Clique-cutsets beyond chordal graphs

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Definition

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Definition

A class $\mathcal{G}$ of graphs is *hereditary* if it is closed under induced subgraphs, that is, if $\forall G \in \mathcal{G}$, all (isomorphic copies of) induced subgraphs of G belong to $\mathcal{G}$.

- $\forall \mathcal{F}$, the class of $\mathcal{F}$-free graphs is hereditary.

- We study several classes of graphs defined by forbidding (as induced subgraphs) certain "Truemper configurations" ($\underbrace{\text{thetas, pyramids, prisms}}_{\text{3PCs}}$, and wheels).

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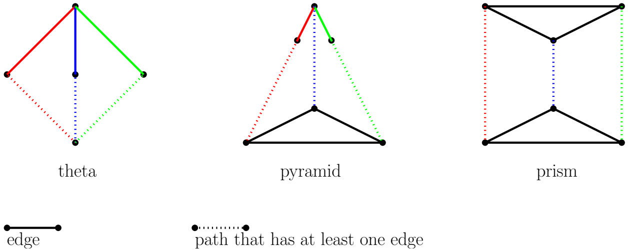
- Truemper configurations have played an important role in the study of complex hereditary graph classes (e.g. perfect graphs, even-hole-free graphs).
  - They sometimes appear as forbidden substructures, and sometimes as configurations around which graphs can be decomposed.
- Let's define Truemper configurations!
  - There are two types: three-path-configurations (3PCs) and wheels.

Definition

A *three-path-configuration* (or *3PC*) is any theta, pyramid, or prism.



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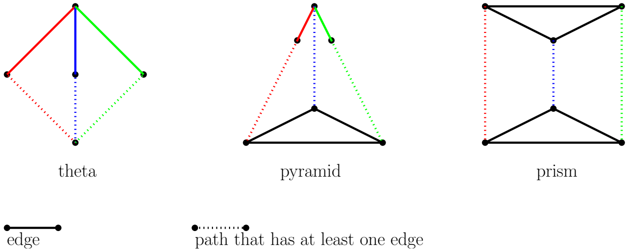
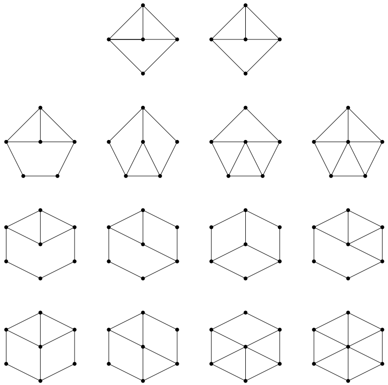


- Every 3PC contains a hole (i.e. induced cycle of length ≥ 4). In fact, every 3PC contains three distinct holes.

Definition

A *wheel* is a graph that consists of a hole[a] and an additional vertex that has at least three neighbors in the hole.

[a] A *hole* is an induced cycle of length ≥ 4.



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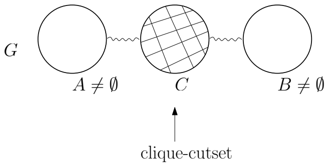
Theorem [Dirac, 1961]

Every chordal graph either is complete or admits a clique-cutset.



Definition

A graph G is *universally signable* if for every prescription of parities to the holes of G, there exists an assignment of zero or one weights to the edges of G s.t. for each hole, the sum of weights of its edges has prescribed parity, and for every triangle, the sum of weights of its edges is odd.

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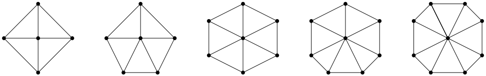
- Chordal graphs are universally signable (we simply assign weight one to each edge).

Theorem [Conforti, Cornuéjols, Kapoor, Vušković, 1997]

A graph is universally signable iff it contains no Truemper configurations. Furthermore, if G is a universally signable graph, then either G is a complete graph or a hole, or G admits a clique-cutset.

Definition

A *universal wheel* is a wheel that consists of a hole and an additional vertex that is adjacent to all the vertices of the hole.



