# The Geodetic Hull Number is Hard for Chordal Graphs

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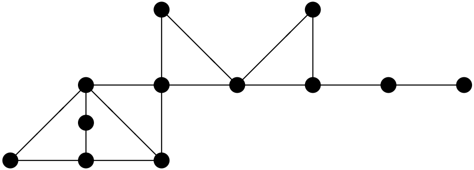
joint work with joint work with S. Bessy, M. Dourado and D. B. Rautenbach

# Geodetic (i.e. Shortest Path) Convexity

- What is the geodetic-convex Hull of a subset $C$ of vertices from $G = (V(G), E(G))$?

The geodetic-convex Hull of $C$ is obtained by iteratively adding to $C^*$ every vertex in the shortest path between two vertices in $C^*$, where initially $C^* = C$.



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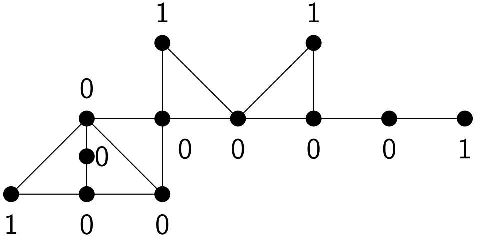


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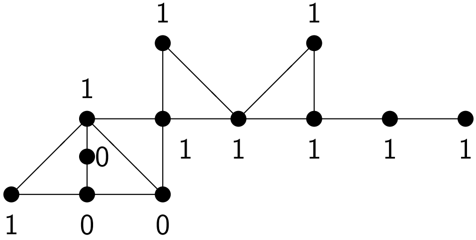


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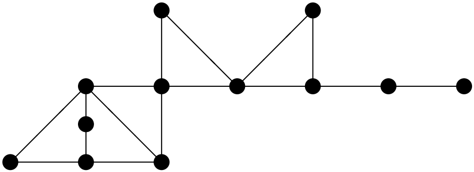
# Geodetic (i.e., Shortest Path) Hull Number

- What is a geodetic-Hull Set and the geodetic-Hull Number $h(G)$ of a graph $G = (V(G), E(G))$?

For a geodetic-Hull Set of $G$ the geodetic-convex Hull equals $V(G)$.

The geodetic-Hull Number is the minimum cardinality of a geodetic-Hull Set in $G$.



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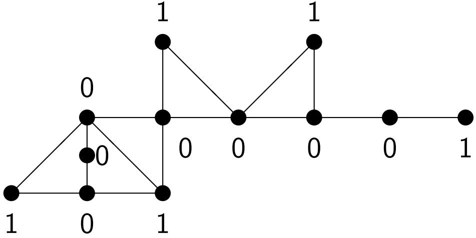


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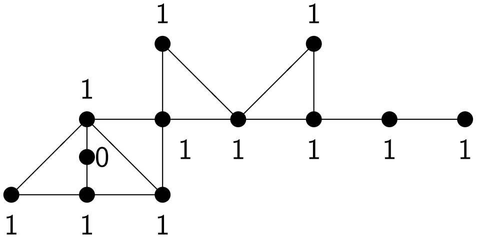


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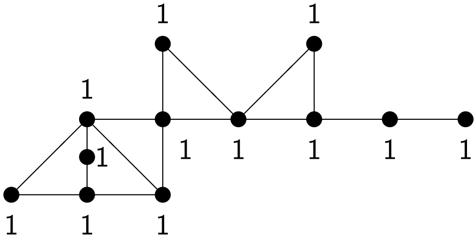


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# In this talk we show...

... that the geodetic hull number is hard for chordal graphs!

That is, as a decision problem:

**Theorem**

*For a given chordal graph $G$, and a given integer $k$, it is NP-complete to decide whether the hull number $h(G)$ of $G$ is at most $k$.*

Clearly the problem is in NP. Now consider the following polynomial reduction.

# Reduction Problem: restricted SATISFIABILITY

Let $\mathcal{C}$ be an instance of SATISFIABILITY consisting of $m$ clauses $C_1, \ldots, C_m$ over $n$ boolean variables $x_1, \ldots, x_n$ such that

- every clause in $\mathcal{C}$ contains at most three literals,
- and, for every variable $x_i$, there are exactly two clauses in $\mathcal{C}$, say $C_{j_i^{(1)}}$ and $C_{j_i^{(2)}}$, that contain the literal $x_i$,
- and exactly one clause in $\mathcal{C}$, say $C_{j_i^{(3)}}$, that contains the literal $\bar{x}_i$,
- and these three clauses are distinct.

