On cliques and bicliques

LAGOS 2017

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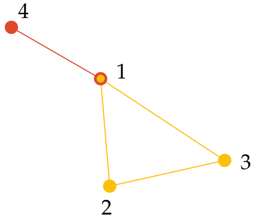
Introduction

Objective

Characterization of graphs G, that **maximize** $|K^2(G)|$.

Definition of clique

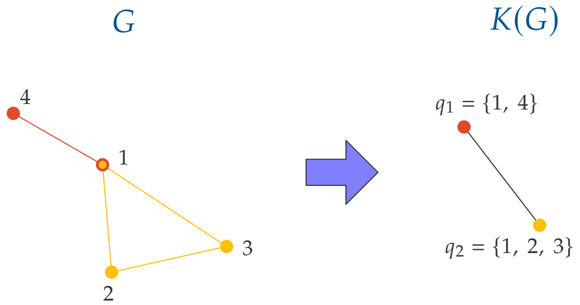
A clique is a maximal complete induced subgraph.



$$q_1 = \{1, 4\}, \quad q_2 = \{1, 2, 3\}.$$

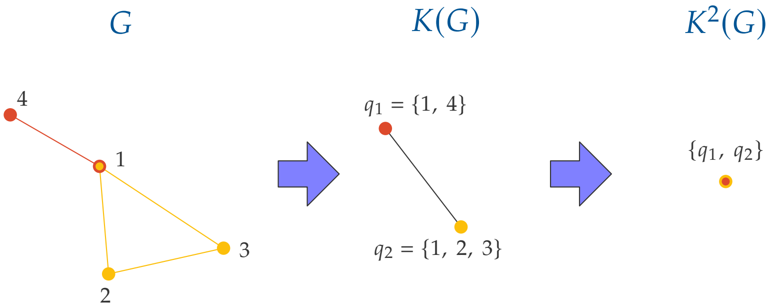
Definition of clique graph $K(G)$

The clique graph $K(G)$ of a graph G is the intersection graph of the set of all cliques.



Second clique graph $K^2(G)$

$$K^2(G) = K(K(G)).$$



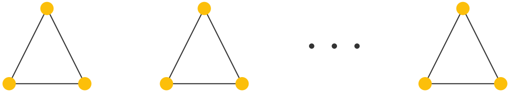
Graphs that maximize $|K(G)|$

The **Moon-Moser** graphs **maximize** $|K(G)|$.

Complement of Moon-Moser Graph, $n = |G|$

If $n \equiv 0 \pmod 3$

$\overline{G} =$



$\frac{n}{3}$ triangles

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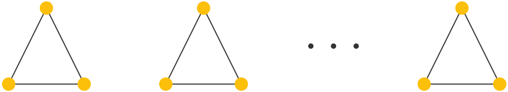
$\overline{G} =$ 

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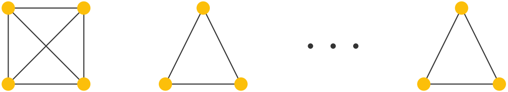
$\overline{G} =$ 

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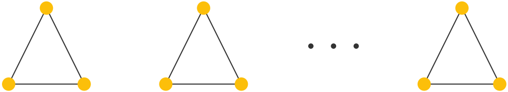
$\overline{G} =$ 

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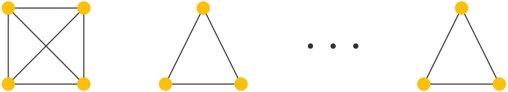
$\overline{G} =$ 

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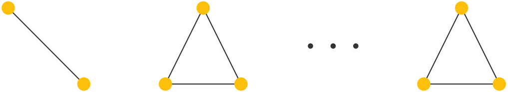
$\overline{G} =$ 

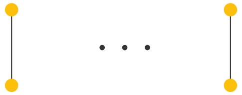
If $n \equiv 2 \pmod 3$

$\frac{(n-2)}{3}$ triangles

Graphs that maximize $|K^2(G)|$

It is **unknown** which graphs **maximize** $|K^2(G)|$.

