Constant Threshold Intersection Graphs of Orthodox Paths in Trees

Claudson F. Bornstein[1] José W. C. Pinto[1,2]
Dieter Rautenbach[3] Jayme Szwarcfiter[1,4]

[1]Federal University of Rio de Janeiro, Brazil
[2]Technological Education Faculty of Rio de Janeiro, Brazil
[3]University of Ulm, Germany
[4]Rio de Janeiro State University, Brazil

2017/09/11

LAGOS 2017

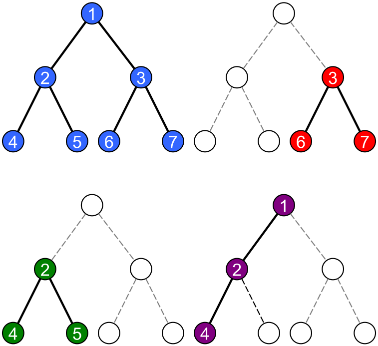
Purpose

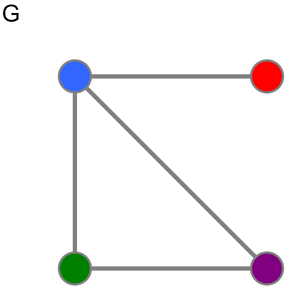
- Present a solution to a problem posed by Golumbic, Lipshteyn and Stern (2008)
- Show that the graphs in ORTH$[3, 2, 3]$ are line graphs of planar graphs

Intersection Graphs of Subtrees of a Tree

A graph G is an Intersection Graph of Subtrees of a Tree if:

- $\exists$ a family of subtrees $\{S_v\}_{v \in V(G)}$ of a host tree T
- $uv \in E(G) \iff S_u \cap S_v \neq \emptyset$





(h,s,t)-representation

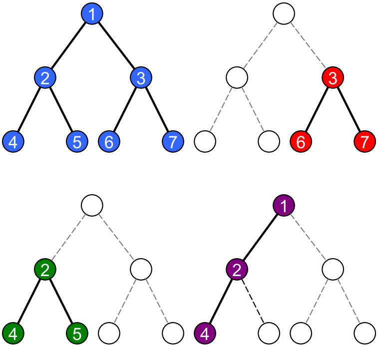
A graph G has an (h,s,t)-representation if $\exists$ a family of subtrees $\{S_v\}_{v \in V(G)}$ of a host tree T, such that:

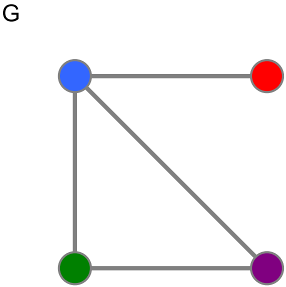
- $\Delta(T) \leq h$
- $\Delta(S_v) \leq s$
- $uv \in E(G) \;\Leftrightarrow\; |S_u \cap S_v| \geq t$

We denote the class of graphs having an (h,s,t)-representation by $[h,s,t]$.

(h,s,t)-representation

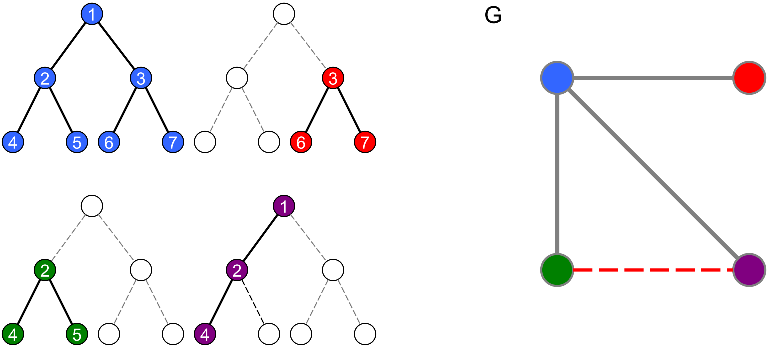
(3 , 3 , 1)-representation





(h,s,t)-representation

(3 , 3 , 3)-representation



orthodox (h, s, t)-representation

- The leaves of each subtree S_v must be leaves of T
- The vertices $u, v \in G$ are adjacent iff:
  S_u, S_v have at least t vertices in common, iff
  S_u, S_v share a leaf of T.

