On the Existence of Critical Clique-Helly Graphs

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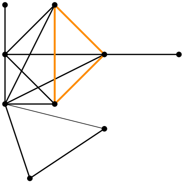
Conjecture [Dourado, Protti and Szwarcfiter]

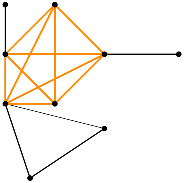
Every clique-Helly graph G (the family of maximal cliques of the graph satisfies the Helly property) contains a vertex v such that $G - v$ is a clique-Helly graph.

Journal of the Brazilian Computer Society . 2006, Vol. 12, Issue 1, pp 7–33 Computational aspects of the Helly property: a survey

A complete set of G is a subset of $V(G)$ inducing a complete subgraph. A clique is a maximal complete set (with respect to the inclusion relation).

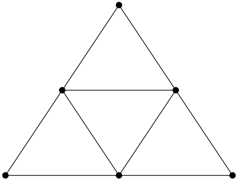


complete

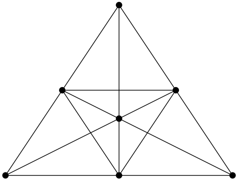


clique

A set family $\mathcal{F}$ satisfies the Helly property if the intersection of all the members of any pairwise intersecting subfamily of $\mathcal{F}$ is non-empty. When the cliques family of G, $\mathcal{C}(G)$, satisfies the Helly property, we say that G is a clique-Helly graph.



Not clique-Helly graph



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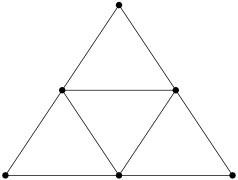

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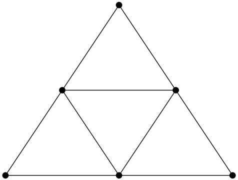

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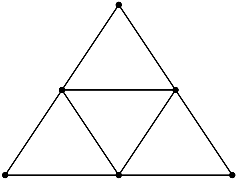

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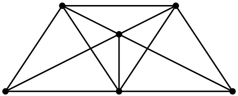


Clique-Helly graph

The way we will build a counterexample to the conjecture:

$$Icosahedron$$

$$\Downarrow$$

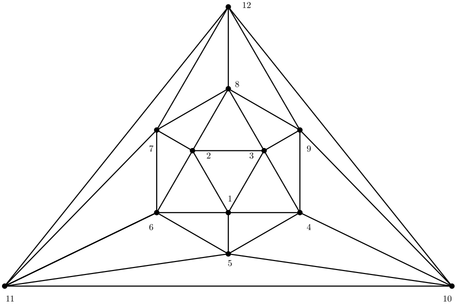
$$Icosahedron \times K_3 \qquad \text{(tensor product)}$$

$$\Downarrow$$

$$K(Icosahedron \times K_3) \qquad \text{(clique graph)}$$

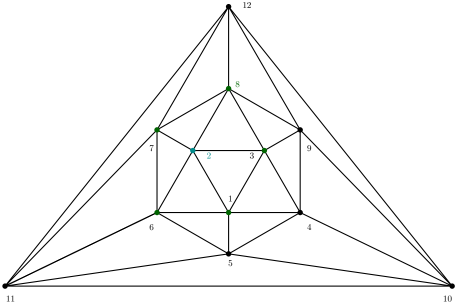
Icosahedron

- The icosahedron I is one of the platonic graphs.



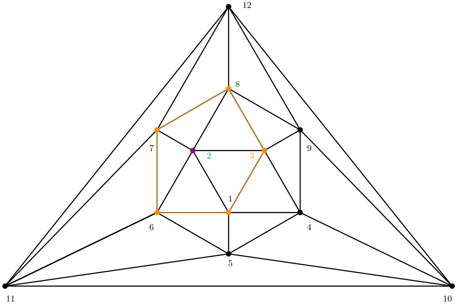
Icosahedron

* Every vertex has degree 5.



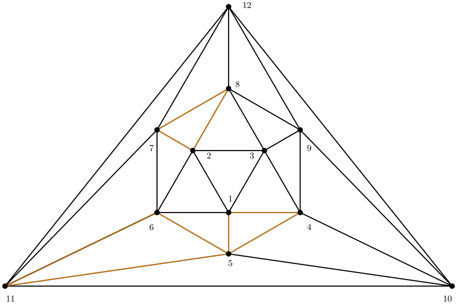
Icosahedron

* The open neighborhood of each vertex induces a C_5.



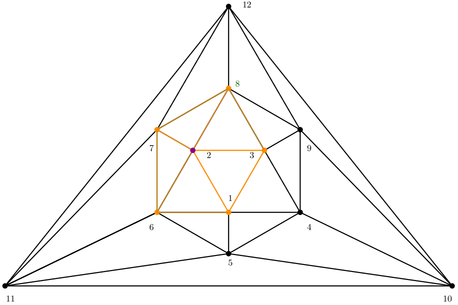
Icosahedron

* The cliques are all triangles.



Icosahedron

* Every vertex is in exactly 5 cliques.



The tensor product $I \times K_3$

- The tensor product $P = I \times K_3$ is the graph with:

- vertices (i, j) where $i \in V(I)$ and $j \in V(K_3)$ and
- two vertices (i, j) and (i', j') adjacent in P if and only if

i is adjacent to i' in I and
j is adjacent to j' in K_3.

The tensor product $I \times K_3$

- Every vertex of P has degree 10.
- The open neighborhood of each vertex of P induces a C_{10}.
- The cliques of P are triangles $\{(i,1),(j,2),(k,3)\}$ for $\{i,j,k\}$ any triangle of I.
- Every vertex of P is in exactly ten cliques.

The clique graph $K(I \times K_3)$

- The clique graph of $I \times K_3$ is the intersection graph of the cliques family of $I \times K_3$.

It has 120 vertices.....

A example of clique graph

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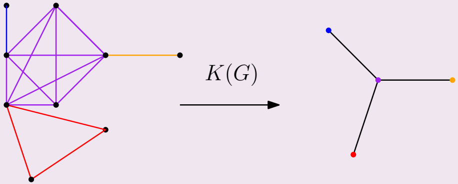

Main result of this work

Theorem

The graph $G = K(I \times K_3)$ is clique-Helly and for each $v \in G$, $G - v$ is not clique-Helly .

Those graphs satisfying the conditions of the theorem are called critical clique-Helly.

$K(I \times K_3)$ is clique-Helly

To prove this we will use

Theorem (Larrión, Neumann-Lara, Pizaña)

If the local girth of the graph G is greater than 6 (i.e. $lg(G) \geq 7$) then $K(G)$ is clique-Helly.

The local girth of G **at a vertex** $v \in V(G)$ is the length of a shortest chordless cycle of the subgraph induced by the open neighborhood of v in G.
And the local girth of G is the minimum of the local girth at all the vertices v.
As we saw in the properties of P the open neighborhood of each vertex induces a C_{10}

$K(I \times K_3) - v$ is not clique-Helly for all $v \in V(K(I \times K_3))$

- Every vertex v of $G = K(I \times K_3)$ represents a clique Q_v of $I \times K_3$, say $Q_v = \{x, y, z\}$.
- Each of these vertices is the center of a 10-wheel.
- Each pair of those wheels have two triangles in common.
- And the total intersection between them is Q_v.
- So, if we remove v of G the corresponding cliques of the wheels are pairwise intersecting with empty total intersection.

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Future work

- Show that there is an infinite family of critical clique-Helly graphs.
- Show that there is an infinite family of critical clique-Helly self clique graphs.

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Thank you!