Inapproximability Ratios for Crossing Number

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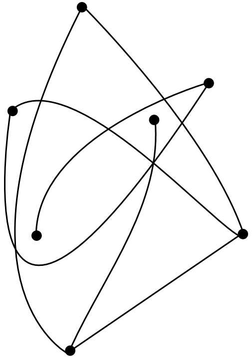
September 14, 2017

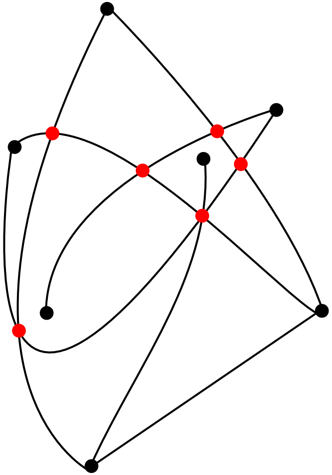
Drawings

- A drawing of a graph G is a function $D : G \to \mathbb{R}^2$ that maps each vertex to a distinct point and each edge $uv \in E(G)$ to an arc connecting $D(u)$ to $D(v)$ that does not contain the image of any other vertex.



Crossing

- A crossing occurs whenever the image of two edges coincide somewhere other than their extremes.



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Crossing Number Problem

Input: A graph G.

Output: A drawing of G whose crossing number is $\text{cr}(G)$.

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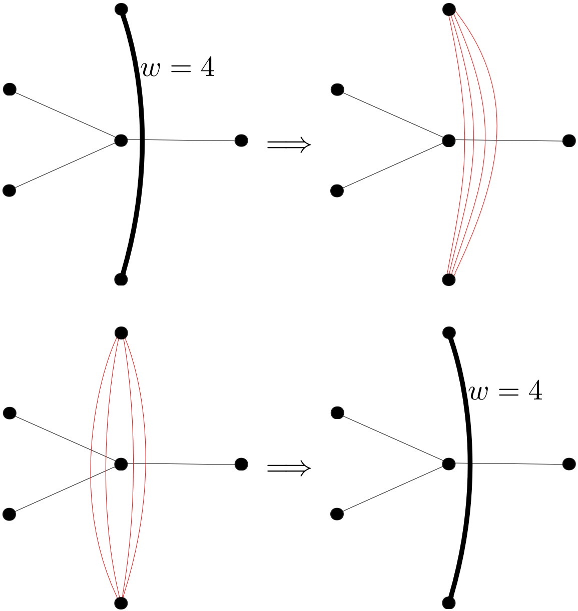
- Here, some precise values for such a c are presented.

Weighted Edges

- For simplicity, weighted graphs will be used in the constructions.
- Despite of this, the results also apply for unweighted graphs.
- For translating a weighted result to the unweighted case, it is enough to substitute each weighted edge $e = uv$, whose weight $w(e)$ is a positive integer, by $w(e)$ paralell paths connecting u to v.

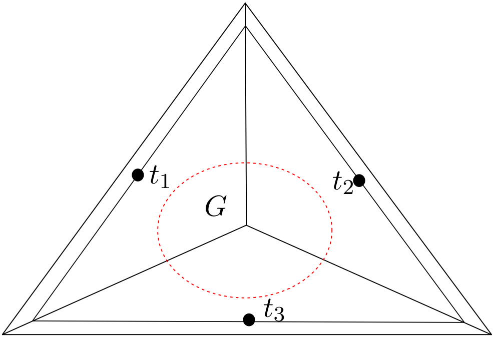
Weighted Edges



Multiway Cut

Input: A graph G and a set $T \subset V(G)$ of terminals.

Output: The minimum set $C \subset E(G)$ such that the vertices in T are disconnected in $G \setminus C$.

