Bounds on Directed star arboricity in some digraph classes

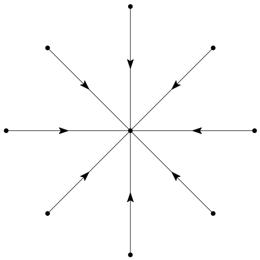
Mourad Baïou, Laurent Beaudou, Vincent Limouzy, Henri Perret du Cray

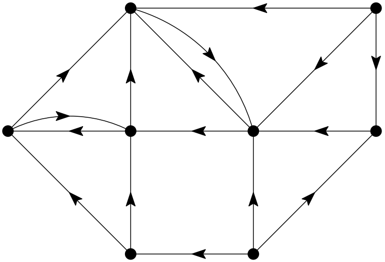
LIMOS, Université Clermont Auvergne

LAGOS 2017

Definitions

A **directed star** is a star with all its arcs oriented towards its center.
A **galaxy** is a set of vertex-disjoint stars.





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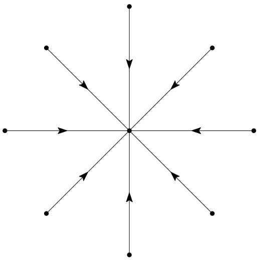

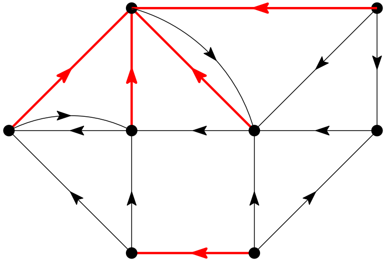

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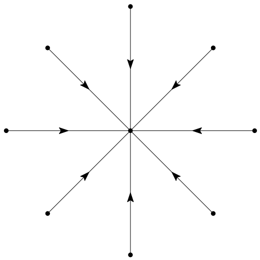

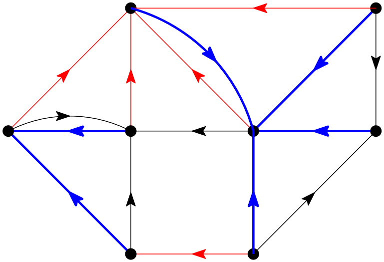

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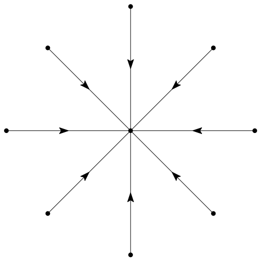

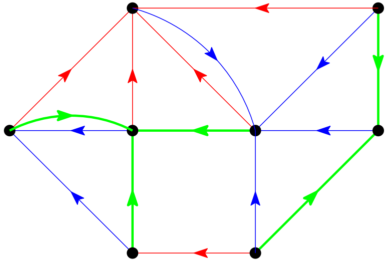

Definitions

Directed star k-coloring problem:

Deciding whether or not there exists a partition of the arcs of a digraph D into k galaxies.

NP-complete for $k \geq 3$ even when restricted to different classes of digraphs (Amini et al. 2010, Baïou et al. 2013).

The **directed star arboricity**, $dst(D)$, of a digraph D, is the minimum number of galaxies needed to cover all the arcs of D.

Known results and Conjectures

O. Amini, F. Havet, F. Huc, and S. Thomassé (2010)

For every digraph D, $dst(D) \leq 2\Delta^+ + 1$.

D. Gonçalves, F. Havet, A. Pinlou, and S. Thomassé (2012)

For every digraph D, $dst(D) \leq \Delta + 1$.

Conjectures (O. Amini, F. Havet, F. Huc, and S. Thomassé (2010))

1. $dst(D) \leq 2\Delta^+$, if $\Delta^+ \geq 2$
2. $dst(D) \leq \Delta$, if $\Delta \geq 3$

$\Delta(D) = max\{d^+(x) + d^-(x), x \in V(D)\}$

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1. $dst(D) \leq 2\Delta^+$, if $\Delta^+ \geq 2$
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- Both conjectures are true when restricted to acyclic digraphs.
- Both conjectures (if true) are tight.
- Conjecture 2 is true for $\Delta = 3$.

k-degenerate digraphs

k-degenerate **graph**

G is empty, or
There is some x with degree at most k such that $(G - x)$ is k-degenerate.