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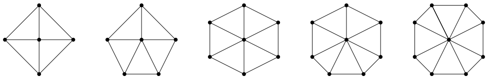
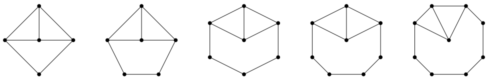


Definition

A *twin wheel* is a wheel that consists of a hole and an additional vertex that is adjacent to three consecutive vertices of the hole, and to no other vertex of the hole.



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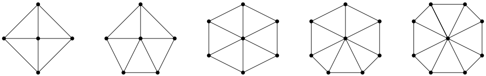

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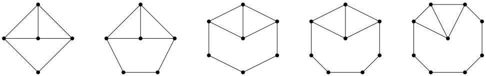

Definition

A *proper wheel* is a wheel that is neither a universal wheel nor a twin wheel.

- $\mathcal{G}_{\mathsf{UT}}$ - class of (3PC, proper wheel)-free graphs
  - The only Truemper configurations that graphs in $\mathcal{G}_{\mathsf{UT}}$ may contain are the universal wheels and twin wheels.

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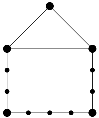
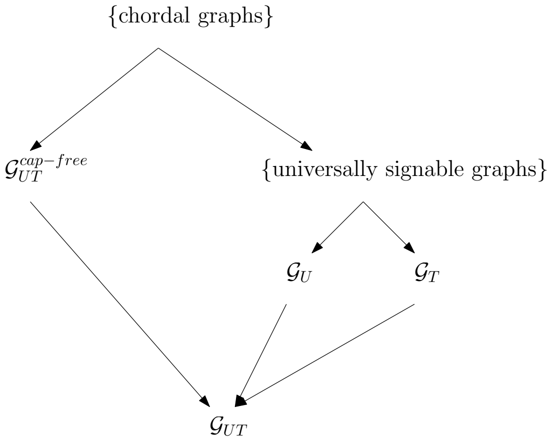
- $\mathcal{G}_{\mathsf{UT}}^{\text{cap-free}}$ - class of (3PC, proper wheel, cap)-free graphs



cap



- Arrows indicate inclusion.

Theorem

Every graph in $\mathcal{G}_{\mathsf{UT}}$ either belongs to $\mathcal{B}_{\mathsf{UT}}$ or admits a clique-cutset.

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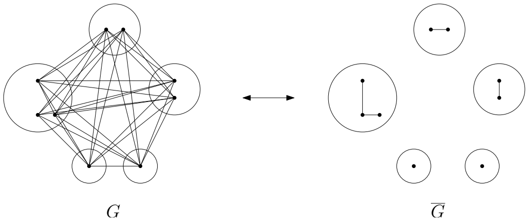
Theorem

Every graph in $\mathcal{G}_{\mathsf{UT}}^{\mathsf{cap\text{-}free}}$ either belongs to $\mathcal{B}_{\mathsf{UT}}^{\mathsf{cap\text{-}free}}$ or admits a clique-cutset.

- Fact: $\mathcal{B}_{\mathsf{UT}} \subseteq \mathcal{G}_{\mathsf{UT}}$, $\mathcal{B}_{\mathsf{U}} \subseteq \mathcal{G}_{\mathsf{U}}$, $\mathcal{B}_{\mathsf{T}} \subseteq \mathcal{G}_{\mathsf{T}}$, $\mathcal{B}_{\mathsf{UT}}^{\mathsf{cap\text{-}free}} \subseteq \mathcal{G}_{\mathsf{UT}}^{\mathsf{cap\text{-}free}}$.

Definition

An *anticomponent* of a graph G is an induced subgraph H of G s.t. $\overline{H}$ is a component of $\overline{G}$. An anticomponent is *trivial* if it has just one vertex, and it is *nontrivial* if it has at least two vertices.



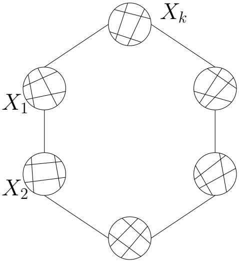
- Anticomponents of a graph G are "complete" to each other in G, i.e. all possible edges between them are present in G.
  - G is the join of its anticomponents.
- Trivial anticomponents together form a clique that is complete to the rest of the graph.