Our conjecture for $|K^2(G)|$ (complement graph of G)

$\overline{G} =$ 

If $|G|$ is even, $d = \frac{|G|}{2}$

$I_2 = \overline{K_2}$

$G = I_2 + \ldots + I_2$ (d times).

$G \cong O_d$

(O_d is the d-dimensional Octahedral graph).

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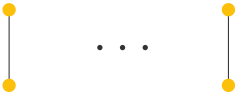
$\overline{G} =$ 

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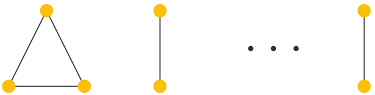
$\overline{G} =$ 

If $|G|$ is odd, $d = \frac{n-3}{2}$

$G = I_3 + O_d$

Bicliques

A new biclique definition

Let $\mathscr{B} = \{(X, Y) \in 2^G \times 2^G \mid x \simeq y, \text{ for every } x \in X \text{ and } y \in Y\}$.

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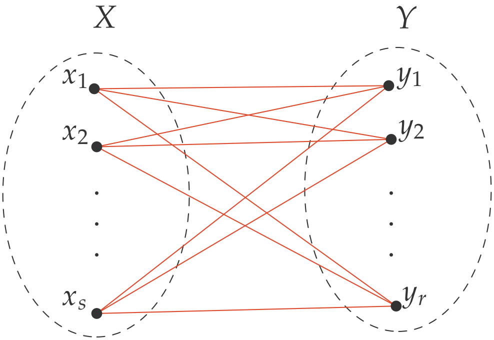

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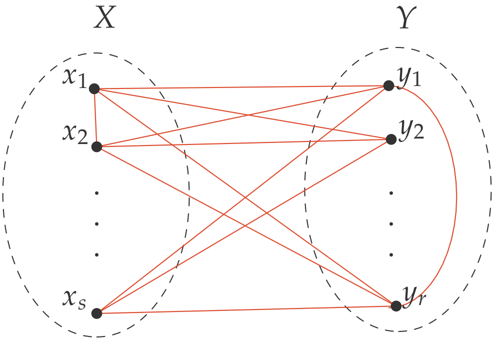

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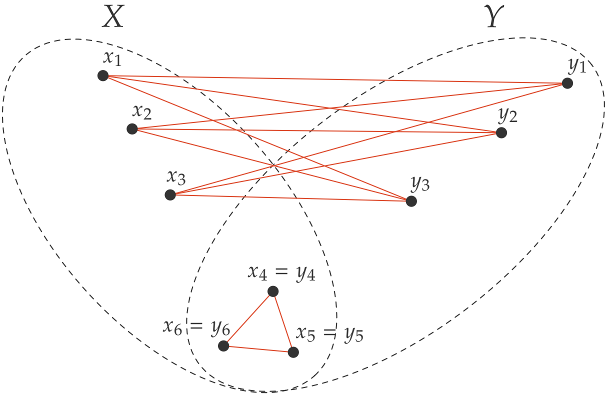
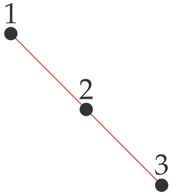


Bicliques (examples)

P_3



$(\{1,2\},\ \{1,2\})$
$(\{2,3\},\ \{2,3\})$
$(\{2\},\ \{1,2,3\})$
$(\{1,2,3\},\ \{2\})$

Biclique graph

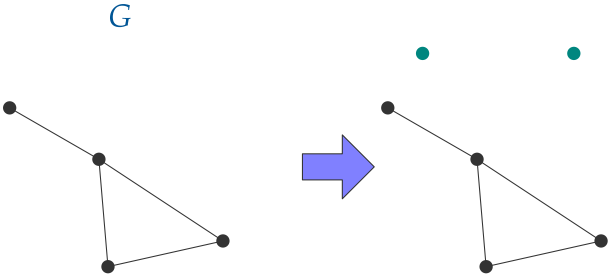
The *biclique graph* $B(G)$ of G is a graph such that

- $V(B(G))$ = bicliques of G
- Two vertices $(X_1, Y_1), (X_2, Y_2) \in B(G)$ are adjacent if and only if:
  - $X_1 \cap X_2 \neq \emptyset$, or
  - $Y_1 \cap Y_2 \neq \emptyset$.