A digraph D is k-**degenerate** if its underlying graph is k-degenerate.

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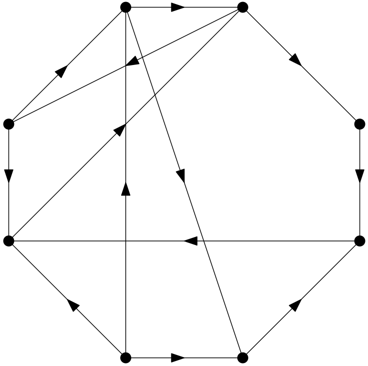

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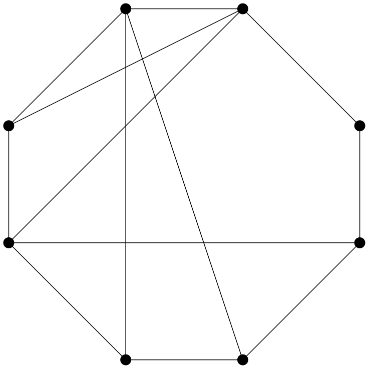

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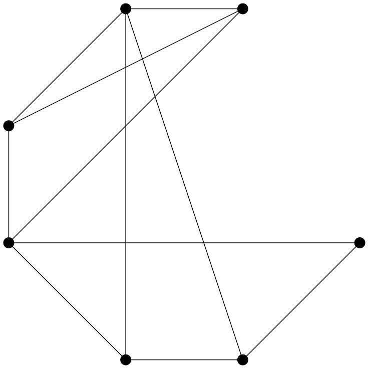

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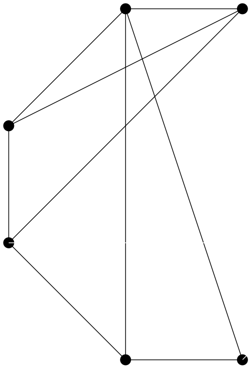

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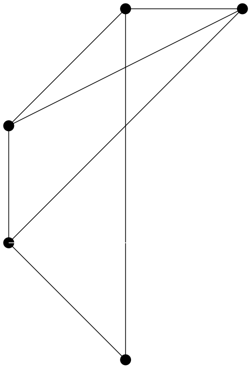

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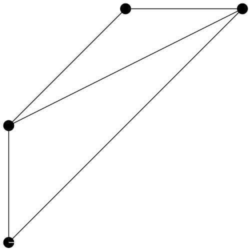

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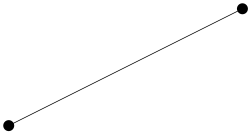

Theorem 1

Let D be a k-degenerate digraph, $dst(D) \leq \Delta^+ + k$.

Sketch of proof

Proof by induction on the number of vertices of D.

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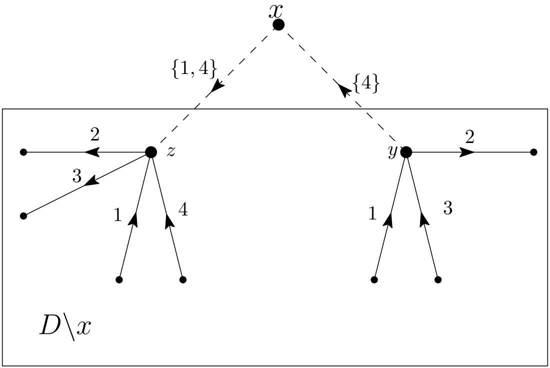
H_n : all k-degenerate oriented graphs on n vertices are $(\Delta^+ + k)$-colorable with at most k colors entering each vertex.

Illustration for $\Delta^+ = 2$ and $k = 2$.



$$d^+(x) = 1$$

Tournaments

A **Tournament** is an orientation of a complete graph.

Theorem 2

Let T be a tournament on n vertices, $n \geq 4$, then $dst(T) \leq \Delta$.

Corollary: if $n \geq 4$, then $dst(T) \leq 2\Delta^+$.