It has been shown in WG 2016 that SATISFIABILITY restricted to such instances is still NP-complete.

Now let k = 4n.

# Gadget $G_i$ for variable $x_i$ with $i \in [n]$

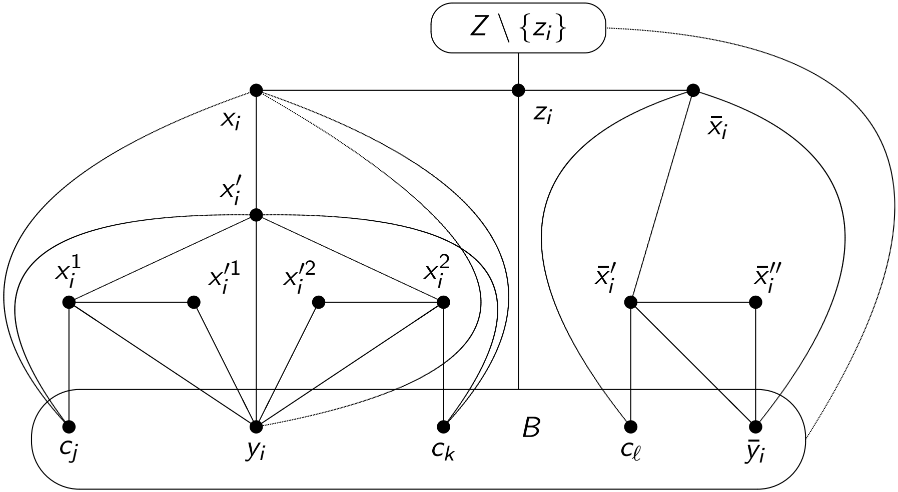



Figure: The vertices and edges added of variable $x_i \in [n]$, where $B \cup Z$ is a clique. (Z $= \{z_i : i \in [n]\}$ and B $= \{c_j : j \in [m]\} \cup \{y_i : i \in [n]\} \cup \{\bar{y}_i : i \in [n]\}$.)

# Observations on the whole Gadget $G$

- The order of $G$ is $12n + m$.
- $G$ is chordal, that is, it admits a *perfect elimination ordering*, which is a linear ordering $v_1, \ldots, v_{12n+m}$ of its vertices such that $v_i$ is simplicial in $G - \{v_1, \ldots v_{i-1}\}$ for every $i$ in $[12n + m]$.
- The eccentricity of $B \cup Z$ is 2.
- The diameter of $G$ is 3.

# First Direction

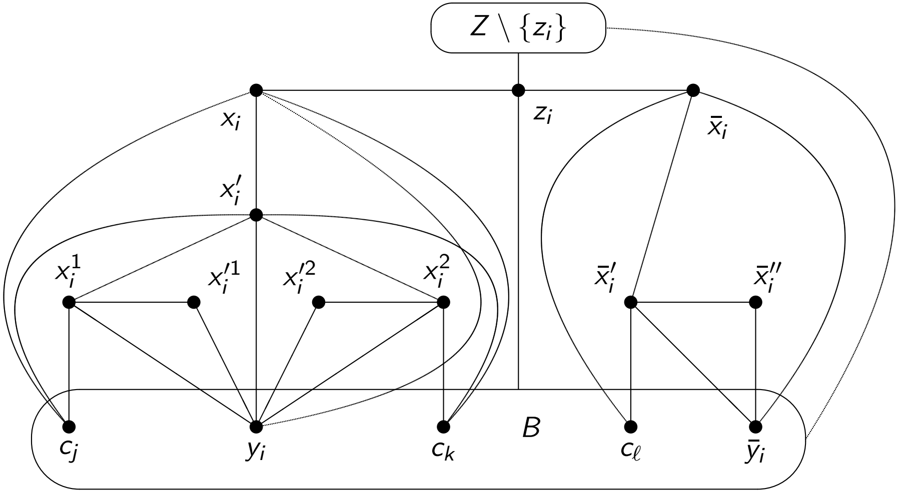
Let $\mathcal{S}$ be a satisfying truth assignment for $\mathcal{C}$.

