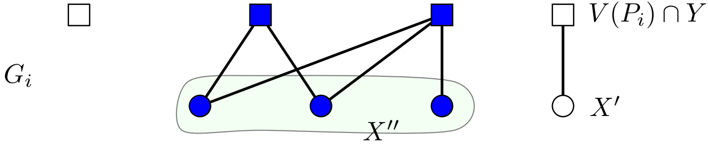

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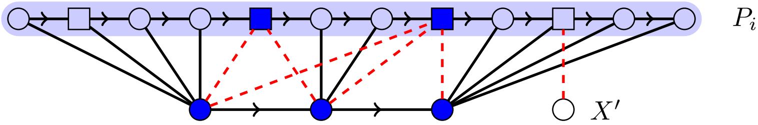

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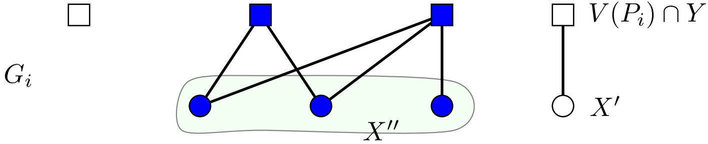

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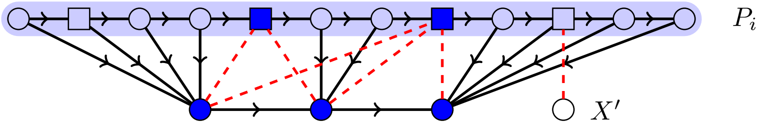

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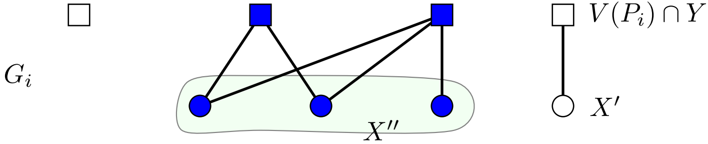

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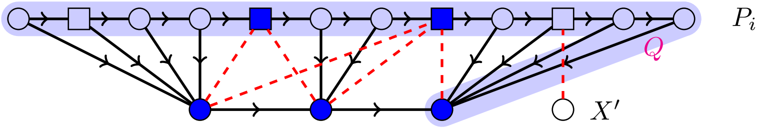

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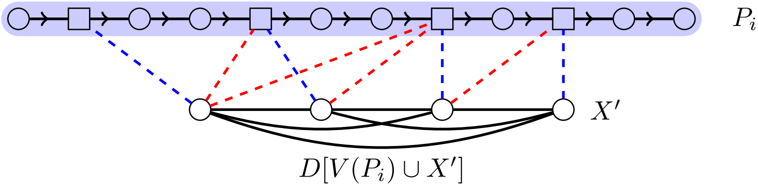
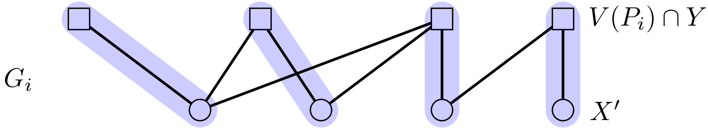
$$||\mathcal{P} - P_i + Q|| > ||\mathcal{P}||$$





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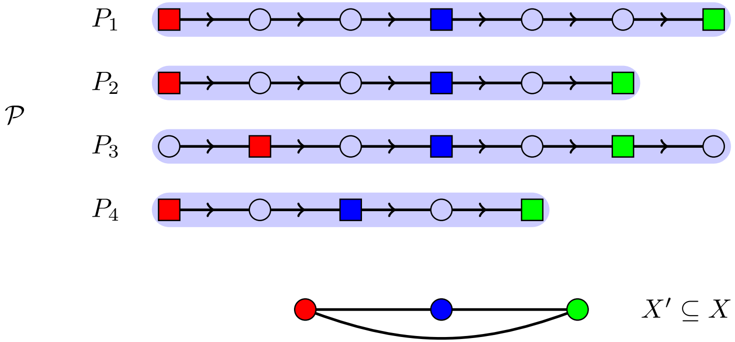

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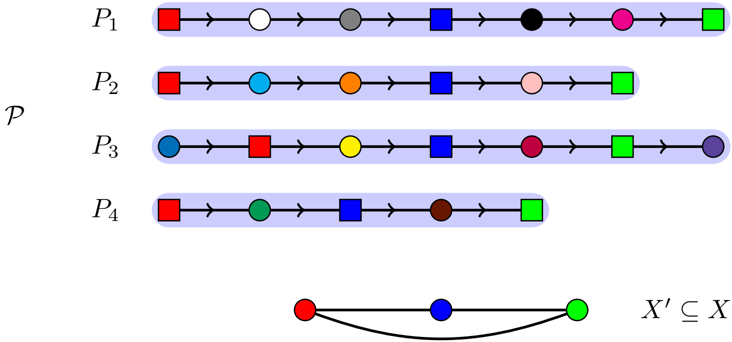
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THANK YOU