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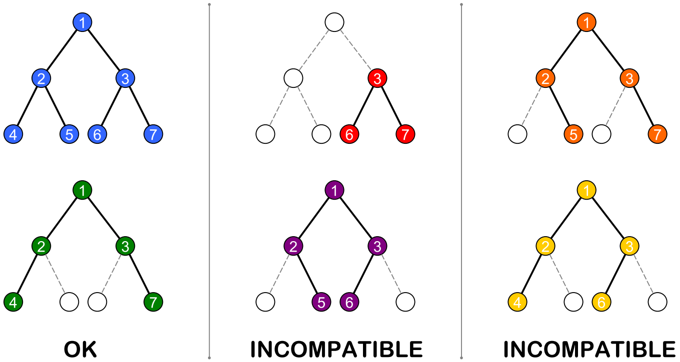
orthodox (3,3,3)-representation



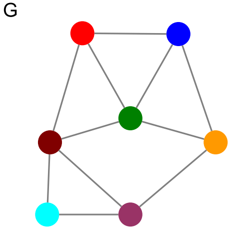
orthodox (h, s, t)-representation

We denote the class of graphs having an orthodox (h, s, t)-representation by $\mathrm{ORTH}[h, s, t]$.

ORTH$[h, 2, t]$ and line graphs

Theorem

Let G be a connected, twin-free graph with $|V(G)| \geq 4$. If G is ORTH[h,2,t] with $h \geq 3$ then G is the line graph of a connected graph H.



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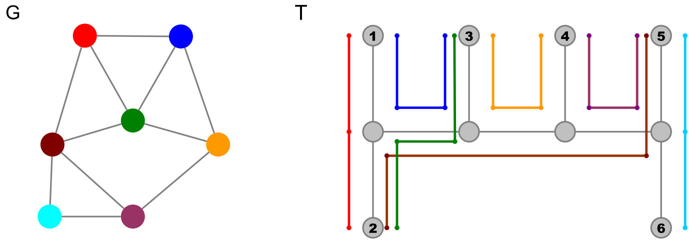

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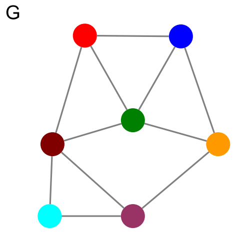

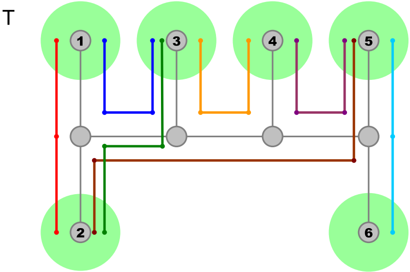

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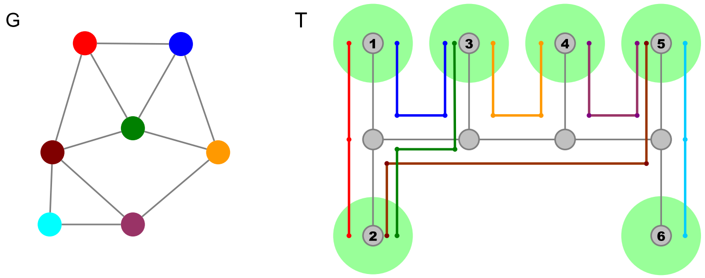

H is unique up to isomorphism

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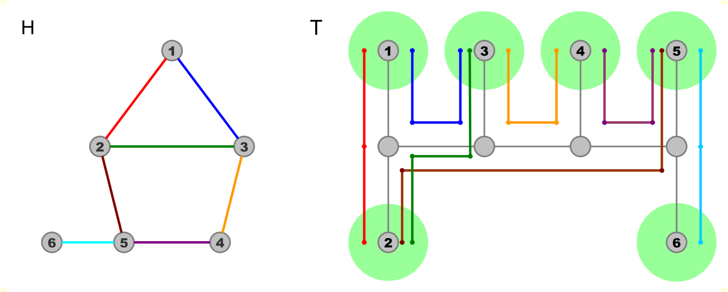

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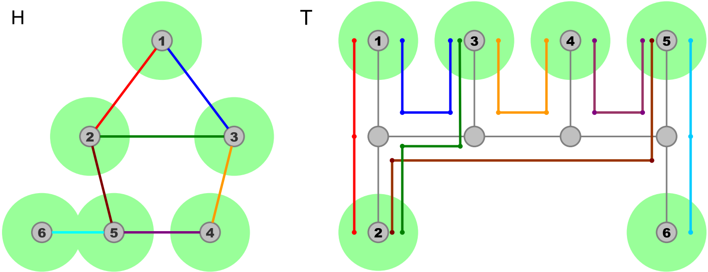

there is a bijection $\phi : V(H) \to \mathcal{L}(T)$, and

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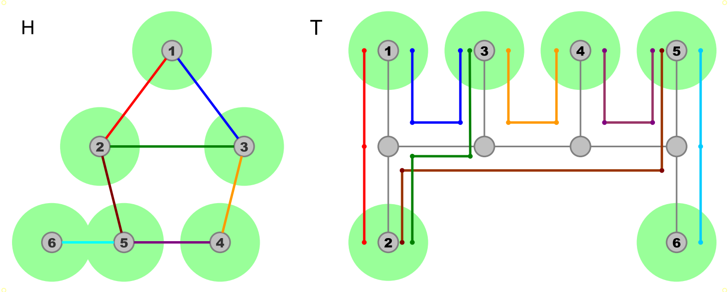