Definition

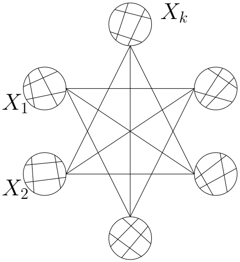
A *k-hyperhole* ($k \geq 4$) is any graph obtained from a k-hole by substituting a complete graph for each vertex of the k-hole.

Definition

A *k-hyperantihole* ($k \geq 4$) is any graph obtained from a k-antihole (i.e. complement of a k-hole) by substituting a complete graph for each vertex of the k-antihole.



k-hyperhole

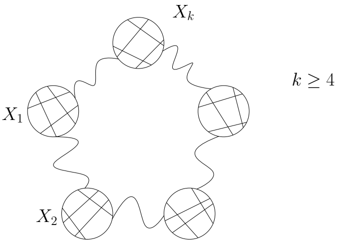


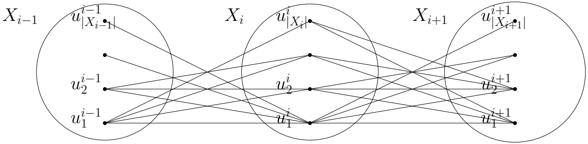
k-hyperantihole

$k \geq 4$

Definition

A *k-ring* ($k \geq 4$) is a graph R whose vertex set can be partitioned into k nonempty sets, say $X_1, \ldots, X_k$, s.t. $\forall i \in \mathbb{Z}_k$, X_i can be ordered as $X_i = \{u^i_1, \ldots, u^i_{|X_i|}\}$ s.t.
$X_i \subseteq N_R[u^i_{|X_i|}] \subseteq \cdots \subseteq N_R[u^i_1] = X_{i-1} \cup X_i \cup X_{i+1}$.



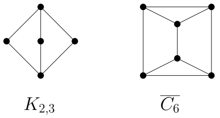


Definition

$\mathcal{B}_{\mathsf{UT}}$ is the class of all graphs G that satisfy at least one of the following:

1. G has exactly one nontrivial anticomponent, and this anticomponent is a ring of length ≥ 5;
2. G is (long hole[a], $K_{2,3}$, $\overline{C_6}$)-free;
3. $\alpha(G) = 2$, and every anticomponent of G is either a 5-hyperhole or a $(C_5, \overline{C_6})$-free graph.

[a] A *long hole* is a hole of length ≥ 5.



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Definition

$\mathcal{B}_{\mathsf{U}}$ is the class of all graphs G that satisfy one of the following:

1. G has exactly one nontrivial anticomponent, and this anticomponent is a long hole[a];
2. all nontrivial anticomponents of G are isomorphic to $\overline{K_2}$.

[a] A *long hole* is a hole of length ≥ 5.

Theorem

Every graph in $\mathcal{G}_{\mathsf{U}}$ either belongs to $\mathcal{B}_{\mathsf{U}}$ or admits a clique-cutset.

Definition

$\mathcal{B}_\mathsf{T}$ is the class of all complete graphs, rings, and 7-hyperantiholes.

Theorem

Every graph in $\mathcal{G}_\mathsf{T}$ either belongs to $\mathcal{B}_\mathsf{T}$ or admits a clique-cutset.

Definition

$\mathcal{B}_{\mathrm{UT}}^{\text{cap-free}}$ is the class of all graphs G that satisfy one of the following:

1. G has exactly one nontrivial anticomponent, and this anticomponent is a hyperhole of length ≥ 6;
2. each anticomponent of G is either a 5-hyperhole or a chordal cobipartite graph.

Theorem

Every graph in $\mathcal{G}_{\mathrm{UT}}^{\text{cap-free}}$ either belongs to $\mathcal{B}_{\mathrm{UT}}^{\text{cap-free}}$ or admits a clique-cutset.