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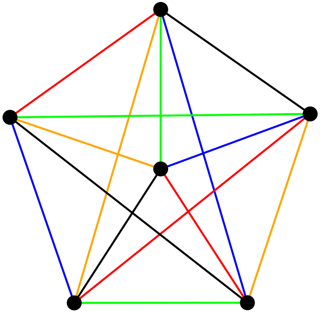
If n is even there is nothing to prove (partition in $n-1$ perfect matchings).



Proof

Theorem 2

Let T be a tournament on n vertices, $n \geq 4$, then $dst(T) \leq \Delta$.

Sketch of the proof:

- remove one vertex,
- color the resulting even sub-tournament.
- extend the coloring using only one additional color.

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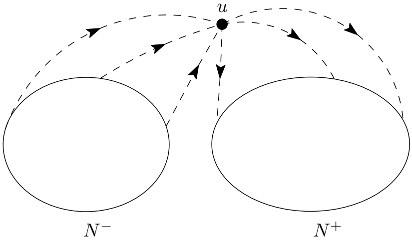

We color the arc entering u with the new color.

$d^+(u) \geq d^-(u)$
$d^-(u) > 0$

Proof

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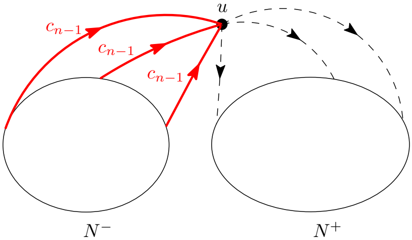
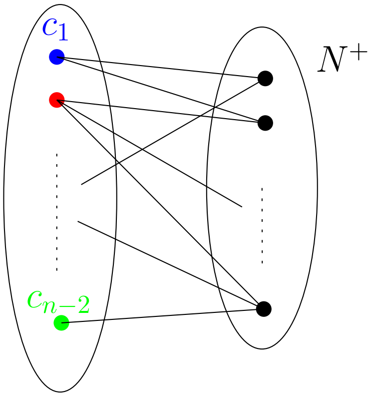


Goal : color the arcs leaving u.

$d^+(u) \geq d^-(u)$
$d^-(u) > 0$

Assign a color to each arc leaving u.



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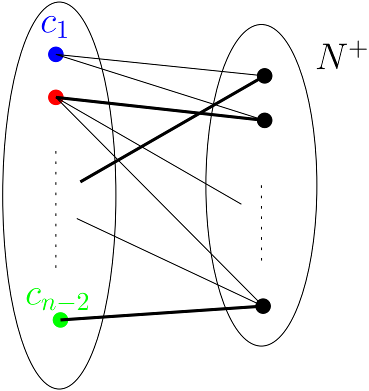
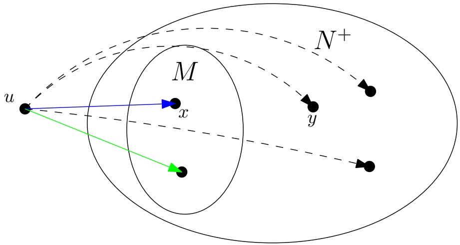


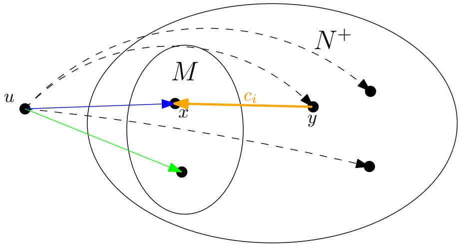
Find a maximum matching.

Color the remaining arcs.



$|N^+| = k$

Color the remaining arcs.



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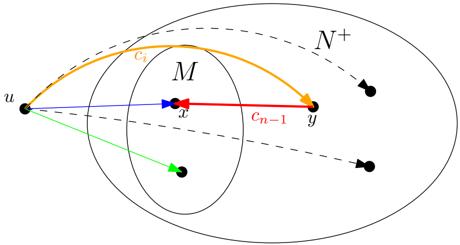

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Thanks for your attention

- O. Amini, F. Havet, F. Huc, et S. Thomassé. WDM and directed star arboricity. Combinatorics, Probability & Computing, 2010.
- D. Gonçalves, F. Havet, A. Pinlou, et S. Thomassé. On spanning galaxies in digraphs. Discrete Applied Mathematics, 2012.