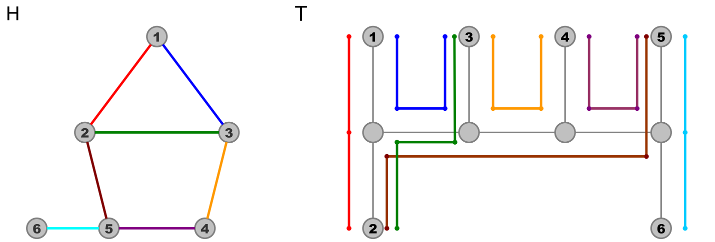
two distinct vertices x and y are adjacent in H if and only if exist a path in T between $\phi(x)$ and $\phi(y)$.

Tree layout

Let H be a graph.
If $\exists$ a tree T and two integers $h \geq 3$ and $t \geq 1$, s.t.

- $\Delta(T) \leq h$,
- $V(H) = \mathcal{L}(T)$, and
- for every two independent edges xy and $x'y'$ of H, the two paths $x - y$ and $x' - y'$ in T share at most $t - 1$ vertices.

Then, T is an (h, t)-$\mathrm{tree\ layout}$ of H.



ORTH$[h, 2, t]$ and (h, t)-tree layout

Theorem

Let G be a connected twin-free line graph of order at least 4, H a connected graph with $L(H) = G$, and $h \geq 3$, $t \geq 1$ integers. Then

$$G \in ORTH[h, 2, t] \;\Leftrightarrow\; H \text{ has an } (h, t)\text{-tree layout.}$$

The Question

Golumbic, Lipshteyn and Stern asked
if ORTH$[\infty, 2, t]$ and ORTH$[3, 2, t]$ coincide or
there is a separating example between these families.

M.C. Golumbic, M. Lipshteyn, M. Stern, *Equivalences and the complete hierarchy of intersection graphs of paths in a tree*, **Discrete Applied Mathematics**, 156, pp. 3203–3215, 2008.

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We find some of these examples.

For each pair $h, t \geq 3$, we exhibit a graph
$G \in (ORTH[h+1, 2, t] \setminus ORTH[h, 2, t])$.

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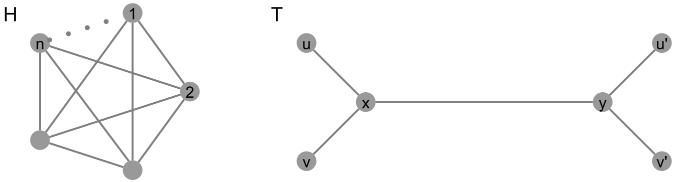
Maximum number of leaves of T

Let H be a complete graph K_n, and T an (h, t)-tree layout of H.



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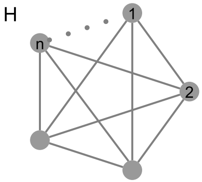
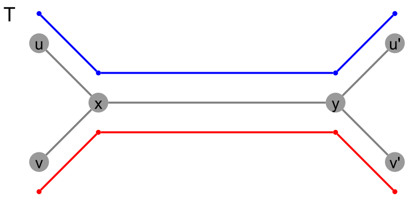
Since H is complete, there is a path between each pair of leaves.





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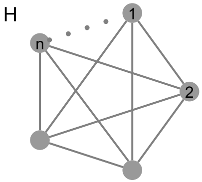

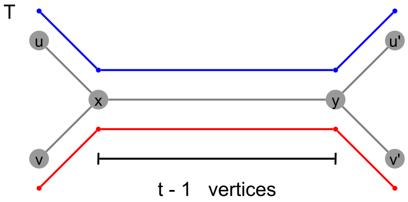

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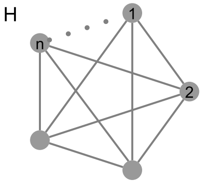
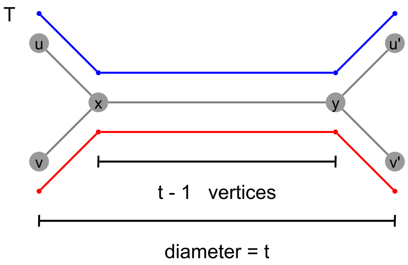
Let $x - y$ be the longest path between two internal vertices. Then, $x - y$ has at most $(t - 1)$ vertices.