Suspensions

The suspension $S(G)$ of a graph G is defined as

$$S(G) = I_2 + G.$$



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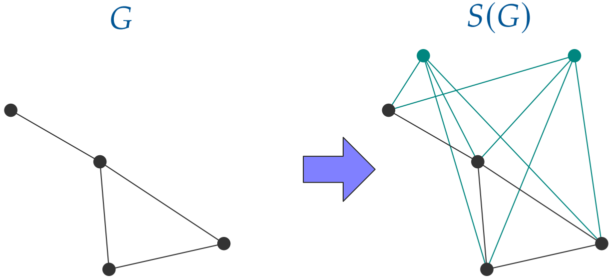

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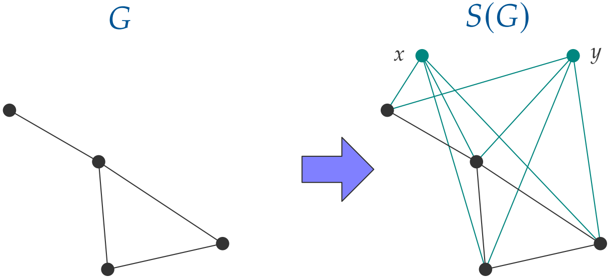
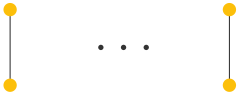


Our conjecture for $|K^2(G)|$ (suspensions)

If $|G| \geq 5$, then we have suspensions in our conjecture.

$\overline{G} =$ 

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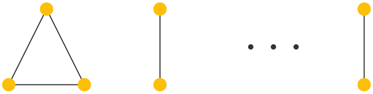
$\overline{G} =$ 

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Characterization of $K^2(S(G))$

Theorem 1

$K^2(S(G)) \cong B(K(G))$.

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Theorem 2

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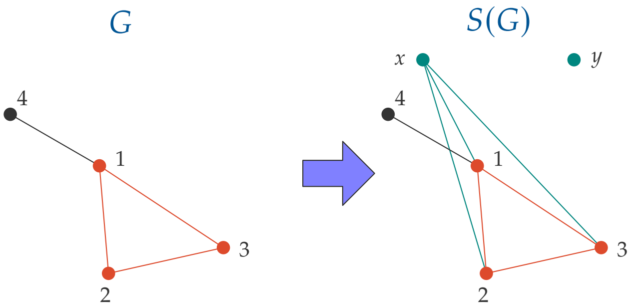
Sketch of the proof:



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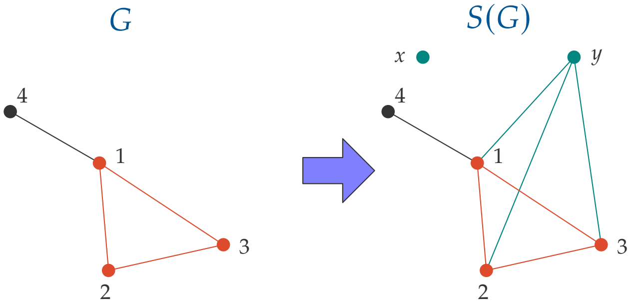

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Characterization of $K^2(S(G))$

Theorem 3

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Define $\tau : K^2(S(G)) \longrightarrow B(K(G))$ by noting that