Let

$$S = \bigcup_{i \in [n]} \left\{x_i'^1, x_i'^2, \bar{x}_i''\right\} \; \cup \bigcup_{i \in [n]:\ x_i \textit{ true in } \mathcal{S}} \left\{x_i\right\} \; \cup \bigcup_{i \in [n]:\ x_i \textit{ false in } \mathcal{S}} \left\{\bar{x}_i\right\}.$$

Clearly, $|S| = 4n = k$.

# First Direction

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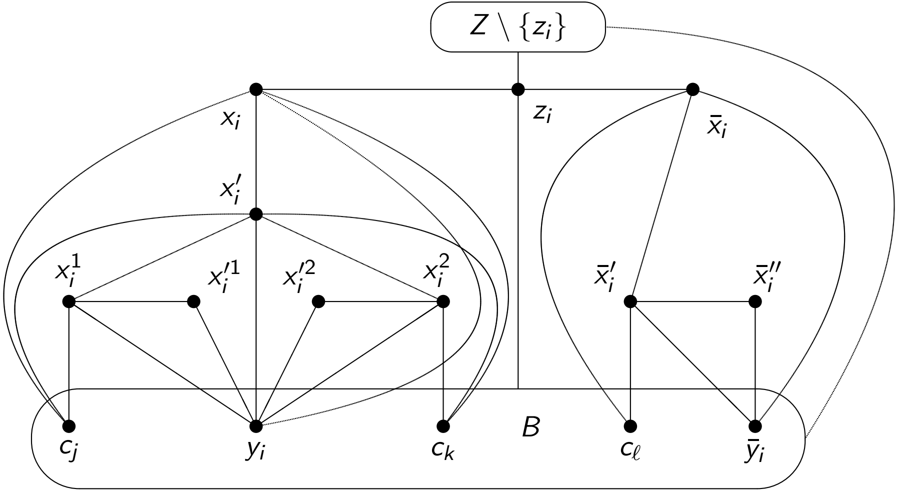
# Second Direction

Let $S$ be a hull set of $G$ of order at most $4n$.

## Claim

*For every $i \in [n]$, the set $\{x_i, z_i, \bar{x}_i\}$ is concave.*

# Second Direction

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# Second Direction

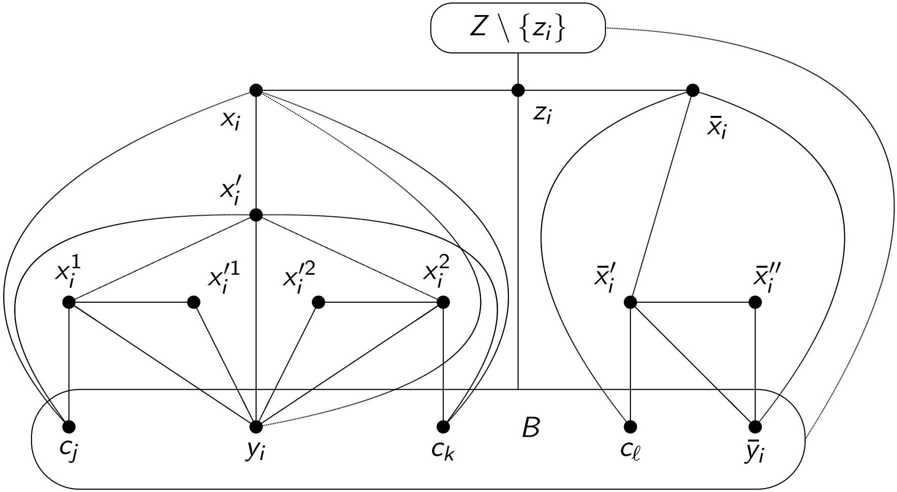
## Claim

*For every* $j \in [m]$*, the set*

$$V_j = \{c_j\} \cup \bigcup_{i \in [n]: j = j_i^{(1)}} \left\{x_i, x_i', x_i^1\right\} \cup \bigcup_{i \in [n]: j = j_i^{(2)}} \left\{x_i, x_i', x_i^2\right\} \cup \bigcup_{i \in [n]: j = j_i^{(3)}} \left\{\bar{x}_i, \bar{x}_i'\right\}$$

*is concave.*

# Second Direction

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# Open Problems

Geodetic Hull Number for $P_k$-free graphs with $5 \leq k \leq 8$

Thank you for the attention!