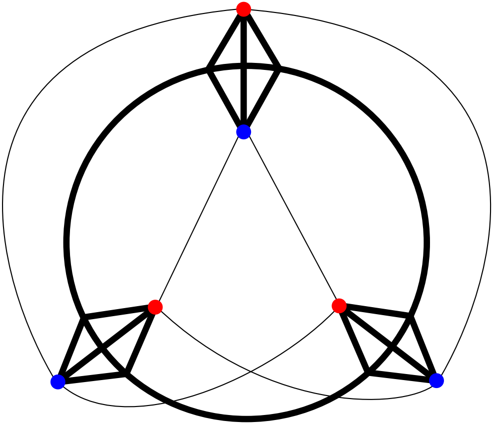


New Reduction

Maximum Cut

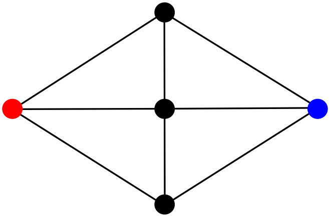
Input: A graph G.

Output: The set $X \subset V(G)$ with largest $\delta(X)$.



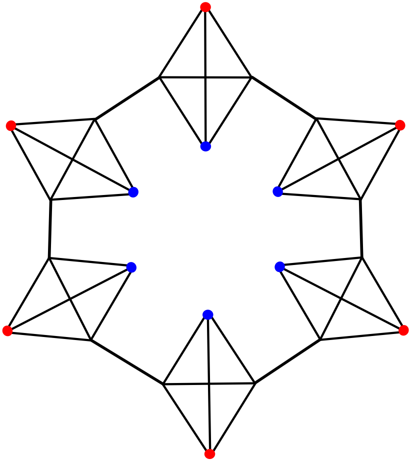
Given a graph G with n vertices and m edges, we construct a new graph G' as follows.

- For each vertex in G, create a gadget composed of five vertices and heavy edges (weighting m^3):



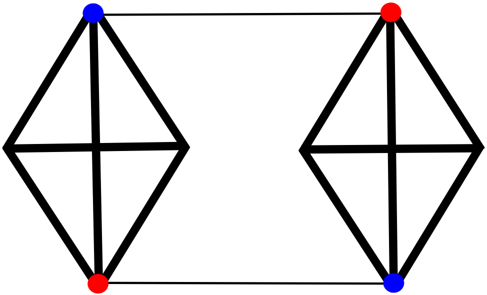
New Reduction

- Seat the gadgets on a circular frame, also made by heavy edges:



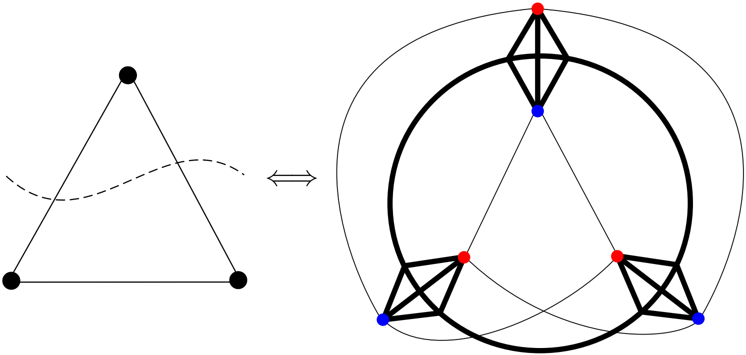
New Reduction

- For every edge $uv \in E(G)$, link the opposite sides of the corresponding gadgets with light edges (with unitary weight):



New Reduction

- Note that solving Crossing Number for the constructed graph is the same as deciding, for each vertex, on which side to put the blue and the red ends.
- Moreover, deciding the side of the blue and red ends is equivalent to solving Maximum Cut on the original graph.



New Reduction

Let:

- $\mathcal{A}_{\text{cr}}$ be a polynomial-time c-approximation algorithm for Crossing Number.
- $\mathcal{A}_{\text{mc}}$ be a polynomial-time approximation for Maximum Cut derived from $\mathcal{A}_{\text{cr}}$.
- $\overline{\text{mc}}(G)$ be the size of the returned by $\mathcal{A}_{\text{mc}}$ applied to G.
- $\overline{\text{cr}}(G)$ be the number of crossings in the drawing returned by $\mathcal{A}_{\text{cr}}$ applied to G'.