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Then

$$\left(\{ q_1, \ldots, q_r \}, \ \{ q'_1, \ldots, q'_r \} \right) \in B(K(G))$$

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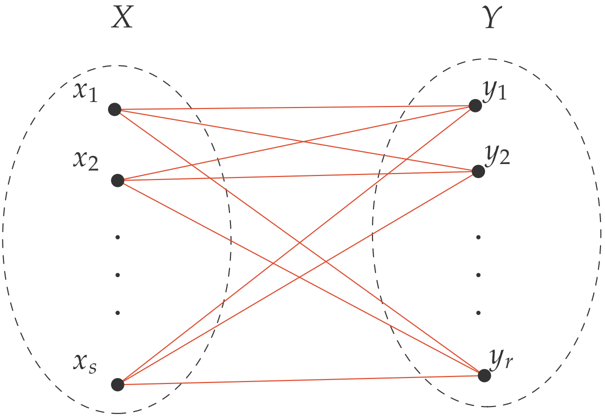

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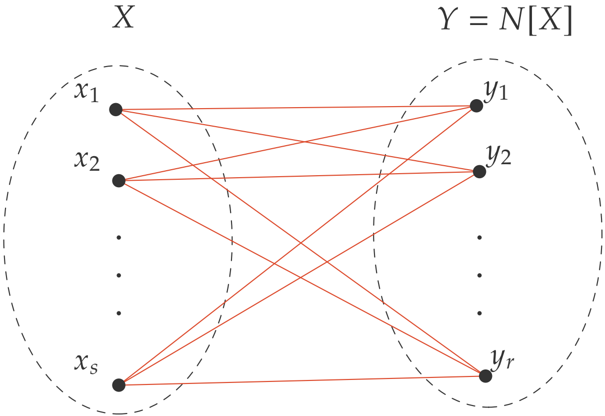

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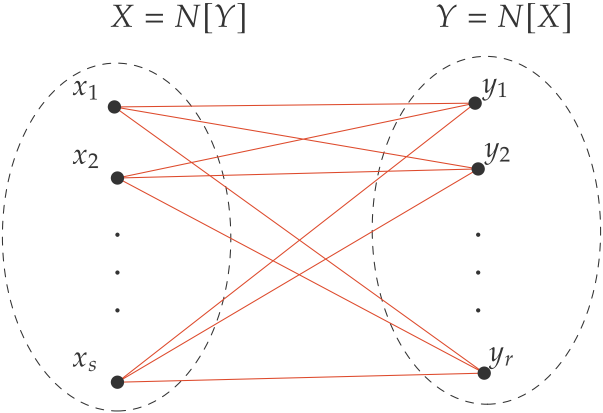

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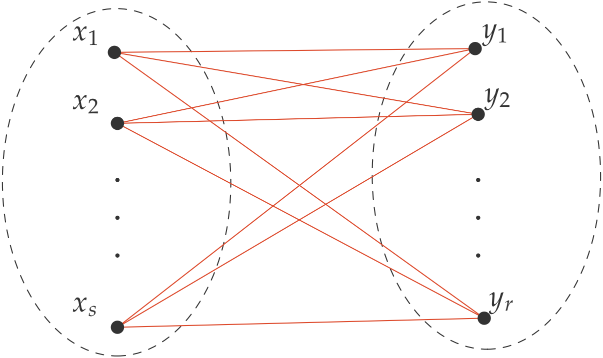
$$\text{Biclique } (X, Y) = (N[N[X]], N[X])$$

$N[Y] = N[N[X]]$ $\qquad$ $Y = N[X]$



The β function

Define $\beta : 2^G \longrightarrow B(G)$ by

$\beta(X) = (N[N[X]], N[X])$.

Observation 1

β is surjective, therefore $|B(G)| \leq 2^{|G|}$.

Graphs that maximize $|B(G)|$ if $|G|$ is even

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5. $n = |G|$ *is even and* $G \cong O_d$ *for* $d = \frac{n}{2}$.