This implies that the diameter of the tree T is at most t.





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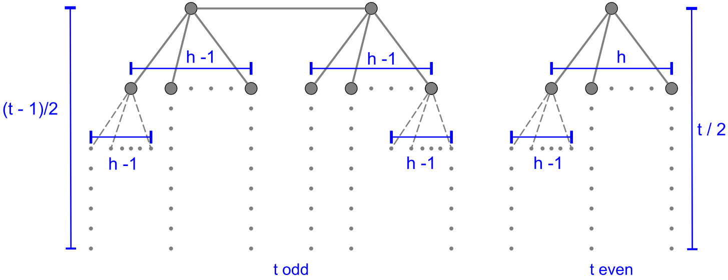

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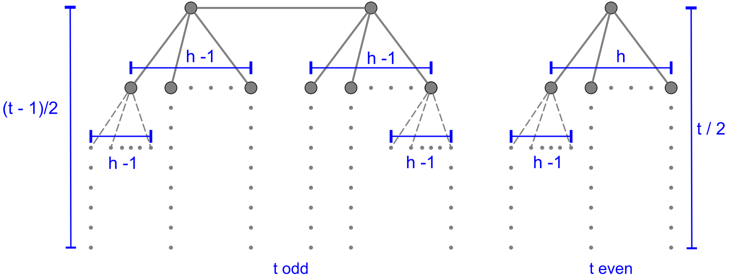

Lemma

Let T be a tree of maximum degree $\leq h$, such that every two leaves are at distance $\leq t$. Then

$$|\mathcal{L}(T)| \leq \begin{cases} 2(h-1)^{\left(\frac{t-1}{2}\right)} & \text{, if } t \text{ is odd,} \\ h(h-1)^{\left(\frac{t}{2}-1\right)} & \text{, if } t \text{ is even.} \end{cases}$$

ORTH$[h+1,2,t]$ and ORTH$[h,2,t]$

We describe some graphs $G \in (\text{ORTH}[h+1,2,t] \setminus \text{ORTH}[h,2,t])$

Which are line graphs of complete graphs, whose orders depend on h and t.

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We describe some graphs $G \in (\mathrm{ORTH}[h+1,2,t] \setminus \mathrm{ORTH}[h,2,t])$

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Theorem

Let h and t be integers with $h \geq 3$ and $t \geq 3$.
If $N = \left\{ n \in \mathbb{N}^* : L(K_n) \in \mathrm{ORTH}[h+1,2,t] \setminus \mathrm{ORTH}[h,2,t] \right\}$, then

$$N = \begin{cases} \left[2(h-1)^{\left(\frac{t-1}{2}\right)} + 1 \ , \ 2h^{\left(\frac{t-1}{2}\right)} \right] & \text{, if } t \text{ is odd, and} \\ \left[h(h-1)^{\left(\frac{t}{2}-1\right)} + 1 \ , \ (h+1)h^{\left(\frac{t}{2}-1\right)} \right] & \text{, if } t \text{ is even.} \end{cases}$$

The largest value of n such that $L(K_n) \in \mathrm{ORTH}[h, 2, t]$

n		t							
		3	**4**	**5**	**6**	**7**	**8**	**9**	**10**
h	**3**	4	6	8	12	16	24	32	48
	4	6	12	18	36	54	108	162	324
	5	8	20	32	80	128	320	512	1.280
	6	10	30	50	150	250	750	1.250	3.750
	7	12	42	72	252	432	1.512	2.592	9.072
	8	14	56	98	392	686	2.744	4.802	19.208
	9	16	72	128	576	1.024	4.608	8.192	36.864
	10	18	90	162	810	1.458	7.290	13.122	65.610

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Example: $L(K_5) \in (\ \mathrm{ORTH}[4, 2, 3] \setminus \mathrm{ORTH}[3, 2, 3]\ )$

ORTH[3, 2, 3] and planar graphs

Lemma

If H is a subdivision of K_5 *and* $G = L(H)$ *then* $G \notin ORTH[3, 2, 3]$

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If H is a subdivision of $K_{3,3}$ and $G = L(H)$ then $G \notin \mathrm{ORTH}[3, 2, 3]$

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Theorem

If a connected twin-free graph G of order ≥ 4 is in $ORTH[3, 2, 3]$, then G is the line graph of a planar graph.

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Such necessary condition is not sufficient.
Example: $K_5 - e$ is planar, but $L(K_5 - e) \notin \mathrm{ORTH}[3,2,3]$

Open questions

- Recognition of the graphs $G \in \mathrm{ORTH}[3, 2, 3]$
- Characterize and determine the complexity of recognizing graphs $G \in \mathrm{ORTH}[3, 3, 3]$

Merci beaucoup !!!