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Lemma 1

$2m^3(m - \overline{\text{mc}}(G)) \leq \overline{\text{cr}}(G')$

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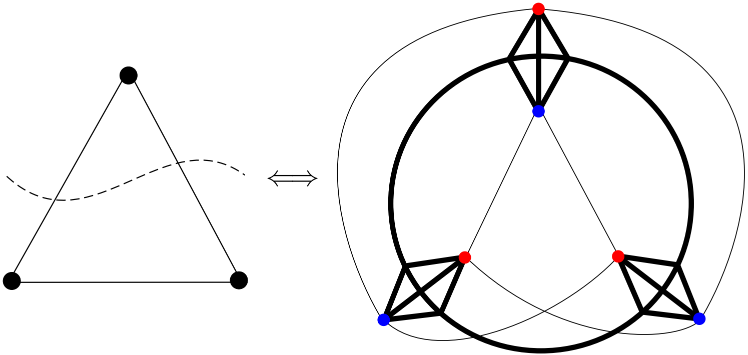

Approximation Ratio of $\mathcal{A}_{\mathrm{mc}}$

- Applying Lemmas 1 and 2:

$$\begin{aligned} 2m^3(m - \overline{\mathrm{mc}}(G)) &\leq \overline{\mathrm{cr}}(G') \\ &\leq c\ \mathrm{cr}(G') \\ &\leq 2cm^3(m - \mathrm{mc}(G)) + c4m^2. \end{aligned}$$

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- Assuming G is not bipartite:

$$\begin{aligned} m - \overline{\mathrm{mc}}(G) &\leq c(m - \mathrm{mc}(G)) + \frac{2c}{m} \\ &\leq c(m - \mathrm{mc}(G)) + \frac{2c(m - \mathrm{mc}(G))}{m} \\ &\leq c\left(1 + \frac{2}{m}\right)(m - \mathrm{mc}(G)) \\ &= c\left(1 + \frac{2}{m}\right)m - c\left(1 + \frac{2}{m}\right)\mathrm{mc}(G). \end{aligned}$$

Approximation Ratio of $\mathcal{A}_{\mathrm{mc}}$

- The last inequality gives:

$$\overline{\mathrm{mc}}(G) \geq c\left(1+\frac{2}{m}\right)\mathrm{mc}(G)+\left(1-c\left(1+\frac{2}{m}\right)\right)m.$$

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- From $\mathrm{mc}(G) \geq \frac{m}{2}$ it follows that:

$$\begin{aligned}\overline{\mathrm{mc}}(G) &\geq c\left(1+\frac{2}{m}\right)\mathrm{mc}(G) + \left(1 - c\left(1+\frac{2}{m}\right)\right)2\mathrm{mc}(G)\\ &= \left(2 - c\left(1+\frac{2}{m}\right)\right)\mathrm{mc}(G).\end{aligned}$$

Results

Theorem

If $\mathcal{A}_{\text{cr}}$ is a constant-factor polynomial c-approximation for Crossing Number, then $c \geq 2 - \frac{16}{17} \approx 1.058824$, assuming $P \neq NP$.

- It is NP-hard to approximate Maximum Cut polynomially by a constant ratio greater than $\frac{16}{17}$ (Håstad, 2001) .
- $\mathcal{A}_{\text{mc}}$ satisfies
$$2 - c\left(1 + \frac{2}{m}\right) \leq \frac{16}{17}.$$
- The inequality must hold for every m. □

- Khot *et al.* (2007) proved that, if $P \neq NP$ and the Unique Games Conjecture is true, then the approximation ratio $\alpha \approx 0.878567$ obtained by the algorithm by Goemans and Williamson is the best possible.

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Theorem

If $\mathcal{A}_{\text{cr}}$ is a constant-factor polynomial c-approximation for Crossing Number, then $c \geq 2 - \alpha \approx 1.121433$, assuming $P \neq NP$ and the UGC.

Thank You!

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