Graphs that maximize $|B(G)|$ if $|G|$ is even, (4) $\Rightarrow$ (5)

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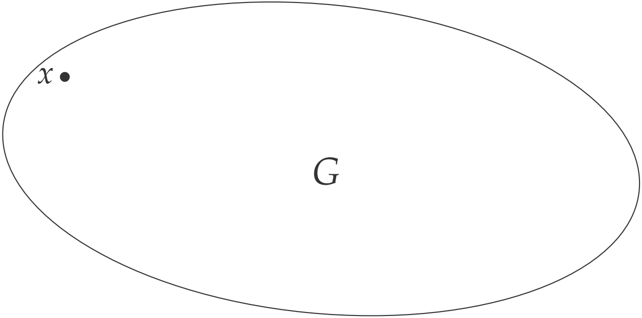

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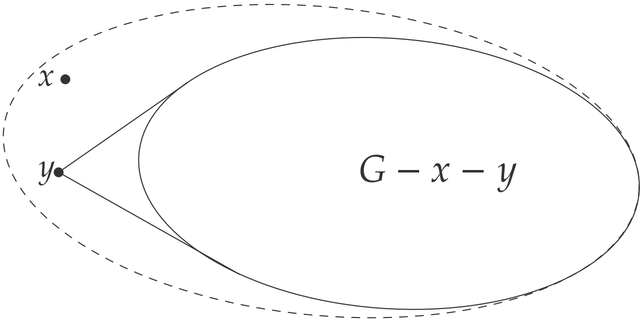

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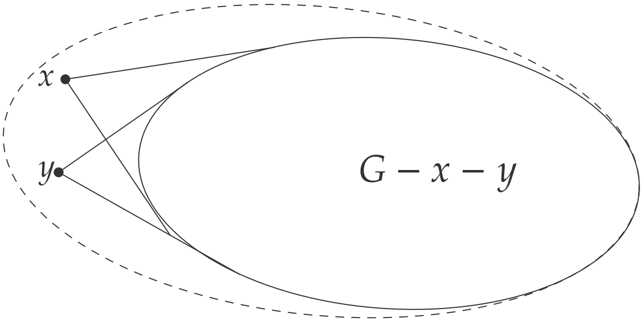

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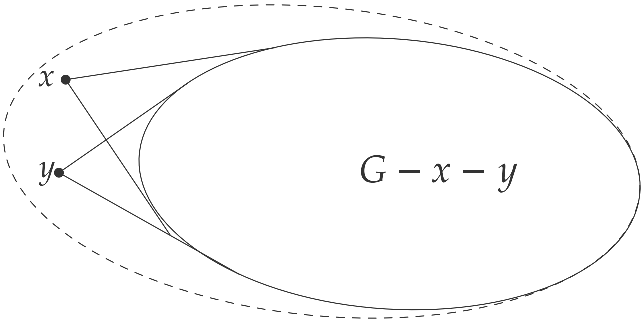
$$G \cong I_2 + (G - x - y)$$



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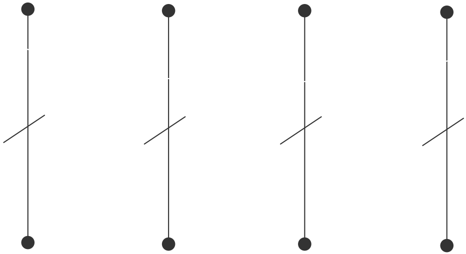

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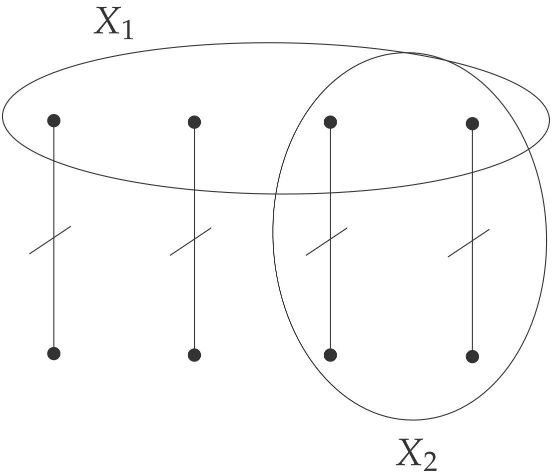