Definition

A hereditary class $\mathcal{G}$ is χ-bounded if $\exists f : \mathbb{N}^+ \to \mathbb{N}^+$ (called a χ-*bounding function* for $\mathcal{G}$), s.t. $\forall G \in \mathcal{G}$, $\chi(G) \leq f(\omega(G))$.

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Theorem [Kühn, Osthus, 2004]

The class of theta-free graphs is χ-bounded.

- The χ-bounding function is superexponential.

Corollary

Classes $\mathcal{G}_{\mathsf{UT}}, \mathcal{G}_{\mathsf{U}}, \mathcal{G}_{\mathsf{T}}, \mathcal{G}_{\mathsf{UT}}^{\text{cap-free}}$ are χ-bounded.

class	χ-bound.	optimal
$\mathcal{G}_{\mathsf{UT}}$	$\chi \leq 2\omega^4$	no
$\mathcal{G}_{\mathsf{U}}$	$\chi \leq \omega + 1$	yes
$\mathcal{G}_{\mathsf{T}}$	$\chi \leq \lfloor \frac{3}{2}\omega \rfloor$	?
$\mathcal{G}_{\mathsf{UT}}^{\text{cap-free}}$	$\chi \leq \lfloor \frac{3}{2}\omega \rfloor$	yes

- Bounds for $\mathcal{G}_{\mathsf{U}}, \mathcal{G}_{\mathsf{T}}, \mathcal{G}_{\mathsf{UT}}^{\text{cap-free}}$ readily follow from our decomposition theorems.
- The bound for $\mathcal{G}_{\mathsf{UT}}$ follows from:
  - Another decomposition theorem for $\mathcal{G}_{\mathsf{UT}}$, which states that every graph in $\mathcal{G}_{\mathsf{UT}}$ either is cap-free (and therefore belongs to $\mathcal{G}_{\mathsf{UT}}^{\text{cap-free}}$) or admits a "small cutset" (the size of the cutset is bounded by a function of the clique number);
  - Our bound for $\mathcal{G}_{\mathsf{UT}}^{\text{cap-free}}$ from the table;
  - The fact that every graph of "large" chromatic number contains a "highly connected" induced subgraph of "large" chormatic number (P., Thomassé, Trotignon, 2016).

class	recognition	MWSSP	MWCP	ColP
$\mathcal{G}_{\mathrm{UT}}$	$O(n^6)$	?	NP-hard	?
$\mathcal{G}_{\mathrm{U}}$	$O(nm)$	$O(nm)$	$O(nm)$[1]	$O(nm)$
$\mathcal{G}_{\mathrm{T}}$	$O(n^3)$	$O(n^2m)$	$O(nm)$	?
$\mathcal{G}_{\mathrm{UT}}^{\text{cap-free}}$	$O(n^5)$	$O(n^3)$	$O(n^3)$	$O(n^3)$

- MWSSP = maximum weight stable set problem
- MWCP = maximum weight clique problem
- ColP = optimal coloring problem

- Our algorithmic results rely on:
  - our structural results;
  - algorithms for clique-cutsets (Tarjan, 1985);
  - the coloring algorithm for hyperholes (Narayanan, Shende, 2001);
  - algorithms for handling chordal graphs.

[1] Aboulker, Charbit, Trotignon, Vušković, 2015.

Definition

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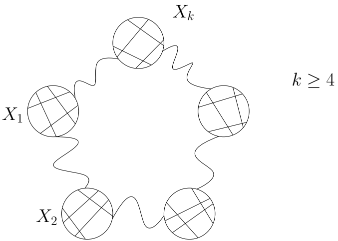

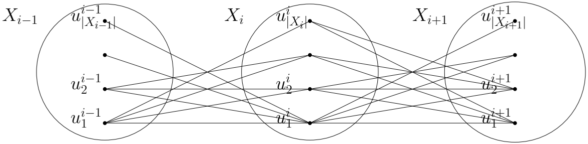

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- Even rings: Yes (easy).
- Odd rings: Unknown.
    - If odd rings are polynomially colorable, then so are graphs in $\mathcal{G}_{\mathsf{T}}$.
    - If coloring odd rings is NP-hard, then coloring even-hole-free graphs is NP-hard.

That's all.

Thanks for listening!

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