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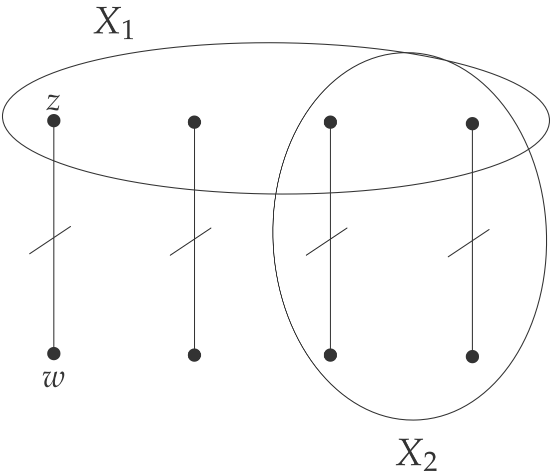

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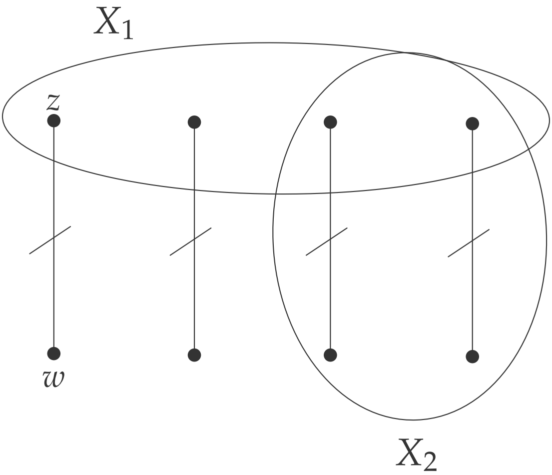
Graphs that maximize $|B(G)|$ if $|G|$ is even, (5) $\Rightarrow$ (1)

$n = |G|$ is even and $G \cong O_d$ for $d = \frac{n}{2}$.



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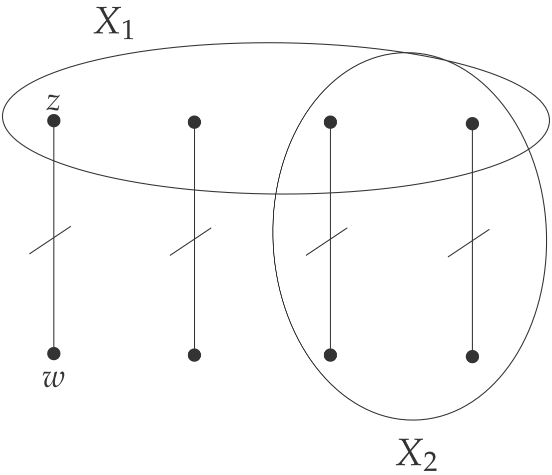

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Graphs that maximize $|B(G)|$ if $|G|$ is odd

The circle product

The *circle product* is defined as

$$G \circ H = \overline{\overline{G} \times \overline{H}}.$$

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Note that if $(g_1, h_1), (g_2, h_2) \in G \circ H$, then:

$$(g_1, h_1) \simeq (g_2, h_2) \Leftrightarrow g_1 \simeq g_2 \text{ in } G \text{ or } h_1 \simeq h_2 \text{ in } H$$

Biclique graph of the circle product

Theorem 5

For any graphs G and H, we have: $B(G + H) \cong B(G) \circ B(H)$.

Sketch of the proof:

Define $\phi : B(G + H) \longrightarrow B(G) \circ B(H)$ by

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$\phi((G_1 \cup H_1, G_2 \cup H_2)) = ((G_1, G_2), (H_1, H_2))$

Bicliques

Lemma 1

Number of bicliques for some basic graph families:

- $|B(K_n)| = 1$.
- $|B(I_n)| = n + 2$, *for* $n \geq 2$.
- $|B(P_n)| = 3n - 3$, *for* $n \geq 4$.
- $|B(C_n)| = 3n + 2$, *for* $n \geq 5$.

Exceptional cases:

- $|B(P_3)| = 4$
- $|B(C_3)| = 1$
- $|B(C_4)| = 16$

Graphs that maximize $|B(G)|$ if $|G|$ is odd

Lemma 2

Let $n = 2d + 3 = |I_3 + O_d|$, *then* $|B(I_3 + O_d)| = \frac{5}{8} \cdot 2^n$

Graphs that maximize $|B(G)|$ if $|G|$ is odd

Lemma 2

Let $n = 2d + 3 = |I_3 + O_d|$, *then* $|B(I_3 + O_d)| = \frac{5}{8} \cdot 2^n$

Proof.

By Theorem 5,

$$B(I_3 + O_d) \cong B(I_3) \circ B(O_d),$$

hence

$$\begin{aligned} |B(I_3 + O_d)| &= |B(I_3) \circ B(O_d)| = |B(I_3)| \cdot |B(O_d)| \\ &= 5 \cdot 2^{2d} = \frac{5}{8} \cdot 2^n. \end{aligned}$$

□

Graphs that maximize $|B(G)|$ if $|G|$ is odd

Theorem 6

Let G be a graph of order $n > 1$, maximizing $|B(G)|$. Then,

- *if $n = 2d$, we have that $G \cong O_d$;*
- *otherwise, $n = 2d + 3$ and $G \cong I_3 + O_d$.*

Graphs that maximize $|B(G)|$ if $|G|$ is odd

Sketch of the proof if $|G| = n = 2d + 3$.

Lemma 3

G can not have twin vertices (i.e. $N[x] \neq N[y]$ for all $x, y \in G$) nor universal vertices (i.e. $N[x] \neq G$ for all $x \in G$).

Lemma 4

If $\overline{G}$ has a vertex of degree r, then $|B(G)| \leq 2^n(\frac{1}{2} + \frac{1}{2^r})$. Hence $\Delta(\overline{G}) \leq 3$.

Lemma 5

If $\overline{G}$ *has a vertex of degree* 3*, then* $|B(G)| \leq \frac{5}{8} \cdot 2^n - 1$*. Hence* $\Delta(\overline{G}) \leq 2$ *and* $\overline{G}$ *is the disjoint union of cycles and paths.*

It follows that

$$G = \overline{P}_{n_1} + \overline{P}_{n_2} + \cdots + \overline{P}_{n_r} + \overline{C}_{m_1} + \overline{C}_{m_2} + \cdots + \overline{C}_{m_s}.$$

Lemma 6

The number of bicliques of the complements of paths satisfies:

1. $|B(\overline{P_2})| = 4 = 2^2$
2. $|B(\overline{P_n})| < \frac{5}{8} \cdot 2^n$ *for* $n \geq 3$.

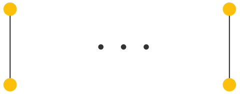
Lemma 7

The number of bicliques of the complements of cycles satisfies:

1. $|B(\overline{C_3})| = 5 = \frac{5}{8} \cdot 2^3$.
2. $|B(\overline{C_n})| < \frac{5}{8} \cdot 2^n$ *for* $n \geq 4$.

Since $I_2 = \overline{P}_2$, $I_3 = \overline{C}_3$ and $O_d = I_2 + I_2 + \cdots + I_2$ (d times), it follows by the previous lemmas and by Theorem 5 that $G \cong I_3 + O_d$, as claimed in Theorem 6.

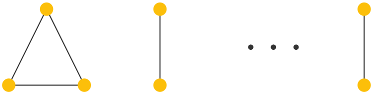
Graphs that maximizes $|B(G)|$

$\overline{G} =$ 

If $|G|$ is even, $d = \frac{|G|}{2}$

$G = I_2 + \ldots + I_2$ (d times).

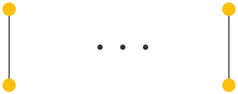
$G \cong O_d$

$\overline{G} =$ 

If $|G|$ is odd, $d = \frac{n-3}{2}$

$G = I_3 + O_d$

Graphs that maximizes $|B(G)|$

$\overline{G} =$ 

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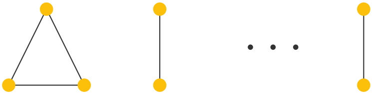
$\overline{G} =$ 

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Remember that $K^2(S(G)) \cong B(K(G))$.

Questions?